Spacecraft and Aircraft Dynamics

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Lecture 9: 6DOF Equations of Motion

In this Lecture we will cover:

Newton's Laws

- $\sum \vec{M}_i = \frac{d}{dt}\vec{H}$
- $\sum \vec{F}_i = m\frac{d}{dt}\vec{v}$

Rotating Frames of Reference

- Equations of Motion in Body-Fixed Frame
- Often Confusing

Review: Coordinate Rotations

Positive Directions

If in doubt, use the right-hand rules.

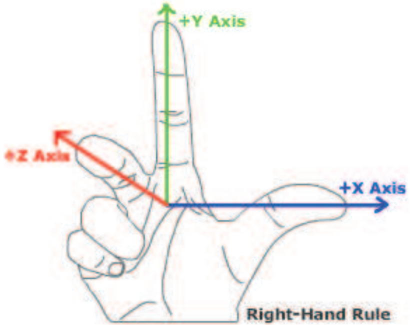


Figure: Positive Directions

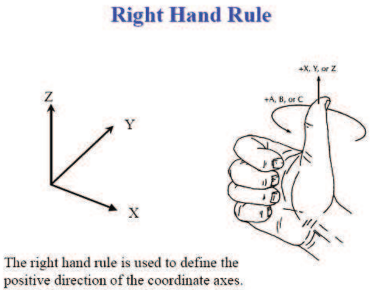
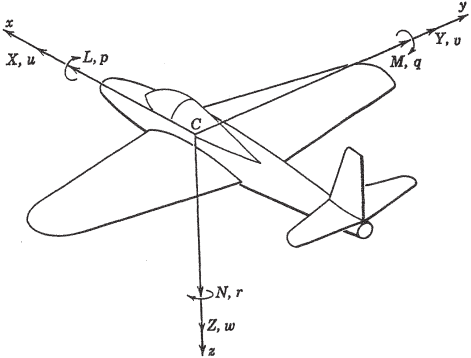


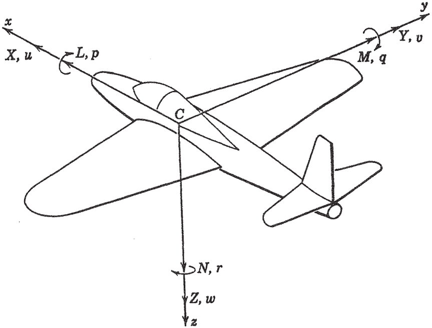
Figure: Positive Rotations



There are 3 basic rotations an aircraft can make:

- Roll = Rotation about x-axis
- Pitch = Rotation about y-axis
- Yaw = Rotation about z-axis
- Each rotation is a one-dimensional transformation.

Any two coordinate systems can be related by a sequence of 3 rotations.

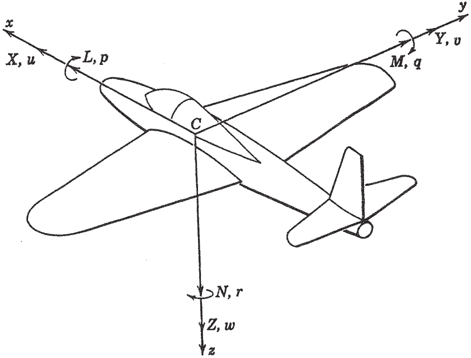


These forces and moments have standard labels. The Forces are:

X	Axial Force	Net Force in the positive x-direction
Y	Side Force	Net Force in the positive y-direction
Z	Normal Force	Net Force in the positive z-direction

Review: Forces and Moments

Moments



The Moments are called, intuitively:

L	Rolling Moment	Net Moment in the positive p-direction
M	Pitching Moment	Net Moment in the positive q-direction
N	Yawing Moment	Net Moment in the positive r-direction

Newton's Second Law tells us that for a particle $F = ma$. In vector form:

$$\vec{F} = \sum_i \vec{F}_i = m \frac{d}{dt} \vec{V}$$

That is, if $\vec{F} = [F_x \; F_y \; F_z]$ and $\vec{V} = [u \; v \; w]$, then

$$F_x = m\frac{du}{dt} \qquad F_x = m\frac{dv}{dt} \qquad F_z = m\frac{dw}{dt}$$

Definition 1.

$\vec{L} = m\vec{V}$ is referred to as **Linear Momentum**.

Newton's Second Law is only valid if $\vec{F}$ and $\vec{V}$ are defined in an *Inertial* coordinate system.

Definition 2.

A coordinate system is **Inertial** if it is not accelerating or rotating.

Using Calculus, this concept can be extended to rigid bodies by integration over all particles.

$$\vec{M} = \sum_i \vec{M}_i = \frac{d}{dt}\vec{H}$$

Definition 3.

Where $\vec{H} = \int (\vec{r}_c \times \vec{v}_c) dm$ is the **angular momentum**.

Angular momentum of a rigid body can be found as

$$\vec{H} = I\vec{\omega}_I$$

where $\vec{\omega}_I = [p,\ q,\ r]^T$ is the angular rotation vector of the body about the center of mass.

- p is rotation about the x-axis.
- q is rotation about the y-axis.
- r is rotation about the z-axis.
- ω_I is defined in an *Inertial* Frame.

The matrix I is the *Moment of Inertia Matrix*.

The moment of inertia matrix is defined as

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

$$I_{xy} = I_{yx} = \int\int\int xydm \qquad I_{xx} = \int\int\int \left(y^2 + z^2\right) dm$$

$$I_{xz} = I_{zx} = \int\int\int xzdm \qquad I_{yy} = \int\int\int \left(x^2 + z^2\right) dm$$

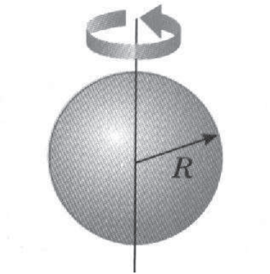
$$I_{yz} = I_{zy} = \int\int\int yzdm \qquad I_{zz} = \int\int\int \left(x^2 + y^2\right) dm$$

So

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} p_I \\ q_I \\ r_I \end{bmatrix}$$

where p_I, q_I and r_I are the rotation vectors as expressed in the inertial frame corresponding to x-y-z.

Examples:



Homogeneous Sphere

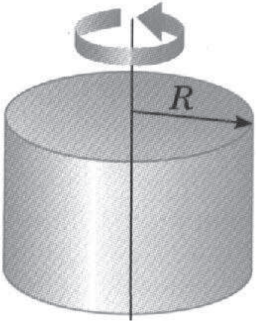
$$I_{sphere} = \frac{2}{5} m r^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ring

$$I_{ring} = m r^2 \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Moment of Inertia

Examples:



Homogeneous Disk

$$I_{disk} = \frac{1}{4}mr^2 \begin{bmatrix} 1+\frac{1}{3}\frac{h}{r^2} & 0 & 0 \\ 0 & 1+\frac{1}{3}\frac{h}{r^2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

F/A-18

$$I = \begin{bmatrix} 23 & 0 & 2.97 \\ 0 & 15.13 & 0 \\ 2.97 & 0 & 16.99 \end{bmatrix} kslug-ft^2$$

Problem:

The Body-Fixed Frame

The moment of inertia matrix, I, is fixed in the body-fixed frame. However, Newton's law only applies for an inertial frame:

$$\vec{M} = \sum_i \vec{M}_i = \frac{d}{dt}\vec{H}$$

If the body-fixed frame is rotating with rotation vector $\vec{\omega}$, then for any vector, $\vec{a}$, $\frac{d}{dt}\vec{a}$ in the inertial frame is

$$\frac{d\vec{a}}{dt}\Big|_I = \frac{d\vec{a}}{dt}\Big|_B + \vec{\omega} \times \vec{a}$$

Specifically, for Newton's Second Law

$$\vec{F} = m\frac{d\vec{V}}{dt}\Big|_B + m\vec{\omega} \times \vec{V}$$

and

$$\vec{M} = \frac{d\vec{H}}{dt}\Big|_B + \vec{\omega} \times \vec{H}$$

Equations of Motion

Thus we have

$$\begin{bmatrix}F_x\\F_y\\F_z\end{bmatrix} = m\begin{bmatrix}\dot u\\ \dot v\\ \dot w\end{bmatrix} + m\det\begin{bmatrix}\hat x & \hat y & \hat z\\ p & q & r\\ u & v & w\end{bmatrix} = m\begin{bmatrix}\dot u + qw - rv\\ \dot v + ru - pw\\ \dot w + pv - qu\end{bmatrix}$$

and

$$\begin{bmatrix}L\\M\\N\end{bmatrix} = \begin{bmatrix}I_{xx} & -I_{xy} & -I_{xz}\\ -I_{yx} & I_{yy} & -I_{yz}\\ -I_{zx} & -I_{zy} & I_{zz}\end{bmatrix}\begin{bmatrix}\dot p\\ \dot q\\ \dot r\end{bmatrix} + \vec\omega \times \begin{bmatrix}I_{xx} & -I_{xy} & -I_{xz}\\ -I_{yx} & I_{yy} & -I_{yz}\\ -I_{zx} & -I_{zy} & I_{zz}\end{bmatrix}\begin{bmatrix}p\\q\\r\end{bmatrix}$$

$$= \begin{bmatrix}I_{xx}\dot p - I_{xy}\dot q - I_{xz}\dot r\\ -I_{xy}\dot p + I_{yy}\dot q - I_{yz}\dot r\\ -I_{xz}\dot p - I_{yz}\dot q + I_{zz}\dot r\end{bmatrix} + \vec\omega \times \begin{bmatrix}pI_{xx} - qI_{xy} - rI_{xz}\\ -pI_{xy} + qI_{yy} - rI_{yz}\\ -pI_{xz} - qI_{yz} + rI_{zz}\end{bmatrix}$$

$$= \begin{bmatrix}I_{xx}\dot p - I_{xy}\dot q - I_{xz}\dot r + q(-pI_{xz} - qI_{yz} + rI_{zz}) - r(-pI_{xy} + qI_{yy} - rI_{yz})\\ -I_{xy}\dot p + I_{yy}\dot q - I_{yz}\dot r - p(-pI_{xz} - qI_{yz} + rI_{zz}) + r(pI_{xx} - qI_{xy} - rI_{xz})\\ -I_{xz}\dot p - I_{yz}\dot q + I_{zz}\dot r + p(-pI_{xy} + qI_{yy} - rI_{yz}) - q(pI_{xx} - qI_{xy} - rI_{xz})\end{bmatrix}$$

Which is too much for any mortal. For aircraft, we have **symmetry** about the x-z plane. Thus $I_{yz} = I_{xy} = 0$. Spacecraft?

Equations of Motion

Reduced Equations

With $I_{xy} = I_{yz} = 0$, we have, in summary:

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = m \begin{bmatrix} \dot{u} + qw - rv \\ \dot{v} + ru - pw \\ \dot{w} + pv - qu \end{bmatrix}$$

and

$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{p} - I_{xz}\dot{r} - qpI_{xz} + qrI_{zz} - rqI_{yy} \\ I_{yy}\dot{q} + p^2 I_{xz} - prI_{zz} + rpI_{xx} - r^2 I_{xz} \\ -I_{xz}\dot{p} + I_{zz}\dot{r} + pqI_{yy} - qpI_{xx} + qrI_{xz} \end{bmatrix}$$

Right now,

- Translational variables (u,v,w) depend on rotational variables (p,q,r).
- Rotational variables (p,q,r) do not depend on translational variables (u,v,w).
  - For aircraft, however, Moment forces (L,M,N) depend on rotational and translational variables.

Question: Consider a sniper firing a rifle due east at the equator. Ignoring gravity and drag, what are the equations of motion of the bullet? Use the North-East-Up local coordinate system. Muzzle velocity: $1000m/s$. Range: $4km$.

Answer: The earth is rotating about its axis at angular velocity $2\pi\frac{rad}{day}$, or $.0000727\frac{rad}{s}$. The rotation is positive about the local North-axis. Thus

$$\vec{\omega} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} .0000727 \\ 0 \\ 0 \end{bmatrix}$$

Since the bullet is in free-flight, there are no forces. Thus the Equations of motion are

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = m \begin{bmatrix} \dot{u} + qw - rv \\ \dot{v} + ru - pw \\ \dot{w} + pv - qu \end{bmatrix} = m \begin{bmatrix} \dot{u} \\ \dot{v} - pw \\ \dot{w} + pv \end{bmatrix}$$

EOMs in Rotating Frame

Example: Snipers

Simplified EOMs: Using $q = r = 0$, we simplify to

$$\dot{u} = 0 \qquad \dot{v} = pw \qquad \dot{w} = -pv.$$

Solution: For initial condition $u(0) = 0$, $v(0) = V$ and $w(0) = 0$ has solution

$$u(t) = 0 \qquad v(t) = v(0)\cos(pt) \qquad w(t) = -v(0)\sin(pt)$$

Since p is very small compared to flight time, we can approximate

$$u(t) = 0 \qquad v(t) = v(0) \qquad w(t) = -v(0)pt$$

Which yields displacement

$$N(t) = 0 \qquad E(t) = v(0)t \qquad U(t) = -\frac{1}{2}v(0)pt^2$$

Conclusion: For a target at range $E(t_i) = 4km$, we have $t_i = 4s$ and hence the error at target is:

$$N(t_i) = 0 \qquad U(t_i) = -\frac{1}{2} * 2000 * .0000727 * 16 = -1.1635m$$

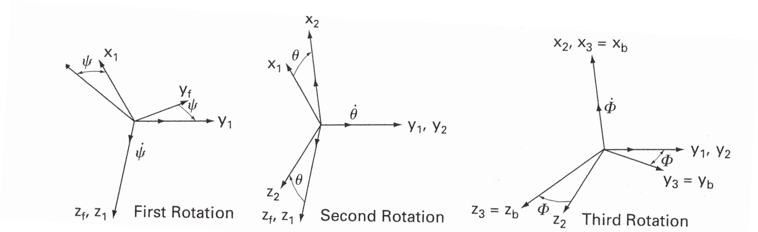
Of course, if we were firing west, the error would be $+1.1635m$.

Euler Angles

Issue: Equations of motion are expressed in the Body-Fixed frame.

Question: How do determine rotation and velocity in the inertial frame. For intercept, obstacle avoidance, etc.

Approach: From Lecture 4, any two coordinate systems can be related through a sequence of three rotations. Recall these transformations are:



Roll Rotation (ϕ) :

$$R_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

Pitch Rotation (θ):

$$R_2(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Yaw Rotation (ψ):

$$R_3(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler Angles

Definition 4.

The term **Euler Angles** refers to the angles of rotation (ψ, θ, ϕ) needed to go from one coordinate system to another using the specific sequence of rotations **Yaw-Pitch-Roll**:

$$\vec{V}_{BF} = R_1(\phi)R_2(\theta)R_3(\psi)\vec{V}_I.$$

NOTE BENE: Euler angles are often defined differently (e.g. 3-1-3). We use the book notation.

The composite rotation matrix can be written

$$R_1(\phi)R_2(\theta)R_3(\psi) =$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This moves a vector

Inertial Frame ⇒ Body-Fixed Frame

Euler Angles

To move a vector

$$\text{Body-Fixed Frame} \Rightarrow \text{Inertial Frame}$$

we need to Invert the Rotations. Rotation matrices are easily inverted, however

$$R_i(\theta)^{-1} = R_i(-\theta)$$

Thus $\vec{V}_I = \left(R_1(\phi)R_2(\theta)R_3(\psi)\right)^{-1} \vec{V}_{BF}$, where

$$\begin{aligned}
&\left(R_1(\phi)R_2(\theta)R_3(\psi)\right)^{-1} = R_3(\psi)^{-1}R_2(\theta)^{-1}R_1(\phi)^{-1} \\
&= R_3(-\psi)R_2(-\theta)R_1(-\phi) \\
&= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\psi\sin\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \cos\theta\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix}
\end{aligned}$$

These transformations now describe a **Roll-Pitch-Yaw**.

Thus to find the inertial velocity vector, we must rotate **FROM** the body-fixed coordinates to the inertial frame:

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} = R_3(-\psi)R_2(-\theta)R_1(-\phi)\begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Euler Angles

The rate of rotation of the Euler Angles can be found by rotating the rotation vector into the inertial frame

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\theta & \cos\theta\cos\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

This transformation can also be reversed as

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Summary

$$X - mgS_\theta = m(\dot{u} + qw - rv)$$
$$Y + mgC_\theta S_\Phi = m(\dot{v} + ru - pw)$$
$$Z + mgC_\theta C_\Phi = m(\dot{w} + pv - qu)$$

Force equations

$$L = I_x\dot{p} - I_{xz}\dot{r} + qr(I_z - I_y) - I_{xz}pq$$
$$M = I_y\dot{q} + rp(I_x - I_z) + I_{xz}(p^2 - r^2)$$
$$N = -I_{xz}\dot{p} + I_z\dot{r} + pq(I_y - I_x) + I_{xz}qr$$

Moment equations

$$p = \dot{\Phi} - \dot{\psi}S_\theta$$
$$q = \dot{\theta}C_\Phi + \dot{\psi}C_\theta S_\Phi$$
$$r = \dot{\psi}C_\theta C_\Phi - \dot{\theta}S_\Phi$$

Body angular velocities in terms of Euler angles and Euler rates

$$\dot{\theta} = qC_\Phi - rS_\Phi$$
$$\dot{\Phi} = p + qS_\Phi T_\theta + rC_\Phi T_\theta$$
$$\dot{\psi} = (qS_\Phi + rC_\Phi)\sec\theta$$

Euler rates in terms of Euler angles and body angular velocities

Velocity of aircraft in the fixed frame in terms of Euler angles and body velocity components

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} = \begin{bmatrix} C_\theta C_\psi & S_\Phi S_\theta C_\psi - C_\Phi S_\psi & C_\Phi S_\theta C_\psi + S_\Phi S_\psi \\ C_\theta S_\psi & S_\Phi S_\theta S_\psi + C_\Phi C_\psi & C_\Phi S_\theta S_\psi - S_\Phi C_\psi \\ -S_\theta & S_\Phi C_\theta & C_\Phi C_\theta \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Conclusion

In this lecture we have covered

Equations of Motion

- How to differentiate Vectors in Rotating Frames
- Derivation of the Nonlinear 6DOF Equations of Motion

Euler Angles

- Definition of Euler Angles
- Using Rotation Matrices to transform vectors
- Derivatives of the Euler angles
  - Relationship to p-q-r in Body-Fixed Frame

Next Lecture

In the next lecture we will cover

Linearized Equations of Motion

- How to linearize the nonlinear 6DOF EOM
- How to linearize the force and moment contributions

Force and Moment Contributions

- The gravity and thrust contributions
- The full linearized equations of motion including forces and moments
- How to decouple into Longitudinal and Lateral Dynamics
  - Reminder on how to create a state-space representation.