Modern Control Systems

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Lecture 15: Small Gain Theorem
A algebra is a vector space with a distributive multiplication operator.

**Definition 1.**

A Banach Algebra, $X$, is a Banach space with an associated mapping $\cdot : X \times X \to X$ such that

1. **Identity:** There exists some $I \in X$ such that

   $F \cdot I = I \cdot F = F$

   for all $F \in X$.

2. **Distributivity:** $F \cdot (G \cdot H) = (F \cdot G) \cdot H$ for all $F, G, H \in X$.

3. **Associativity:** $F \cdot (G + H) = F \cdot G + F \cdot H$ for all $F, G, H \in X$.

4. $F \cdot (\alpha G) = (\alpha F) \cdot G$ for all $F, G \in X$ and $\alpha \in \mathbb{R}$.

5. **Submultiplicative Inequality:**

   $\|F \cdot G\| \leq \|F\| \|G\|$
Some algebras have extra properties

- **Inverse Property (Group):** For any \( F \in X \), there exists a \( F^{-1} \in X \) such that

\[
F \cdot F^{-1} = F^{-1} \cdot F = I
\]

- **Commutative Algebra, Abelian Group:** \( F \cdot G = G \cdot F \)

**Square Matrices:**

- Matrix multiplication using the \( \bar{\sigma} \) norm.

**Vectors:**

- Pointwise addition/multiplication **only**.

**Linear Operators:** \( \mathcal{L}(X) \)

- Using the composition operation and induced norm.
Some elements of a Banach Algebra may have an inverse.

**Definition 2.**

For $J \in X$, where $X$ is a Banach Algebra, we say $K$ is the **inverse** of $J$ if $K \in X$ and

$$J \cdot K = K \cdot J = I$$

If such a $K$ exists, we say $J$ is **invertible**.

Note that if the inverse exists, it is unique. **Observation:** The feedback interconnection yields

$$y = (I + GK)^{-1}Gu$$

**Question:** Given $G$ and $K$, how to tell whether $(I + GK)$ is invertible?

- How to tell whether anything is invertible?

**Answer:** Spectral Theory

- When is $\lambda I - A$ invertible?
The simplest form of spectral theory.

**Theorem 3 (Small Gain Theorem).**

Suppose $B$ is a Banach Algebra and $Q \in B$. If $\|G\| < 1$, then $(I - Q)^{-1}$ exists and furthermore

$$(I - Q)^{-1} = \sum_{k=0}^{\infty} Q^k$$

Clearly holds for $B = \mathbb{R}$ since

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r} = (1 - r)^{-1}$$
Small Gain Theorem

Proof.

Relatively Simple: Show $\sum_{k=0}^{\infty} Q^k$ converges and that it is the inverse.

- To show that the sequence $T_i = \sum_{k=0}^{i} Q^k$ converges, we show it is Cauchy.
- Suppose $m > n$. Then

$$||T_m - T_n|| = || \sum_{k=n+1}^{m} Q^k || \leq \sum_{k=n+1}^{m} ||Q^k|| \quad \text{Triangle Inequality}$$

$$\leq \sum_{k=n+1}^{m} ||Q||^k \quad \text{Submultiplicative Inequality}$$

$$= ||Q||^{n+1} \sum_{k=0}^{m-n-1} ||Q||^k$$

$$\leq ||Q||^{n+1} \sum_{k=0}^{\infty} ||Q||^k$$

$$= \frac{1}{1 - ||Q||} \quad \text{Scalar Power Series}$$
Small Gain Theorem

Proof.

• Thus for \( m > n \)

\[
\lim_{n \to \infty} \|T_m - T_n\| = \lim_{n \to \infty} \|Q\|^{n+1} \frac{1}{1 - \|Q\|} = 0
\]

since \( \|Q\| < 1 \).

• Since the sequence is Cauchy and the space Banach, \( \sum_{k=0}^{\infty} Q^k \in B \)

We have shown that \( \sum_{k=0}^{\infty} Q^k \in B \). Now we show that is is the inverse.

• Start by showing it is a right inverse

\[
(I - Q) \sum_{k=0}^{\infty} Q^k = \sum_{k=0}^{\infty} Q^k - \sum_{k=1}^{\infty} Q^k = Q^0 + \sum_{k=1}^{\infty} Q^k - \sum_{k=1}^{\infty} Q^k = I
\]

• Showing \( \sum_{k=1}^{\infty} Q^k (I - Q) = I \) is identical.

• Thus \( (I - Q)^{-1} \) exists, so \( (I - Q) \) is invertible.
Recall the feedback interconnection:

\[ y = (I + GK)^{-1}Gu \]

**Question:** When is \((I + GK)^{-1}G \in \mathcal{L}(L_\infty)\)?

**Answer:** When \(\|GK\| < 1\).

- Using Banach Algebra of Composition

\[ \|GK\| \leq \|G\|\|K\| \]

- We can require \(\|K\| < \frac{1}{\|G\|}\) or \(\|K\| < 1\) and \(\|G\| \leq 1\).

- Can also express as \(\int_0^\infty \|Ce^{As}B\| \, ds \leq 1\)

**Note:** Small Gain Theorem works for **ANY** Banach space (unusual).

- For most results we need a Hilbert space.
Small Gain Theorem

Example

Take the Banach Algebra of square matrices. \((\mathbb{R}^{n \times n}, \|\cdot\| = \bar{\sigma}(\cdot))\).

- Let \(Q = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\), with norm \(\bar{\sigma}(Q) = \frac{1}{2}\).
- By small gain \((I - Q)^{-1}\) exists. Further,

\[(I - Q)^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}^{-1} = \sum_{k=0}^{\infty} Q^k\]

- Now

\[Q = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \quad Q^2 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \quad Q^3 = \begin{bmatrix} 0 & \frac{1}{8} \\ \frac{1}{8} & 0 \end{bmatrix}, \quad Q^4 = \begin{bmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{16} \end{bmatrix}\]

- We conclude that

\[Q = \begin{bmatrix} \frac{1}{4^k} & \frac{1}{2} \frac{1}{4^k} \\ \frac{1}{2} \frac{1}{4^k} & \frac{1}{4^k} \end{bmatrix} = \frac{1}{4^k} \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}\]

- Thus

\[\sum_{k=0}^{\infty} Q^k = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \sum_{k=0}^{\infty} \frac{1}{4^k} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \frac{1}{1 - \frac{1}{4}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \frac{4}{3} \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix}\]
Unfortunately, the small gain theorem is conservative.

- Let
  \[ Q = \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix} \]

- Then \( \bar{\sigma}(Q) = 10 \), yet
  \[
  (I - Q)^{-1} = \begin{bmatrix} 1 & -10 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}^{-1}
  \]

- Furthermore, \( (I - Q)^{-1} = \sum_{k=0}^{\infty} Q^k \).
Spectral Theorem

An extension of the concept of eigenvalues.

- with important differences.

**Definition 4.**

Let $B$ be a Banach Space and $M \in \mathcal{L}(B)$. The **Spectrum** of $M$ is:

$$
\sigma(M) := \{ \lambda \in \mathbb{C} : (\lambda I - M) \text{ is not invertible in } \mathcal{L}(B) \}
$$

The **Spectral Radius** is

$$
\rho(M) := \max \{|\lambda| : \lambda \in \sigma(M)\}
$$

**Fact:** $\sigma(M)$ is non-empty and closed.

- all you can say.
Spectral Theorem

Consider a multiplication operator.

\[ M : u(t) \mapsto e^{-t}u(t) \]

**Theorem 5.**

\[ M \in \mathcal{L}(L_\infty[0, \infty)) \text{ and } \sigma(M) = [0, 1]. \]

**Proof.**

First we show that \([0, 1] \subset \sigma(M)\).

- Since \(((\lambda I - M)u)(t) = (\lambda - e^{-t})u(t)\), we can construct the inverse

  \[ \left( (\lambda I - M)^{-1}u \right)(t) = \frac{1}{\lambda - e^{-t}}u(t) \]

- If \(\lambda \in [0, 1]\), then \(\lim_{t \to -\infty} \log(\lambda) \frac{1}{\lambda - e^{-t}} = \infty\).
- Therefore, \((\lambda I - M)^{-1}\) is unbounded.
- Thus \([0, 1] \subset \sigma(M)\).
Proof.

Now we show that if \( \lambda \not\in [0, 1] \), then \( \lambda \not\in \sigma(M) \).

- If \( \lambda < 0 \), then
  \[
  \left| ((\lambda I - M)^{-1}u)(t) \right| = \frac{1}{\lambda - e^{-t}u(t)} \leq \frac{1}{|\lambda|} |u(t)|
  \]

- If \( \lambda > 1 \), then
  \[
  \left| ((\lambda I - M)^{-1}u)(t) \right| = \frac{1}{\lambda - e^{-t}u(t)} \leq \frac{1}{|\lambda + 1|} |u(t)|
  \]