LMI Methods in Optimal and Robust Control

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Lecture 20: LMI/SOS Tools for the Study of Hybrid Systems

Stability Concepts

There are several classes of problems for which we would like to prove stability:

- Stability under Arbitrary Switching
 - Sometimes called Differential Inclusions
 - Similar to Time-Varying uncertainty, but discrete

 $\dot{x}(t) \in \{A_1 x(t), A_2 x(t)\}$

- Stability under State-Dependent Switching
 - Alternatively, Does there exist a stabilizing switching law?
 - Often used to stabilize systems such as the inverted pendulum.
- Stability under Arbitrary Switching with Dwell-Time restrictions
 - Places a lower limit on $\tau_{i+1} \tau_i$.
 - Often used for hysteresis (e.g. Thermostat problem)

There are three tools we will use

- Quadratic Stability
- Common non-quadratic Lyapunov functions
 - ▶ or Multiple (state-dependent) Lyapunov functions (w/ continuity)
- Switched Lyapunov functions.

2 Non-intuitive Facts About Switched Stability

Fact 1: Stability of each subsystem $\{A_1, A_2\}$ does not guarantee stability under arbitrary switching.



Figure: 2 Unstable Systems (a,b) can be Stabilized (c) or Destabilized (d)

Fact 2: Smart switching can stabilize two unstable subsystems $\{A_1, A_2\}$.



Figure: 2 Stable Systems (a,b) can be Stabilized (c) or Destabilized (d)

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Quadratic Stability

Theorem 1.

The switched system $\dot{x}(t) \in \{A_1x(t), A_2x(t)\}$ is stable under arbitrary switching if there exists some P > 0 such that

 $\begin{aligned} A_1^T P + P A_1 < 0 & \quad \text{and} \\ A_2^T P + P A_2 < 0 & \end{aligned}$



This implies that BOTH A_1 and A_2 are Hurwitz (Necessity).

• But A_1 and A_2 Hurwitz is not Sufficient.

For example, consider

$$A_1 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & -10 \\ .1 & -1 \end{bmatrix}$$

- A_1 and A_2 are both Hurwitz.
- $\dot{x}(t) \in \{A_1x(t), A_2x(t)\}$ is stable under arbitrary switching
- There is no common quadratic Lyapunov function for A_1 and A_2 !

Quadratic Stability and Commuting Matrices

Commuting Matrices: If A_1 , A_2 are Hurwitz and commute, we have quadratic Stability.

- Sufficient, not necessary for quadratic stability.
- Also for larger sets, $\{A_i\}$ if all pairs commute.
- Easier to check than the LMI

Simply Test if

 A_i is Hurwitz

and

$$A_i A_j - A_j A_i = 0 \qquad \forall i, j.$$

Common Non-Quadratic Lyapunov Functions

Theorem 2.

The switched system $\dot{x}(t) \in \{f_1(x(t)), f_2(x(t))\}$ is stable under arbitrary switching if there exists some $V(x) > \alpha \|x\|^2$ such that

 $abla V(x)^T f_1(x) < 0 \quad \forall x \quad \text{and}$ $abla V(x)^T f_1(x) < 0 \quad \forall x$

Converse Result: If $\dot{x} = f_p(x(t))$ is asymptotically stable and f_p is locally Lipschitz, then there exists a common Lyapunov function.

• If $\dot{x} \in \{A_i x(t)\}_i$, then V can be chosen to have the form

$$V(x) = \max_{i} (c_i^T x)^2$$

However, this is difficult to enforce, and an easier approach is to use SOS: Find V such that ∇

$$egin{aligned} V(x) &- \epsilon(\sum_i x_i^6) & ext{ is SOS} \ &-
abla V(x)^T f_i(x) & ext{ is SOS for all } i \end{aligned}$$

Common Non-Quadratic Lyapunov Functions

Example: Consider the System (Boyd)

$$\dot{x}(t) \in \mathbf{Co}\{A_1, A_2\}$$

 $A_1 = \begin{bmatrix} -100 & 0\\ 0 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 8 & -9\\ 120 & -18 \end{bmatrix}$

This system is stable, but not Quadratically Stable.

• Stability can be proven using the Lyapunov function

$$V(x) := \max\{x^T P_1 z, x^T P_2 x\}$$
$$P_1 = \begin{bmatrix} 14 & -1\\ -1 & 1 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0 & 0\\ 0 & 1 \end{bmatrix}$$

Stability under State-Dependent Switching

Multiple Lyapunov Functions with Continuity

Quadratic Stability is mostly useless for Analysis of state-dependent switching.

• Since it establishes stability under arbitrary switching

Consider a simple model:

$$\dot{x}(t) = \begin{cases} f_i(x(t)), & x(t) \in D_i. \end{cases}$$

where the D_i are disjoint except at the Guard sets $G_{(i,j)}$ for $(i,j) \in E$.

• Recall $E := \{e_i\}$ is the set of possible transitions from domain i to j.

Theorem 3.

Suppose there exist Lyapunov functions $V_i(x)$ such that $V_i(x) \ge \epsilon ||x||^2$ for all i and

$$abla V_i(x)^T f_i(x) \le 0$$
 for all $x \in D_i$, $i = 1, \cdots, k$

and

$$V_i(x) = V_j(x) \qquad \text{for all } x \in G_{(i,j)} \text{for all } i,j \ : \ (i,j) \in E$$

Basically
$$V(x) = \begin{cases} V_i(x), & x \in D_i \end{cases}$$

is a common, continuous, non-quadratic Lyapunov function.

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Multiple Lyapunov Functions with Continuity

Consider the System:

$$\dot{x}(t) = \begin{cases} A_1 x(t), & \text{if } x_1 \ge 0\\ A_2 x(t), & \text{otherwise.} \end{cases}$$
$$A_1 = \begin{bmatrix} \gamma & -1\\ 2 & \gamma \end{bmatrix}, \qquad A_2 = \begin{bmatrix} \gamma & -2\\ 1 & \gamma \end{bmatrix}$$

where $\gamma < 0$ implies each subsystem is stable



• There is no global common quadratic Lyapunov function. However, if we define

$$P_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad P_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

Then $V_1(x) = x^T P_1 x = x^T P_2 x = V_2(x)$ when $x_1 = 0$.

• Furthermore, $A_1^T P_1 + P_1 A_1 < 0$ and $A_2^T P_2 + P_2 A_2 < 0$.

Multiple Lyapunov Functions with Continuity

The S-Procedure... Again

Our goal is to find $V_1(x) = x^T P_1 x$ and $V_2(x) = x^T P_2 x$ with $P_1, P_2 > 0$ such that

$$\begin{aligned} x^T P_1 x - x^T P_2 x &= 0 \qquad \forall x \in \{x \ : \ x_1 = 0 \\ x^T (A_1^T P_1 + P_1 A_1) x &\leq 0 \qquad \forall x \in \{x \ : \ x^T S x \leq 0\} \\ x^T (A_2^T P_2 + P_2 A_2) x &\leq 0 \qquad \forall x \in \{x \ : \ x^T S x \geq 0\} \end{aligned}$$

Of course, we could use the P-Satz...

- But lets put away the big guns for now...
- Lets consider the constraints separately.

Instead, recall the S-procedure:

S-Procedure: $x^T P x \ge 0$ for all x such that $x^T S x \ge 0$ if there exists a $\tau > 0$ such that $P - \tau S \ge 0$. So...

$$-x^{T}(A_{1}^{T}P_{1} + P_{1}A_{1})x \ge 0 \qquad \forall x \in \{x : x^{T}Sx \le 0\}$$

if there exists $\tau > 0$ such that

$$A_1^T P_1 + P_1 A_1 + \tau S < 0$$

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Multiple Lyapunov Functions with Continuity

The S-Procedure... Nullstellensatz Form!

Now, lets examine the constraint

$$x^T P_1 x - x^T P_2 x = 0$$
 $\forall x \in \{x : c^T x = 0\}$

lf

$$P_1 - P_2 + tc^T + ct^T = 0$$

Then

$$x^{T}(P_{1} - P_{2})x = x^{T}(P_{1} - P_{2} + tc^{T} + ct^{T})x = 0$$
 when $c^{T}x = 0$

Therefore, if we can divide the guard set into lines, then we can enforce continuity

Consider: $x : x_1 x_2 = 0$

• This can be represented as the union of $x_1 = 0$ and $x_2 = 0$.

Stabilization by State-Dependent Switching

Theorem 4.

Suppose there exists some $\alpha \in [0,1]$ such that

 $\alpha A_1 + (1-\alpha)A_2$ is Hurwitz

then there exists a state-dependent switching law which quadratically stabilizes the systems.

Note: This is also *Necessary* for **Quadratic** Stabilization. **Switching Law:** To test the condition, we find some P > 0 such that

$$\alpha(A_1^T P + PA_1) + (1 - \alpha)(A_2^T P + PA_2) < 0$$

This is bilinear in α and P but can be done by gridding α .

• Since $\alpha > \text{and } (1 - \alpha) > 0$ this implies that for any x, either

$$x^T(A_1^TP+PA_1)x<0\qquad \text{or}\qquad x^T(A_1^TP+PA_1)x<0$$

Then choose the switching law

$$\dot{x}(t) = \begin{cases} A_1 x(t) & x^T (A_1^T P + P A_1) x < 0\\ A_2 x(t) & \text{otherwise} \end{cases}$$

Stabilization by State-Dependent Switching

Multiple Subsystems

Can be extended to multiple subsystems:

• Find λ_i such that $\sum_i \lambda_i = 1$, $\lambda_i \ge 0$ and

 $\sum_{i} \alpha_{i} A_{i} \qquad \text{is Hurwitz}$

• No subsystem need be stable.



Arbitrary Switching with Dwell-Time Restrictions

Multiple Lyapunov Functions without Continuity

We can extend the concept of multiple Lyapunov functions to relax continuity.

- Associate one Lyapunov function to each mode.
- Require that each function is decreasing at sequential points of activation.

Theorem 5.

Let each mode $\dot{x} = f_q(x)$ be globally asymptotically stable with Lyapunov functions V_q . The switched system is stable if for every execution (I, T, p, C)

$$V_q(x(\tau_j)) - V_q(x(\tau_k)) < 0$$

for every $i, j \in I$ such that $p_i = p_j = q$ and $p_k \neq q$ for any i < k < j.



Arbitrary Switching with Dwell-Time Restrictions



This only works if we can:

- Bound the decrease during interval $T_i = [\tau_i, \tau_{i+1}]$
- Bound the increase during intervals T_{i+1}, \cdots, T_{j-1}

Controlling Uncontrollable Systems by Switching

The Inverted Pendulum

The inverted pendulum cannot be stabilized by continuous feedback.

- The Domain (circle) is not a contractible set.
 - Requires a continuous function with $H(0,\theta) = \theta$ and $H(1,\theta) = 0$.
 - A Smooth path from any point to the origin.

Model:

$$\ddot{x} = u$$
$$J\ddot{\theta} = mgl\sin\theta - ml\cos\theta u$$

Instead, we define 2 control laws

- 1. Energy maximization when θ is large (bottom)
- 2. Linearized Control when θ is small (top)





The Inverted Pendulum

Energy Maximization

To get to the top, pendulum needs 2mgl of Energy. Ignoring the cart, the Energy of the System is

$$E = \frac{1}{2}J\dot{\theta}^2 + mgl\left(1 + \cos\theta\right)$$

Taking the derivative

 $\dot{E}=-ml\dot{\theta}\cos\theta u$

The input which maximizes this energy gain is

 $u(t) = \mathsf{sat}(u)\mathsf{sign}(\dot{\theta}\cos\theta)$

where sat(u) is the maximum acceleration.

- Discontinuous, due to sign function.
- $\dot{E} = 0$ if $\cos \theta = 0$
- May require multiple swings.

The Inverted Pendulum

Multiple Swings and Multiple Pendula

The Controller is:

$$u(t) = \mathsf{sat}(u)\mathsf{sign}(\dot{\theta}\cos\theta)$$

If Energy 2mgl is not achieved prior to $|\theta| = \frac{\pi}{2}$, we need multiple swings.

- Controller reverses when $\dot{\theta} = 0$.
- Don't Forget Linear Control at the top!

Multi-Swing Geometry:

Link: Double Inverted Pendulum Link: Triple Inverted Pendulum







Conclusion!

Stuff we didn't get to:

- Systems with Delay.
- Control of PDE Systems.
- The rest of Nonlinear Control, Hybrid Systems, etc.

For more details on these and other topics, consult the recommended texts and references therein.



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