Spacecraft and Aircraft Dynamics

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Lecture 12: Lateral Dynamics and Summary

Aircraft Dynamics

Lecture 12

In this Lecture we will cover:

Lateral Dynamics:

- Finding dimensional coefficients from non-dimensional coefficients
- Eigenvalue Analysis
- Approximate modal behavior
  - Spiral mode
  - Roll Mode
  - Dutch Roll mode

Combined Terms

$$\Delta\dot{v} = \Delta\phi g \cos\theta_0 - u_0 \Delta r + Y_v \Delta v + Y_p \Delta p + Y_r \Delta r + Y_{\delta_r} \delta_r$$

$$\Delta\dot{p} = \frac{I_{zz}}{I_{xx}I_{zz} - I_{xz}^2} \left(L_v \Delta v + L_p \Delta p + L_r \Delta r + L_{\delta_r} \delta_r + L_{\delta_a} \delta_a\right) + \frac{I_{xz}I_{xx}}{I_{xx}I_{zz} - I_{xz}^2} \left(N_v \Delta v + N_p \Delta p + N_r \Delta r + N_{\delta_r} \delta_r + N_{\delta_a} \delta_a\right)$$

$$\Delta\dot{r} = \frac{I_{xz}I_{zz}}{I_{xx}I_{zz} - I_{xz}^2} \left(L_v \Delta v + L_p \Delta p + L_r \Delta r + L_{\delta_r} \delta_r + L_{\delta_a} \delta_a\right) + \frac{I_{xx}}{I_{xx}I_{zz} - I_{xz}^2} \left(N_v \Delta v + N_p \Delta p + N_r \Delta r + N_{\delta_r} \delta_r + N_{\delta_a} \delta_a\right)$$

$$\Delta\dot{\phi} = \Delta p + \Delta r \tan\theta_0$$

Force Coefficients

Force/Moment Coefficients can be found in Table 3.6 of Nelson

TABLE 3.6
Summary of lateral directional derivatives

$Y_\beta = \dfrac{QSC_{y\beta}}{m}$ (ft/s^2) or (m/s^2)	$N_\beta = \dfrac{QSbC_{n\beta}}{I_z}$ (s^{-2})	$L_\beta = \dfrac{QSbC_{l_\beta}}{I_x}$ (s^{-2})
$Y_p = \dfrac{QSbC_{y_p}}{2mu_0}$ (ft/s) (m/s)	$N_p = \dfrac{QSb^2C_{n_p}}{2I_zu_0}$ (s^{-1})	$L_p = \dfrac{QSb^2C_{l_p}}{2I_xu_0}$ (s^{-1})
$Y_r = \dfrac{QSbC_{y_r}}{2mu_0}$ (ft/s) or (m/s)	$N_r = \dfrac{QSb^2C_{n_r}}{2I_zu_0}$ (s^{-1})	$L_r = \dfrac{QSb^2C_{l_r}}{2I_xu_0}$ (s^{-1})
$Y_{\delta a} = \dfrac{QSC_{y\delta_a}}{m}$ (ft/s^2) or (m/s^2)	$Y_{\delta_r} = \dfrac{QSC_{y\delta_r}}{m}$ (ft/s^2) or (m/s^2)	
$N_{\delta_a} = \dfrac{QSbC_{n\delta_a}}{I_z}$ (s^{-2})	$N_{\delta_r} = \dfrac{QSbC_{n\delta_r}}{I_z}$ (s^{-2})	
$L_{\delta_a} = \dfrac{QSbC_{l\delta_a}}{I_x}$ (s^{-2})	$L_{\delta_r} = \dfrac{QSbC_{l\delta_r}}{I_x}$ (s^{-2})	

Nondimensional Force Coefficients

Nondimensional Force/Moment Coefficients can be found in Table 3.4 of Nelson

TABLE 3.4

Equations for estimating the lateral stability coefficients

	Y-force derivatives	Yawing moment derivatives	Rolling moment derivatives
β	$C_{y_\beta} = -\eta \frac{S_v}{S} C_{L_{\alpha_v}} \left(1 + \frac{d\sigma}{d\beta}\right)$	$C_{n_\beta} = C_{n_{\beta wf}} + \eta_v V_v C_{L_{\alpha_v}} \left(1 + \frac{d\sigma}{d\beta}\right)$	$C_{l_\beta} = \left(\frac{C_{l_\beta}}{\Gamma}\right)\Gamma + \Delta C_{l_\beta}$ (see Figure 3.11)
p	$C_{y_p} = C_L \frac{AR + \cos\Lambda}{AR + 4\cos\Lambda} \tan\Lambda$	$C_{n_p} = -\frac{C_L}{8}$	$C_{l_p} = -\frac{C_{L_\alpha}}{12} \frac{1 + 3\lambda}{1 + \lambda}$
r	$C_{y_r} = -2\left(\frac{l_v}{b}\right)(C_{y_\beta})_{\text{tail}}$	$C_{n_r} = -2\eta_v V_v \left(\frac{l_v}{b}\right) C_{L_{\alpha_v}}$	$C_{l_r} = \frac{C_L}{4} - 2\frac{l_v}{b}\frac{z_v}{b} C_{y\beta_{\text{tail}}}$
δ_a	0	$C_{n_{\delta_a}} = 2KC_{L_0} C_{l_{\delta_a}}$ (see Figure 3.12)	$C_{l_{\delta_a}} = \frac{2C_{L_\alpha}\tau}{Sb}\int_{y_1}^{y_2} cy\, dy$
δ_r	$C_{y_{\delta_r}} = \frac{S_v}{S}\tau C_{L_{\alpha_v}}$	$C_{n_{\delta_r}} = -V_v \eta_v \tau C_{L_{\alpha_v}}$	$C_{l_{\delta_r}} = \frac{S_v}{S}\left(\frac{z_v}{b}\right)\tau C_{L_{\alpha_v}}$

AR	Aspect ratio
b	Wingspan
C_{L_0}	Reference lift coefficient
C_{L_α}	Airplane lift curve slope
$C_{L_{\alpha_w}}$	Wing lift curve slope
$C_{L_{\alpha_t}}$	Tail lift curve slope
$\bar{c}$	Mean aerodynamic chord
K	empirical factor
l_v	Distance from center of gravity to vertical tail aerodynamic center
V_v	Vertical tail volume ratio
S	Wing area
S_v	Vertical tail area
z_v	Distance from center of pressure of vertical tail to fuselage centerline
Γ	Wing dihedral angle
Λ	Wing sweep angle
η_v	Efficiency factor of the vertical tail
λ	Taper ratio (tip chord/root chord)
$\frac{d\sigma}{d\beta}$	Change in sidewash angle with a change in sideslip angle

State-Space

$$\begin{bmatrix}\Delta\dot{v}\\ \Delta\dot{p}\\ \Delta\dot{r}\\ \Delta\dot{\phi}\end{bmatrix} = \begin{bmatrix} Y_v & Y_p & Y_r - u_0 & g\cos\theta_0 \\ \frac{L_v}{I'} + I'_{zx} I_{zz} N_v & \frac{L_p}{I'} + I'_{zx} I_{zz} N_p & \frac{L_r}{I'} + I'_{zx} I_{zz} N_r & 0 \\ \frac{N_v}{I'} + I'_{zx} I_{xx} L_v & \frac{N_p}{I'} + I'_{zx} I_{xx} L_p & \frac{N_r}{I'} + I'_{zx} I_{xx} L_r & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{bmatrix} \begin{bmatrix}\Delta v\\ \Delta p\\ \Delta r\\ \Delta\phi\end{bmatrix}$$

$$+ \begin{bmatrix} Y_{\delta_r} & Y_{\delta_a} \\ \frac{L_{\delta_r}}{I'} + I'_{zx} I_{zz} N_{\delta_r} & \frac{L_{\delta_a}}{I'} + I'_{zx} I_{zz} N_{\delta_a} \\ \frac{L_{\delta_r}}{I'} + I'_{zx} I_{zz} N_{\delta_r} & \frac{L_{\delta_a}}{I'} + I'_{zx} I_{zz} N_{\delta_a} \\ 0 & 0 \end{bmatrix} \begin{bmatrix}\Delta\delta_r\\ \Delta\delta_a\end{bmatrix}$$

$$I' := 1 - \frac{I_{zx}^2}{I_{xx} I_{zz}}, \qquad I'_{zx} := \frac{I_{zx}}{I_x I_z I'}$$

C172: $V_0 = 132kt$, $5000ft$.

$$\begin{bmatrix}\Delta\dot{\beta}\\ \Delta\dot{p}\\ \Delta\dot{r}\\ \Delta\dot{\phi}\end{bmatrix} = \begin{bmatrix}-.1473 & -.0014 & -.9918 & .1470\\ -28.749 & -12.4092 & 2.5346 & 0\\ 10.119 & -.3817 & -1.2597 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}\Delta\beta\\ \Delta p\\ \Delta r\\ \Delta\phi\end{bmatrix}$$

We use $\Delta\beta \cong \frac{\Delta v}{u_0}$.

Using the Matlab command `[u,V] = eigs(`A_{lat}`)`, we find the eigenvalues as

Spiral Mode

$$\lambda_1 = -.011$$

Exponential decay, not oscillating.

Roll Mode

$$\lambda_2 = -12.43$$

Exponential decay, not oscillating.

Dutch Roll Mode

$$\lambda_{3,4} = -.686 \pm 3.306\imath$$

Exponential decay, oscillating.

Spiral Mode

$$\lambda_1 = -.011$$

with Eigenvector

$$\nu_1 = \begin{bmatrix} .0176 \\ -.011 \\ .1458 \\ 1 \end{bmatrix}$$

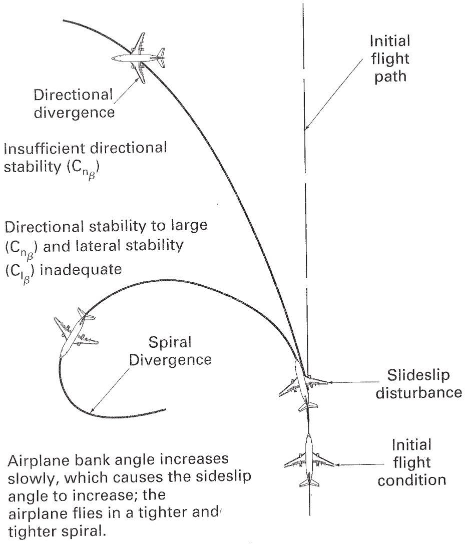
Natural Frequency: N/A

Damping Ratio: $d = .011$
Period: N/A
Half-Life: $\gamma = 91s$

Slow convergence in coupled ϕ and r.

State-Space

Spiral Mode



State-Space

Spiral Mode

Roll Mode

$$\lambda_2 = -12.43$$

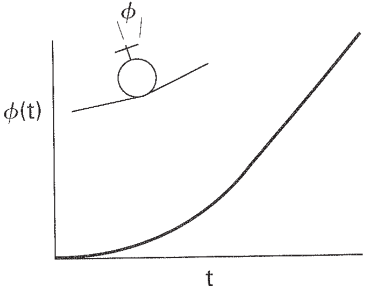
with Eigenvector

$$\nu_2 = \begin{bmatrix} .0036 \\ 1 \\ .0309 \\ -.0804 \end{bmatrix}$$

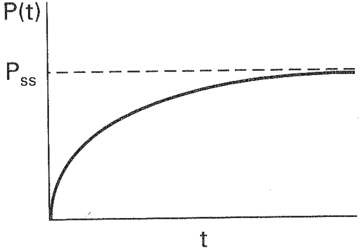
Natural Frequency: N/A

Damping Ratio: $d = -12.43$
Period: N/A
Half-Life: $\gamma = .08s$

Fast convergence in roll rate, p.



ϕ
ϕ(t)
t



P(t)
P_ss
t

Dutch Roll Mode

$$\lambda_{3,4} = -.686 \pm 3.306\imath \qquad \omega_{DR} = 3.4 \qquad d_{DR} = .2$$

and Eigenvectors

$$\nu_{3,4} = \begin{bmatrix} .0954 \\ -.2150 \\ .7569 \\ -.1357 \end{bmatrix} \pm \begin{bmatrix} .2175 \\ .5127 \\ .1856 \\ .0932 \end{bmatrix} \imath$$

Natural Frequency: $\omega_n = 3.4 rad/s$

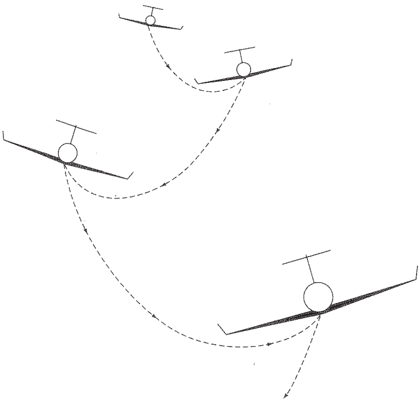
Damping Ratio: $d = .2$
Period: $\tau = 1.848s$
Half-Life: $\gamma = 1.01s$

Complex coupling in β, p, r and ϕ.

- Exacerbated by high-mounted wings (Counter with anhedral)
- Swept wings and dihedral contribute
- Most commonly compensated mode.
- Yaw damper is typical response.

State-Space

Dutch Roll Mode



Now that we know that lateral dynamics have three modes:

- Spiral Mode
- Roll Mode
- Dutch Roll Mode

Spiral Mode:

- Motion in r and β
- Set $\Delta\dot{p} = \Delta p = 0$ and solve for $\Delta\beta$.
- Use EOM for dynamics of Δr

Roll Mode:

- motion only in roll, Δp

Dutch Roll Mode:

- Coupled motion in β, r

Spiral Mode Approximation

By setting $\Delta\dot{p} = \Delta p = 0$ and solving for β, we get

$$\Delta\beta = \frac{L_r}{L_\beta}\Delta r$$

Plug that into EOM for Δr and we get

$$\Delta\dot{r} = -\frac{L_r N_\beta - L_\beta N_r}{L_\beta}\Delta r.$$

Which obviously has

$$\lambda_1 = -\frac{L_r N_\beta - L_\beta N_r}{L_\beta}$$

- L_β is negative for dihedral.
- N_r is negative for a yaw-damper.

Spiral Mode Approximations

Examples: NAVION and C172

NAVION:
Approximate Eigenvalue:

$$\lambda_{roll} = -.144$$

True Eigenvalue:

$$\lambda_{roll} = -.00877$$

C172:
Approximate Eigenvalue:

$$\lambda_{roll} = -.36$$

True Eigenvalue:

$$\lambda_{roll} = -.011$$

Not Encouraging.

Roll Mode Approximation

By ignoring all variables except p, we get the simple dynamics

$$\Delta\dot{p} = L_p \Delta p + L_{\delta_a} \delta_a$$

Which obviously has

$$\lambda_2 = L_p$$

- Typically quite stable.
- If unstable, something is very wrong.

Roll Mode Approximations

Examples: NAVION and C172

NAVION:
Approximate Eigenvalue:

$$L_p = -12.41$$

True Eigenvalue:

$$\lambda_{roll} = -12.43$$

C172:
Approximate Eigenvalue:

$$L_p = -8.4$$

True Eigenvalue:

$$\lambda_{roll} = -8.435$$

So, generally good agreement.

Dutch Roll Mode Approximation

If dihedral (L_β) is small, then we can neglect rolling motion

- L_β hurts spiral mode but helps Dutch Roll.

$$\Delta\dot{\beta} = \frac{Y_\beta}{u_0}\Delta\beta + \left(\frac{Y_r}{u_0} - 1\right)\Delta r + \frac{Y_{\delta_a}}{u_0}\delta_a + \frac{Y_{\delta_r}}{u_0}\delta_r$$
$$\Delta\dot{r} = N_\beta\Delta\beta + N_r\Delta r + N_{\delta_a}\delta_a + N_{\delta_r}\delta_r$$

- Also assume $\frac{I_{zx}}{I_{zz}}$ is negligible.

Ignoring the inputs, we can write a state-space description as:

$$\begin{bmatrix}\Delta\dot{\beta}\\ \Delta\dot{r}\end{bmatrix} = \begin{bmatrix}\frac{Y_\beta}{u_0} & \left(\frac{Y_r}{u_0} - 1\right)\\ N_\beta & N_r\end{bmatrix}\begin{bmatrix}\Delta\beta\\ \Delta r\end{bmatrix}.$$

Dutch Roll Mode Approximation

Starting with state-space:

$$\begin{bmatrix} \Delta\dot{\beta} \\ \Delta\dot{r} \end{bmatrix} = \begin{bmatrix} \frac{Y_\beta}{u_0} & \left(\frac{Y_r}{u_0} - 1\right) \\ N_\beta & N_r \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta r \end{bmatrix},$$

we find the characteristic equation

$$\det(sI - A) = s^2 - \left(N_r + \frac{Y_\beta}{u_0}\right) s + \left(N_\beta + \frac{Y_\beta N_r}{u_0} - \frac{Y_r N_\beta}{u_0}\right),$$

which yields the approximate formulae:

$$\omega_{DR} = \sqrt{\frac{1}{u_0}\left(Y_\beta N_r + N_\beta u_0 - N_\beta Y_r\right)}$$

$$d_{DR} - \frac{N_r + \frac{Y_\beta}{u_0}}{\omega_{DR}}$$

Approximation is generally OK approximation for ω, but not for d.

Dutch Roll Mode Approximations

Example: NAVION

Approximate Period:

$$\omega_{DR} = 2.98s$$

True Period:

$$\omega_{DR} = 2.69s$$

Approximate Damping:

$$d_{DR} = 1.35s$$

True Damping:

$$d_{DR} = 1.42rad/s$$

So, generally good, but not as good. Why?

Conclusion

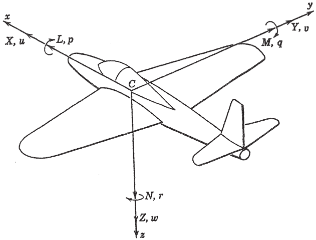
In this lecture, we covered:

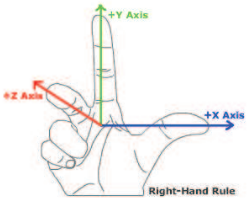
- Eigenvalue analysis of the lateral dynamics
- How to identify
  - Spiral Mode Eigenvlaues/Motion
  - Roll Mode Eigenvlaues/Motion
  - Dutch Roll Mode Eigenvlaues/Motion
- Modal Approximations
  - Spiral, Roll and Dutch Roll Modes
  - Formulae for natural frequency
  - Formulae for damping ratio

Review: Static Stability

Coordinates

- The origin is the center of mass.
- The x-axis points toward the front of the aircraft.
- The z-axis points down.
- The y-axis is perpendicular to the $x - z$ plane.
- Use the "right-hand rule" to define y





Lift Equation: For α measured from zero lift line.

$$C_L = \left(C_{L\alpha,wf} + \eta \frac{S_t}{S} C_{L\alpha,t}(1 - \frac{d\epsilon}{d\alpha}) \right) \alpha$$

Review: Static Stability

Longitudinal Stability

Moment Equation:

$$C_m = C_{m0} + C_{m\alpha}\alpha + C_{m\delta_e}$$

Expressions for $C_{m\alpha}$ and $C_{m\delta}$

$$C_{M0,wf} + \eta V_H C_{L\alpha,t}(\varepsilon_0 + i_{wf} - i_t)$$

$$C_{m\alpha} = C_{L\alpha,wf}\left(\frac{X_{CG}}{\bar{c}} - \frac{X_{AC,wf}}{\bar{c}}\right) - \eta V_H C_{L\alpha,t}\left(1 - \frac{d\varepsilon}{d\alpha}\right)$$

$$C_{m\delta_e} = -\eta V_H C_{L\alpha,t}\tau$$

- Stable if $C_{m\alpha} < 0$
- $\alpha_{eq} = -\frac{C_{m0}}{C_{m\alpha}}$
- Neutral point when $C_{m\alpha} = 0$

$$C_n = C_{n\beta}\beta + C_{n\delta_r}\delta_r$$

where

$$C_{n\delta_r} = -\eta_v V_v C_{L\alpha,v}\tau,$$

- Stable if $C_{n\beta} > 0$
- Use plots/data to find $C_{n\beta}$

$$C_l = C_{l\beta}\beta + C_{l\delta_a}\delta_a$$

where

$$C_{l\delta_a} \cong 2C_{L\alpha}\tau\frac{y_c}{b}$$

- Stable if $C_{l\beta} < 0$
- Use plots/data to find $C_{l\beta}$

$$\lambda = \lambda_R \pm \lambda_I \imath$$

1. **Natural Frequency:**
$$\omega_n = \sqrt{\lambda_R^2 + \lambda_I^2}$$
2. **Damping Ratio:**
$$d = -\frac{\lambda_R}{\omega_n}$$
3. **Period:**
$$\tau = \frac{2\pi}{\omega_n}$$
4. **Half Life:**
$$\gamma = \frac{.693}{|\lambda_R|}$$

Short Period Mode:

$$\lambda_{3,4} = \frac{1}{2}(M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0}) \pm \frac{1}{2}\sqrt{(M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0})^2 - 4(m_q\frac{Z_\alpha}{u_0} - M_\alpha)}$$

- **Natural Frequency:**

$$\omega_{sp} = M_q\frac{Z_\alpha}{u_0}$$

- **Damping Ratio:**

$$d_{sp} = -\frac{1}{2}\frac{M_1 + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0}}{\omega_{sp}}$$

Long Period Mode:

$$\lambda_{1,2} = \frac{X_u \pm \sqrt{X_u^2 + 4\frac{Z_u}{u_0}g}}{2}$$

- **Natural Frequency:**

$$\omega_{lp} = \sqrt{-\frac{Z_u g}{u_0}}$$

- **Damping Ratio:**

$$d_{lp} = -\frac{X_u}{2\omega_{lp}}$$

Spiral Mode:

$$\lambda_{spiral} = -\frac{L_r N_\beta - L_\beta N_r}{L_\beta}$$

Roll Mode:

$$\lambda_{roll} = L_p$$

Dutch Roll:

$$\omega_{DR} = \sqrt{\frac{1}{u_0}\left(Y_\beta N_r + N_\beta u_0 - N_\beta Y_r\right)}$$

$$d_{DR} - \frac{N_r + \frac{Y_\beta}{u_0}}{\omega_{DR}}$$