Lecture 8: Impulsive Orbital Maneuvers
Introduction

In this Lecture, you will learn:

Coplanar Orbital Maneuvers
- Impulsive Maneuvers
  - $\Delta v$
- Single Burn Maneuvers
- Hohmann transfers
  - Elliptic
  - Circular

**Numerical Problem**: Suppose we are in a circular parking orbit at an altitude of 191.34km and we want to raise our altitude to 35,781km. Describe the required orbital maneuvers (time and $\Delta v$).
Changing Orbits

Suppose we have designed our ideal orbit.

- We have chosen $a$, $e$, $i$, $\Omega$, $\omega$
- For now, we don’t care about $f$
  - Lambert’s Problem
- Don’t care about efficiency

Question:
- Given an object with position, $\vec{r}$ and velocity $\vec{v}$.
- How to move the object into a desired orbit?

Unchanged, the object will remain in initial orbit indefinitely.
Impulsive Orbit Maneuvers

Orbit maneuvers are made through changes in velocity.

- $\vec{r}$ and $\vec{v}$ determine orbital elements.
- For fixed $\vec{r}$, changes in $\vec{v}$ map to changes in orbital elements.
  - Set of achievable orbits is limited.
  - Only 3 degrees of freedom.
  - Orbit must pass through $\vec{r}$. 
Impulsive Orbit Maneuvers

Velocity change is caused by thrust.

- For constant thrust, $F$,

$$v(t) = v(0) + \frac{F}{m} \Delta t$$

- for a desired $\Delta v$, the time needed is

$$\Delta t = \frac{m \Delta v}{F}$$

The change in position is

$$\Delta \vec{r}(t) = \frac{m \Delta v^2}{F}$$

- For fixed $\Delta v$, if $\frac{m}{F}$ is small, the $\Delta \vec{r}$ is small
- We will assume $\Delta \vec{r} = 0$
The $\Delta v$ Maneuver

$\Delta v$ refers to the difference between the initial and final velocity vectors.

A $\Delta v$ maneuver can:
- Raise/lower the apogee/perigee
- A change in inclination
- Escape
- Reduction/Increase in period
- Change in RAAN
- Begin a 2+ maneuver sequence of burns.
  - Creates a Transfer Orbit.

We’ll start by talking about coplanar maneuvers.
**Single Burn Coplanar Maneuvers**

**Definition 1.**

Coplanar Maneuvers are those which do not alter $i$ or $\Omega$.

**Example: Simple Tangential Burn**

- For maximum efficiency, a burn must occur at $0^\circ$ flight path angle
  - $\dot{r} = 0$
- Tangential burns can occur at perigee and apogee
Example: Parking Orbits

Suppose we launch from the surface of the earth.

- This creates an initial elliptic orbit which will re-enter.
- To circularize the orbit, we plan on using a burn at apogee.

Problem: We are given $a$ and $e$ of the initial elliptic orbit. Calculate the $\Delta v$ required at apogee to circularize the orbit.
Example: Parking Orbits

Solution: At apogee, we have that

\[ r_a = a(1 + e) \]

From the vis-viva equation, we can calculate the velocity at apogee.

\[ v_a = \sqrt{\mu \left( \frac{2}{r_a} - \frac{1}{a} \right)} = \sqrt{\frac{\mu}{a} \left( \frac{1 - e}{1 + e} \right)} \]

However, for a circular orbit at the same point, we calculate from vis-viva

\[ v_c = \sqrt{\frac{\mu}{r_a}} = \sqrt{\frac{\mu}{a(1 + e)}} \]

Therefore, the \( \Delta v \) required to circularize the orbit is

\[ \Delta v = v_c - v_a = \sqrt{\frac{\mu}{a(1 + e)}} - \sqrt{\frac{\mu}{a} \left( \frac{1 - e}{1 + e} \right)} = \frac{\mu}{a(1 + e)} \left( 1 - \sqrt{1 - e} \right) \]

- It is unusual to launch directly into the desired orbit.
- Instead we use the parking orbit while waiting for more complicated orbital maneuvers.
Most orbits cannot be achieved using a single burn.

**Definition 2.**

- The **Initial Orbit** is the orbit we want to leave.
- The **Target Orbit** is the orbit we want to achieve.
- The **Transfer Orbit** is an orbit which intersects both the initial orbit and target orbit.

**Step 1:** Design a transfer orbit \((a,e,i,\text{ etc.})\).

**Step 2:** Calculate \(\vec{V}_{tr,1}\) at the point of intersection with initial orbit.

**Step 3:** Calculate initial burn to maneuver into transfer orbit.

\[
\Delta v_1 = \vec{V}_{tr,1} - \vec{V}_{init}
\]
Step 4: Calculate \( \vec{v}_{tr,2} \) at the point of intersection with target orbit.

Step 5: Calculate velocity of the target orbit, \( \vec{v}_{fin} \), at the point of intersection with transfer orbit.

Step 6: Calculate the final burn to maneuver into target orbit.

\[
\Delta v_2 = \vec{v}_{fin} - \vec{v}_{tr,2}
\]
Transfer Orbits

There are many orbits which intersect both the initial and target orbits. However, there are some constraints.

Consider

- Circular initial orbit of radius \( r_2 \)
- Circular target orbit of radius \( r_1 \)

Obviously, the transfer orbit must satisfy

\[
    r_p = \frac{p}{1 + e} \leq r_1
\]

and

\[
    r_a = \frac{p}{1 - e} \geq r_2
\]
Occasionally, we want to arrive at

- A certain point in the target orbit, $\vec{r}_2$
- at a certain time, $t_f$

Finding the necessary transfer orbit is Lambert’s Problem.

Primary Applications are:

- Targeting
- Rendez-vous

We are skipping the section on Lambert’s problem.
The cost of a transfer orbit can be calculated using kinetic energy arguments

\[ E_{cost} = \frac{||\Delta v_1||^2 + ||\Delta v_2||^2}{2} \]

Of course, this doesn’t tell us how good the transfer orbit is.

- The energy difference between 2 orbits must come from somewhere.

\[ \Delta E_{min} = -\frac{\mu}{2a_2} + \frac{\mu}{2a_1} \]

- The closer \( E_{cost} \) is to \( E_{min} \), the more efficient the transfer
- More on this effect later
The Hohmann Transfer

The Hohmann transfer is the energy-optimal two burn maneuver between any two coaxial elliptic orbits.

- Proposed by Hohmann (1925)
  - Why?
- Proven for circular orbits by Lawden (1952)
- Proven for coaxial ellipses by Thompson (1986)
We will first consider the circular case.

Theorem 3 (The Hohmann Conjecture).

The energy-optimal transfer orbit between two circular orbits of radii \( r_1 \) and \( r_2 \) is an elliptic orbit with

\[
\begin{align*}
    r_p &= r_1 \quad \text{and} \quad r_a &= r_2
\end{align*}
\]

This yields the orbital elements of the transfer orbit \((a, e)\) as

\[
\begin{align*}
    a &= \frac{r_1 + r_2}{2} \quad \text{and} \quad e = 1 - \frac{r_p}{a} = \frac{r_2 - r_1}{r_2 + r_1}
\end{align*}
\]
The Hohmann Transfer

To calculate the required $\Delta v_1$ and $\Delta v_2$, the initial velocity is the velocity of a circular orbit of radius $r_1$

$$v_{init} = \sqrt{\frac{\mu}{r_1}}$$

The required initial velocity is that of the transfer orbit at perigee. From the vis-viva equation,

$$v_{trans,p} = \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{a}} = \sqrt{2\mu} \sqrt{\frac{1}{r_1} - \frac{1}{r_1 + r_2}} = \sqrt{2\mu} \frac{r_2}{r_1(r_1 + r_2)}$$

So the initial $\Delta v_1$ is

$$\Delta v_1 = v_{trans,p} - v_{init} = \sqrt{2\mu} \frac{r_2}{r_1(r_1 + r_2)} - \sqrt{\frac{\mu}{r_1}} = \sqrt{\frac{\mu}{r_1}} \left( \sqrt{\frac{2r_2}{(r_1 + r_2)}} - 1 \right)$$

The velocity of the transfer orbit at apogee is

$$v_{trans,a} = \sqrt{\frac{2\mu}{r_2} - \frac{\mu}{a}} = \sqrt{2\mu} \frac{r_1}{r_2(r_1 + r_2)}$$
The Hohmann Transfer

The required velocity for a circular orbit at apogee is

\[ v_{fin} = \sqrt{\frac{\mu}{r_2}} \]

So the final \( \Delta v_2 \) is

\[ \Delta v_2 = v_{fin} - v_{trans,a} = \sqrt{\frac{\mu}{r_2}} - \sqrt{2\mu \frac{r_1}{r_2(r_1 + r_2)}} = \sqrt{\frac{\mu}{r_2}} \left( 1 - \sqrt{\frac{2r_1}{(r_1 + r_2)}} \right) \]

Thus we conclude to raise a circular orbit from radius \( r_1 \) to radius \( r_2 \), we use

\[ \Delta v_1 = \sqrt{\frac{\mu}{r_1}} \left( \sqrt{\frac{2r_2}{(r_1 + r_2)}} - 1 \right) \]

\[ \Delta v_2 = \sqrt{\frac{\mu}{r_2}} \left( 1 - \sqrt{\frac{2r_1}{(r_1 + r_2)}} \right) \]
Hohmann Transfer Illustration
The Hohmann Transfer

Transfer Time

The Hohmann transfer is optimal

- Only for impulsive transfers
  - Continuous Thrust is not considered
- Only for two impulse transfers
  - A three impulse transfer can be better
  - Bi-elliptics are better

The transfer time is simply half the period of the orbit. Hence

\[ \Delta t = \frac{\tau}{2} = \pi \sqrt{\frac{a^3}{\mu}} \]

\[ = \pi \sqrt{\frac{(r_1 + r_2)^3}{2\mu}} \]

The Hohmann transfer is also the Maximum Time 2-impulse Transfer.

- Always a tradeoff between time and efficiency
- Bielliptic Transfers extend this tradeoff.
Numerical Example

Problem: Suppose we are in a circular parking orbit at an altitude of 191.34km and we want to raise our altitude to 35,781km. Describe the required orbital maneuvers (time and Δv).

Solution: We will use a Hohmann transfer between circular orbits of

\[ r_1 = 191.35\, km + 1\, ER = 1.03\, ER \quad \text{and} \quad r_2 = 35781\, km + 1\, ER = 6.61\, ER \]

The initial velocity is

\[ v_i = \sqrt{\frac{\mu}{r_1}} = .985 \frac{ER}{TU} \]

The transfer ellipse has \( a = \frac{r_1 + r_2}{2} = 3.82\, ER \). The velocity at perigee is

\[ v_{trans,1} = \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{a}} = 1.296 \frac{ER}{TU} \]

Thus the initial Δv is \( \Delta v_1 = 1.296 - .985 = .315 \frac{ER}{TU} \).
Numerical Example

The velocity at apogee is

\[ v_{trans,1} = \sqrt{\frac{2\mu}{r_2}} - \frac{\mu}{a} = 0.202 \frac{ER}{TU} \]

However, the required velocity for a circular orbit at radius \( r_2 \) is

\[ v_f = \sqrt{\frac{\mu}{r_2}} = 0.389 \frac{ER}{TU} \]

Thus the final \( \Delta v \) is \( \Delta v_2 = 0.389 - 0.202 = 0.182 \frac{ER}{TU} \). The second \( \Delta v \) maneuver should be made at time

\[ t_{fin} = \pi \sqrt{\frac{a^3}{\mu}} = 23.45TU = 5.256hr \]

The total \( \Delta v \) budget is \( 0.497ER/TU \).
The Elliptic Hohmann Transfer

The Hohmann transfer is also energy optimal for coaxial elliptic orbits.

The only ambiguity is whether to make the initial burn at perigee or apogee.

- Need to check both cases
- Often better to make initial burn at perigee
  - Due to Oberth Effect
Summary

This Lecture you have learned:

Coplanar Orbital Maneuvers
- Impulsive Maneuvers
  - $\Delta v$
- Single Burn Maneuvers
- Hohmann transfers
  - Elliptic
  - Circular

Next Lecture: Oberth Effect, Bi-elliptics, Out-of-plane maneuvers.