Spacecraft and Aircraft Dynamics

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Lecture 10: Linearized Equations of Motion

In this Lecture we will cover:

Linearization of 6DOF EOM

- Linearization of Motion
- Linearization of Forces
  - Discussion of Coefficients

Longitudinal and Lateral Dynamics

- Omit Negligible Terms
- Decouple Equations of Motion

Review: 6DOF EOM

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = m \begin{bmatrix} \dot{u} + qw - rv \\ \dot{v} + ru - pw \\ \dot{w} + pv - qu \end{bmatrix}$$

and

$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{p} - I_{xz}\dot{r} - qpI_{xz} + qrI_{zz} - rqI_{yy} \\ I_{yy}\dot{q} + p^2 I_{xz} - prI_{zz} + rpI_{xx} - r^2 I_{xz} \\ -I_{xz}\dot{p} + I_{zz}\dot{r} + pqI_{yy} - qpI_{xx} + qrI_{xz} \end{bmatrix}$$

Linearization

Consider our collection of variables:

$$X,\, Y,\, Z,\, p,\, q,\, r,\, L,\, M,\, N,\, u,\, v,\, w... \qquad \text{also don't forget } \phi, \theta,\, \psi.$$

To Linearize:
Step 1: Choose Equilibrium point:
$X_0,\, Y_0,\, Z_0,\, p_0,\, q_0,\, r_0,\, L_0,\, M_0,\, N_0,\, u_0,\, v_0,\, w_0$.

Step 2: Substitute.

$$\begin{array}{ccc}
u(t) = u_0 + \Delta u(t) & v(t) = v_0 + \Delta v(t) & w(t) = u_0 + \Delta w(t) \\
p(t) = p_0 + \Delta p(t) & q(t) = q_0 + \Delta q(t) & r(t) = r_0 + \Delta r(t) \\
X(t) = X_0 + \Delta X(t) & Y(t) = Y_0 + \Delta Y(t) & Z(t) = Z_0 + \Delta Z(t) \\
L(t) = L_0 + \Delta L(t) & M(t) = M_0 + \Delta M(t) & N(t) = N_0 + \Delta N(t)
\end{array}$$

Step 3: Eliminate small nonlinear terms. e.g.

$$\Delta u(t)^2 = 0, \qquad \Delta u(t)\Delta r(t) = 0, \qquad \text{etc.}$$

For steady flight, let

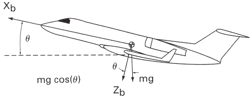
$$\begin{aligned} u_0 &\neq 0 & v_0 &= 0 & w_0 &= 0 \\ p_0 &= 0 & q_0 &= 0 & r_0 &= 0 \\ X_0 &\neq 0 & Y_0 &= 0 & Z_0 &\neq 0 \\ L_0 &= 0 & M_0 &= 0 & N_0 &= 0 \end{aligned}$$

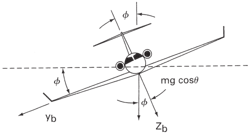
Other Equilibrium Factors

- Weight

Weight: Pitching Motion changes force distribution.

$$X_0 = mg\sin\theta_0$$
$$Z_0 = mg\cos\theta_0$$





Linearization

Step 2: Substitute into EOM

We use trig identities and small angle approximations ($\Delta\theta$ small):

$$\begin{aligned}\sin(\theta_0 + \Delta\theta) &= \sin\theta_0 \cos\Delta\theta + \cos\theta_0 \sin\Delta\theta \\ &\cong \sin\theta_0 + \Delta\theta \cos\theta_0\end{aligned}$$

$$\begin{aligned}\cos(\theta_0 + \Delta\theta) &= \cos\theta_0 \cos\Delta\theta - \sin\theta_0 \sin\Delta\theta \\ &\cong \cos\theta_0 - \Delta\theta \sin\theta_0\end{aligned}$$

Linearization

Step 2: Substitute into EOM

Substituting into EOM, and ignoring 2nd order terms, we get

$$\begin{aligned}
\Delta\dot{u} + \Delta\theta g \cos\theta_0 &= \frac{\Delta X}{m} \\
\Delta\dot{v} - \Delta\phi g \cos\theta_0 + u_0 \Delta r &= \frac{\Delta Y}{m} \\
\Delta\dot{w} + \Delta\theta g \sin\theta_0 - u_0 \Delta q &= \frac{\Delta Z}{m} \\
\Delta\dot{p} &= \frac{I_{zz}}{I_{xx}I_{zz} - I_{xz}^2}\Delta L + \frac{I_{xz}}{I_{xx}I_{zz} - I_{xz}^2}\Delta N \\
\Delta\dot{q} &= \frac{\Delta M}{I_{yy}} \\
\Delta\dot{r} &= \frac{I_{xz}}{I_{xx}I_{zz} - I_{xz}^2}\Delta L + \frac{I_{xx}}{I_{xx}I_{zz} - I_{xz}^2}\Delta N
\end{aligned}$$

Note these are coupled with θ, ϕ.

We include expressions for θ, ϕ.

$$\Delta\dot{\theta} = \Delta q$$
$$\Delta\dot{\phi} = \Delta p + \Delta r \tan\theta_0$$
$$\Delta\dot{\psi} = \Delta r \sec\theta_0$$

For steady-level flight, $\theta_0 = 0$, so we can simplify

$$\Delta\dot{\theta} = \Delta q$$
$$\Delta\dot{\phi} = \Delta p$$
$$\Delta\dot{\psi} = \Delta r$$

which is what we will mostly do.

Linearization

Step 2: Substitute into EOM

We can also express the equations for translational motion

$$\Delta\dot{x} = \delta u \cos\theta_0 - u_0 \Delta\theta \sin\theta_0 + \Delta w \sin\theta_0$$
$$\Delta\dot{y} = u_0 \Delta\psi \cos\theta_0 + \Delta v$$
$$\Delta\dot{z} = -\delta u \sin\theta_0 - u_0 \Delta\theta \cos\theta_0 + \Delta w \cos\theta_0$$

So now we have 12 equations and 12 variables.

But Wait **But Wait!!! There's More!!!**

Recall the forces and moments depend on motion and controls: e.g. $\Delta X(u, v, w, \cdots, \delta_e, \delta_t)$.

- More Variables!
- More Nonlinear Terms!

Out of $(u, v, w, \dot{u}, \dot{v}, \dot{w}, p, q, r, \delta_a, \delta_e, \delta_r, \delta_T)$, we make the restrictive assumptions on form (Why?):

- $\Delta X(\Delta u, \Delta w, \delta_e, \delta_T)$
- $\Delta Y(\Delta v, \Delta p, \Delta r, \delta_r)$
- $\Delta Z(\Delta u, \Delta w, \Delta \dot{w}, \Delta q, \delta_e, \delta_T)$
- $\Delta L(\Delta v, \Delta p, \Delta r, \delta_r, \delta_a)$
- $\Delta M(\Delta u, \Delta w, \Delta \dot{w}, \Delta q, \delta_e, \delta_T)$
- $\Delta N(\Delta v, \Delta p, \Delta r, \delta_r, \delta_a)$

where we have following new variables

- δ_T - Throttle control input.
- δ_e - Elevator control input.
- δ_a - Aileron control input.
- δ_r - Rudder control input.

It could be worse (θ, ψ, ϕ). Reality is worse.

Linearization

Force Contribution

To linearize the forces/moments we use first-order derivative approximations:

$$\begin{aligned}
\Delta X &= \frac{\partial X}{\partial u}\Delta u + \frac{\partial X}{\partial w}\Delta w + \frac{\partial X}{\partial \delta_e}\delta_e + \frac{\partial X}{\partial \delta_T}\delta_T \\
\Delta Y &= \frac{\partial Y}{\partial v}\Delta v + \frac{\partial Y}{\partial p}\Delta p + \frac{\partial Y}{\partial r}\Delta r + \frac{\partial Y}{\partial \delta_r}\delta_r \\
\Delta Z &= \frac{\partial Z}{\partial u}\Delta u + \frac{\partial Z}{\partial w}\Delta w + \frac{\partial Z}{\partial \dot{w}}\Delta \dot{w} + \frac{\partial Z}{\partial q}\Delta q + \frac{\partial Z}{\partial \delta_e}\delta_e + \frac{\partial Z}{\partial \delta_T}\delta_T \\
\Delta L &= \frac{\partial L}{\partial v}\Delta v + \frac{\partial L}{\partial p}\Delta p + \frac{\partial L}{\partial r}\Delta r + \frac{\partial L}{\partial \delta_r}\delta_r + \frac{\partial L}{\partial \delta_a}\delta_a \\
\Delta M &= \frac{\partial M}{\partial u}\Delta u + \frac{\partial M}{\partial w}\Delta w + \frac{\partial M}{\partial \dot{w}}\Delta \dot{w} + \frac{\partial M}{\partial q}\Delta q + \frac{\partial M}{\partial \delta_e}\delta_e + \frac{\partial M}{\partial \delta_T}\delta_T \\
\Delta N &= \frac{\partial N}{\partial v}\Delta v + \frac{\partial N}{\partial p}\Delta p + \frac{\partial N}{\partial r}\Delta r + \frac{\partial N}{\partial \delta_r}\delta_r + \frac{\partial N}{\partial \delta_a}\delta_a
\end{aligned}$$

We have a notation for the partial derivatives:

$$\frac{1}{m}\frac{\partial X}{\partial u} = X_u \qquad \frac{1}{m}\frac{\partial X}{\partial w} = X_w \qquad \frac{1}{m}\frac{\partial X}{\partial \delta_e} = X_{\delta_e} \qquad \frac{1}{m}\frac{\partial X}{\partial \delta_T} = X_{\delta_T}$$

$$\frac{1}{m}\frac{\partial Y}{\partial v} = Y_v \qquad \frac{1}{m}\frac{\partial Y}{\partial p} = Y_p \qquad \frac{1}{m}\frac{\partial Y}{\partial r} = Y_r \qquad \frac{1}{m}\frac{\partial Y}{\partial \delta_r} = Y_{\delta_r}$$

$$\frac{1}{m}\frac{\partial Z}{\partial u} = Z_u \qquad \frac{1}{m}\frac{\partial Z}{\partial w} = Z_w \qquad \frac{1}{m}\frac{\partial Z}{\partial \dot{w}} = Z_{\dot{w}} \qquad \frac{1}{m}\frac{\partial Z}{\partial q} = Z_q \qquad \frac{1}{m}\frac{\partial Z}{\partial \delta_e} = Z_{\delta_e}$$

$$\frac{1}{I_{zz}}\frac{\partial L}{\partial v} = L_v \qquad \frac{1}{I_{zz}}\frac{\partial L}{\partial p} = L_p \qquad \frac{1}{I_{zz}}\frac{\partial L}{\partial r} = L_r \qquad \frac{1}{I_{zz}}\frac{\partial L}{\partial \delta_r} = L_{\delta_r} \qquad \frac{1}{I_{zz}}\frac{\partial L}{\partial \delta_a} = L_{\delta_a}$$

$$\frac{1}{I_{yy}}\frac{\partial M}{\partial u} = M_u \qquad \frac{1}{I_{yy}}\frac{\partial M}{\partial w} = M_w \qquad \frac{1}{I_{yy}}\frac{\partial M}{\partial \dot{w}} = M_{\dot{w}} \qquad \frac{1}{I_{yy}}\frac{\partial M}{\partial q} = M_q \qquad \frac{1}{I_{yy}}\frac{\partial M}{\partial \delta_e} = M_{\delta_e}$$

$$\frac{1}{I_{xx}}\frac{\partial N}{\partial v} = N_v \qquad \frac{1}{I_{xx}}\frac{\partial N}{\partial p} = N_p \qquad \frac{1}{I_{xx}}\frac{\partial N}{\partial r} = N_r \qquad \frac{1}{I_{xx}}\frac{\partial N}{\partial \delta_r} = N_{\delta_r} \qquad \frac{1}{I_{xx}}\frac{\partial N}{\partial \delta_a} = N_{\delta_a}$$

Longitudinal Dynamics

Although we now have many equations, we notice that some of them decouple:

$$\Delta\dot{u} + \Delta\theta g\cos\theta_0 = \frac{\Delta X}{m}$$
$$\Delta\dot{w} + \Delta\theta g\sin\theta_0 - u_0\Delta q = \frac{\Delta Z}{m}$$
$$\Delta\dot{q} = \frac{\Delta M}{I_{yy}}$$
$$\Delta\dot{\theta} = \Delta q$$

where $\frac{1}{m}\Delta X = X_u\Delta u + X_w\Delta w + X_{\delta_e}\delta_e + X_{\delta_T}\delta_T$

$$\frac{1}{m}\Delta Z = Z_u\Delta u + Z_w\Delta w + \frac{\partial Z}{\partial\dot{w}}\Delta\dot{w} + Z_q\Delta q + \frac{\partial Z}{\partial\delta_e}\delta_e + Z_{\delta_T}\delta_T$$
$$\frac{1}{I_{yy}}\Delta M = M_u\Delta u + M_w\Delta w + M_{\dot{w}}\Delta\dot{w} + M_q\Delta q + M_{\delta_e}\delta_e + M_{\delta_T}\delta_T$$

and also,

$$\Delta\dot{x} = \delta u\cos\theta_0 - u_0\Delta\theta\sin\theta_0 + \Delta w\sin\theta_0$$
$$\Delta\dot{z} = -\delta u\sin\theta_0 - u_0\Delta\theta\cos\theta_0 + \Delta w\cos\theta_0$$

Simplified Longitudinal Dynamics

Note $\ddot{\theta} = \dot{q}$

$$\begin{aligned}
\Delta\dot{u} + \Delta\theta g \cos\theta_0 &= \frac{\Delta X}{m} \\
\Delta\dot{w} + \Delta\theta g \sin\theta_0 - u_0 \Delta\dot{\theta} &= \frac{\Delta Z}{m} \\
\Delta\ddot{\theta} &= \frac{\Delta M}{I_{yy}}
\end{aligned}$$

where

$$\begin{aligned}
\frac{1}{m}\Delta X &= X_u \Delta u + X_w \Delta w + X_{\delta_e}\delta_e + X_{\delta_T}\delta_T \\
\frac{1}{m}\Delta Z &= Z_u \Delta u + Z_w \Delta w + Z_{\dot{w}}\Delta\dot{w} + Z_q \Delta\dot{\theta} + Z_{\delta_e}\delta_e + Z_{\delta_T}\delta_T \\
\frac{1}{I_{yy}}\Delta M &= M_u \Delta u + M_w \Delta w + M_{\dot{w}}\Delta\dot{w} + M_q \Delta\dot{\theta} + M_{\delta_e}\delta_e + M_{\delta_T}\delta_T
\end{aligned}$$

Simplified Longitudinal Dynamics

Combining:

$$\begin{aligned}
\Delta\dot{u} + \Delta\theta g \cos\theta_0 &= X_u \Delta u + X_w \Delta w + X_{\delta_e}\delta_e + X_{\delta_T}\delta_T \\
\Delta\dot{w} + \Delta\theta g \sin\theta_0 - u_0 \Delta\dot{\theta} &= Z_u \Delta u + Z_w \Delta w + Z_{\dot{w}} \Delta\dot{w} + Z_q \Delta\dot{\theta} + Z_{\delta_e}\delta_e + Z_{\delta_T}\delta_T \\
\Delta\ddot{\theta} &= M_u \Delta u + M_w \Delta w + M_{\dot{w}} \Delta\dot{w} + M_q \Delta\dot{\theta} + M_{\delta_e}\delta_e + M_{\delta_T}\delta_T
\end{aligned}$$

Homework: Put in state-space form. **Hint:** Watch out for $\dot{w}$.

Lateral Dynamics

The rest of the equations are also decoupled:

$$\Delta\dot{v} - \Delta\phi g \cos\theta_0 + u_0 \Delta r = \frac{\Delta Y}{m}$$

$$\Delta\dot{p} = \frac{I_{zz}}{I_{xx}I_{zz} - I_{xz}^2}\Delta L + \frac{I_{xz}}{I_{xx}I_{zz} - I_{xz}^2}\Delta N$$

$$\Delta\dot{r} = \frac{I_{xz}}{I_{xx}I_{zz} - I_{xz}^2}\Delta L + \frac{I_{xx}}{I_{xx}I_{zz} - I_{xz}^2}\Delta N$$

$$\Delta\dot{\phi} = \Delta p + \Delta r \tan\theta_0$$

where

$$\frac{1}{m}\Delta Y = Y_v \Delta v + Y_p \Delta p + Y_r \Delta r + Y_{\delta_r}\delta_r$$

$$\frac{1}{I_{zz}}\Delta L = L_v \Delta v + L_p \Delta p + L_r \Delta r + L_{\delta_r}\delta_r + L_{\delta_a}\delta_a$$

$$\frac{1}{I_{xx}}\Delta N = N_v \Delta v + N_p \Delta p + N_r \Delta r + N_{\delta_r}\delta_r + N_{\delta_a}\delta_a$$

and also

$$\Delta\dot{y} = u_0 \Delta\psi \cos\theta_0 + \Delta v$$

Alternative Representation

Longitudinal equations

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - X_u\right)\Delta u - X_w\,\Delta w + (g\cos\theta_0)\,\Delta\theta = X_{\delta_e}\,\Delta\delta_e + X_{\delta_T}\,\Delta\delta_T$$

$$-Z_u\,\Delta u + \left[(1 - Z_{\dot{w}})\frac{\mathrm{d}}{\mathrm{d}t} - Z_w\right]\Delta w - \left[(u_0 + Z_q)\frac{\mathrm{d}}{\mathrm{d}t} - g\sin\theta_0\right]\Delta\theta = Z_{\delta_e}\,\Delta\delta_e + Z_{\delta_T}\,\Delta\delta_T$$

$$-M_u\,\Delta u - \left(M_{\dot{w}}\frac{\mathrm{d}}{\mathrm{d}t} + M_w\right)\Delta w + \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} - M_q\frac{\mathrm{d}}{\mathrm{d}t}\right)\Delta\theta = M_{\delta_e}\,\Delta\delta_e + M_{\delta_T}\,\Delta\delta_T$$

Lateral equations

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - Y_v\right)\Delta v - Y_p\,\Delta p + (u_0 - Y_r)\,\Delta r - (g\cos\theta_0)\,\Delta\phi = Y_{\delta_r}\,\Delta\delta_r$$

$$-L_v\,\Delta v + \left(\frac{\mathrm{d}}{\mathrm{d}t} - L_p\right)\Delta p - \left(\frac{I_{xz}}{I_x}\frac{\mathrm{d}}{\mathrm{d}t} + L_r\right)\Delta r = L_{\delta_a}\Delta\delta_a + L_{\delta_r}\Delta\delta_r$$

$$-N_v\,\Delta v - \left(\frac{I_{xz}}{I_z}\frac{\mathrm{d}}{\mathrm{d}t} + N_p\right)\Delta p + \left(\frac{\mathrm{d}}{\mathrm{d}t} - N_r\right)\Delta r = N_{\delta_a}\,\Delta\delta_a + N_{\delta_r}\,\Delta\delta_r$$

Conclusion

Today you learned

Linearized Equations of Motion:

- Linearized Rotational Dynamics
- Linearized Force Contributions
- New Force Coefficients

Equations of Motion Decouple

- Longitudinal Dynamics
- Lateral Dynamics

Conclusion

Next class we will cover:

Longitudinal Dynamics:

- Finding dimensional Coefficients from non-dimensional coefficients
- Approximate modal behaviour
  - short period mode
  - phugoid mode