Advanced Topics and Applications

Matthew M. Peet
Arizona State University

Lecture 04: Advanced Topics and Applications

Overview

Exploration of LMI's and SOS has just begun

- A unifying framework for optimization
- Applications limited only by imagination

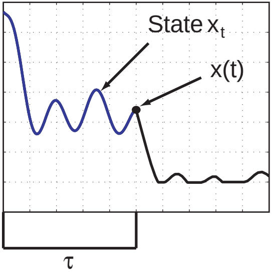
Today we examine a few high-profile areas of SOS research.

1. Control of Delay Systems
2. Control of PDE Systems

Functional Differential Equations

$$\dot{x}(t) = f(x_t)$$

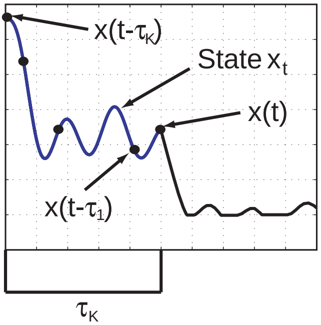
$$x_t(\theta) := x(t+\theta) \qquad \theta \in [-\tau, 0]$$



- Here $x(t) \in \mathbb{R}^n$ and $f : \mathcal{C}_\tau \to \mathbb{R}^n$.
- $x_t \in \mathcal{C}_\tau$ is the full state of the system at time t.
- $x(t) \in \mathbb{R}^n$ is the present state of the system at time t.
- $\mathcal{C}_\tau$ is the space of continuous functions defined in the interval $[-\tau, 0]$.

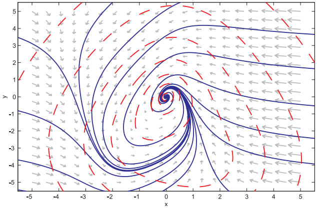
Time-Delay Systems

$$\dot{x}(t) = f(x(t), x(t-\tau_1), \ldots, x(t-\tau_K))$$



- Assume f is a polynomial.

Question: Is the System Stable?



Consider the functional differential equation

$$\dot{x}(t) = f(x_t) \tag{1}$$

Lyapunov Theory: System 1 is stable if there exists some function $V : \mathcal{C}_\tau \to \mathbb{R}$ for which the following holds for all $\phi \in \mathcal{C}_\tau$.

$$\beta\|\phi\| \geq V(\phi) \geq \epsilon\|\phi(0)\|_2$$
$$\dot{V}(\phi) \leq 0$$

Here $\dot{V}(x)$ is the derivative of the functional along trajectories of the system.

Consider: A System of Linear Differential Equations with Discrete Delays

$$\dot{x}(t) = \sum_{i=1}^{m} A_i x(t - \tau_i)$$

- Here $x(t) \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n \times n}$.
- We say the system has K delays, $\tau_i > \tau_{i-1}$ for $i = 1, \ldots, K$ and $\tau_0 = 0$

Question: Is the System Stable?

Consider: A System of Linear Differential Equations with Discrete Delays

$$\dot{x}(t) = \sum_{i=0}^{m} A_i x(t - h_i)$$

Problem: Stability
Given specific $A_i \in \mathbb{R}^{n \times n}$ and $h_i \in \mathbb{R}^+$,
and arbitrary initial condition x_0,
does $\lim_{t \to \infty} x(t) = 0$?

Converse Lyapunov Theorem

Definition 1.

We say that V is a complete quadratic functional if can be represented as:

$$V(\phi) = \int_{-\tau_K}^{0} \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix} d\theta + \int_{-\tau_K}^{0} \int_{-\tau_K}^{0} \phi(\theta) N(\theta, \omega) \phi(\omega) d\theta d\omega$$

Theorem 2.

If a linear time-delay system is asymptotically stable, then there exists a complete quadratic functional, V, and $\eta > 0$ such that for all $\phi \in \mathcal{C}_\tau$

$$V(\phi) \geq \eta \|\phi(0)\|^2 \qquad \textit{and} \qquad \dot{V}(\phi) \leq -\eta \|\phi(0)\|^2$$

Note: Furthermore, M and R can be taken to be continuous everywhere except possibly at points $\theta, \eta = -\tau_i$ for $i = 1, \ldots, K-1$.

Problem Statement

We would like to construct polynomials M and N such that

$$V(\phi) = \int_{-\tau_K}^{0} \begin{bmatrix}\phi(0)\\ \phi(\theta)\end{bmatrix}^T M(\theta) \begin{bmatrix}\phi(0)\\ \phi(\theta)\end{bmatrix} d\theta + \int_{-\tau_K}^{0}\int_{-\tau_K}^{0} \phi(\theta) N(\theta,\omega)\phi(\omega) d\theta d\omega \geq \epsilon\|\phi(0)\|^2$$

and

$$\dot{V}(\phi) = \int_{-\tau_K}^{0} \begin{bmatrix}\phi(-\tau_0)\\ \vdots \\ \phi(-\tau_K)\\ \phi(\theta)\end{bmatrix}^T D(\theta) \begin{bmatrix}\phi(-\tau_0)\\ \vdots \\ \phi(-\tau_K)\\ \phi(\theta)\end{bmatrix} d\theta + \int_{-\tau_K}^{0}\int_{-\tau_K}^{0} \phi(\theta) L(\theta,\omega)\phi(\omega) d\theta d\omega \leq 0$$

Where D and L are polynomials defined by the derivative.

Stability of Linear Differential Equations with Delay

$$V(x) = \int_{-h}^{0} \begin{bmatrix} x(0) \\ x(s) \end{bmatrix}^T M(s) \begin{bmatrix} x(0) \\ x(s) \end{bmatrix} ds + \int_{-h}^{0}\int_{-h}^{0} x(s)N(s,t)x(t)dsdt$$

Problem: Find M and N so that for all $x \in \mathcal{C}[-h,0]$:

$$V(x) > 0$$
$$\dot{V}(x) < 0$$

Question:

Is the problem finite-dimensional? Can we test positivity?

Positive Quadratic Functionals

Consider the complete quadratic functional.

$$V(\phi) = \int_{-\tau_K}^{0} \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix} d\theta + \int_{-\tau_K}^{0} \int_{-\tau_K}^{0} \phi(\theta) N(\theta,\omega) \phi(\omega) d\theta d\omega$$

The complete quadratic Lyapunov functional is **positive** if

- $M \geq_{\mathbf{1}} 0$,
- $R \geq_{\mathbf{2}} 0$.

Definition 3.

$M \geq_{\mathbf{1}} 0$ if for all $\phi \in \mathcal{C}_\tau$

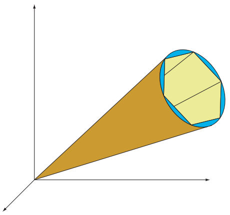
$$\int_{-\tau_K}^{0} \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix} d\theta \geq 0$$

Definition 4.

$N \geq_{\mathbf{2}} 0$ if for all $\phi \in \mathcal{C}_\tau$

$$\int_{-\tau_K}^{0} \int_{-\tau_K}^{0} \phi(\theta) N(\theta,\omega) \phi(\omega) d\theta d\omega \geq 0$$

Searching for Positive Quadratic Functionals



- $\geq_1$ and $\geq_2$ define convex cones.
- Q: How can we represent $\geq_1$ and $\geq_2$ for polynomials using SDP?

Note: Even for matrices, determining positivity on a subset is difficult. e.g. Matrix Copositivity

Representing the Cone $\geq_1$

Theorem 5.

For a given M, the following are equivalent

1. $M \geq_1 \epsilon I$ *for some* $\epsilon > 0$.
2. *There exists a function T and $\epsilon' > 0$ such that*

$$\int_{-\tau_K}^{0} T(\theta) d\theta = 0 \quad \textit{and} \quad M(\theta) + \begin{bmatrix} T(\theta) & 0 \\ 0 & 0 \end{bmatrix} \succeq \epsilon' I$$

Computationally Semi-Tractable:

- Assume M and T are polynomials.
- For the 1-D case, Σ_s is exact.
- The constraint $\int_{-h}^{0} T(s) ds = 0$ is linear

$$\geq_1 \to \Sigma_s \to \text{SDP}$$

Example: Positive Multipliers

$$
\begin{aligned}
M(\theta) &= \begin{bmatrix} -2\theta^2+2 & \theta^3-\theta \\ \theta^3-\theta & \theta^4+\theta^2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ \theta & 0 \\ 0 & \theta \\ 0 & \theta^2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \theta & 0 \\ 0 & \theta \\ 0 & \theta^2 \end{bmatrix} + \begin{bmatrix} 3\theta^2-1 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ \theta & 0 \\ 0 & \theta \\ 0 & \theta^2 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \theta & 0 \\ 0 & \theta \\ 0 & \theta^2 \end{bmatrix} + \begin{bmatrix} 3\theta^2-1 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \theta & \theta^2 \\ 1 & -\theta \end{bmatrix}^T \begin{bmatrix} \theta & \theta^2 \\ 1 & -\theta \end{bmatrix} + \begin{bmatrix} 3\theta^2-1 & 0 \\ 0 & 0 \end{bmatrix} \geq_{\mathbf{1}} 0
\end{aligned}
$$

Since

$$\int_{-1}^{0} (3\theta^2-1)d\theta = 0$$

Positivity of Part 2

Theorem 6 (The Cone $N \geq_2 0$).

Suppose $N(s,t)$ is a polynomial of degree $2d$. Then the following are equivalent:

- $$\int_{-h}^{0}\int_{-h}^{0} x(s)^T N(s,t) x(t) ds dt \geq 0 \qquad \text{for all } x \in \mathcal{C}$$
- *There exists a $Q \geq 0$ such that*
$$N(s,t) + N(t,s)^T = Z_d(s)^T Q Z_d(t)$$

Computationally Semi-Tractable:

- SDP constraint on Q
- Assumes N is polynomial

Example: Positive Integral Operators

If

$$R(\theta,\omega) = \begin{bmatrix} 1-\omega-\theta+2\theta\omega & 1-\theta-\theta\omega^2 \\ 1-\omega-\theta^2\omega & 1+\theta^2\omega^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \theta & 0 \\ 0 & 1 \\ 0 & \theta^2 \end{bmatrix}^T \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 2 & -1 & -1 \\ 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \omega & 0 \\ 0 & 1 \\ 0 & \omega^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \theta & 0 \\ 0 & 1 \\ 0 & \theta^2 \end{bmatrix}^T \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \omega & 0 \\ 0 & 1 \\ 0 & \omega^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\theta & 1 \\ -\theta & \theta^2 \end{bmatrix}^T \begin{bmatrix} 1-\omega & 1 \\ -\omega & \omega^2 \end{bmatrix} \geq_{\mathbf{2}} 0$$

Then

$$\int_{-\tau}^{0}\int_{-\tau}^{0} x(\theta)^T R(\theta,\omega) x(\omega) d\theta d\omega = \int_{-\tau}^{0}\int_{-\tau}^{0} x(\theta)^T G(\theta)^T G(\omega) x(\omega) d\theta d\omega$$

$$= \int_{-\tau}^{0} x(\theta)^T G(\theta)^T d\theta \int_{-\tau}^{0} G(\omega) x(\omega) d\omega = K^T K \geq 0$$

Discontinuous Elements

Theorem 7.

Suppose $M(\theta,\omega)$ *is discontinuous at* $\theta,\omega=-\tau_i$. *Then* $N\ge_2 0$ *if and only if there exists some continuous* $R\ge_2 0$ *such that*

$$N(\theta,\omega)=N_{ij}(\theta,\omega) \qquad \text{for all } \ \theta\in I_i, \quad \omega\in I_j$$

$$N_{ij}(\theta,\omega)=R_{ij}\left(\frac{\tau_K}{\Delta_i}\theta+\tau_{i-1}\frac{\tau_K}{\Delta_i},\frac{\tau_K}{\Delta_j}\omega+\tau_{j-1}\frac{\tau_K}{\Delta_j}\right)$$

$$R(\theta,\omega)=\begin{bmatrix} R_{11}(\theta,\omega) & \dots & R_{1K}(\theta,\omega)\\ \vdots & & \vdots \\ R_{K1}(\theta,\omega) & \dots & R_{KK}(\theta,\omega)\end{bmatrix}$$

Where $I_i=[-\tau_i,-\tau_{i-1}]$ *and* $\Delta_i=\tau_i-\tau_{i-1}$.

Computationally Tractable

The Derivative of Positive Quadratic Functionals

If $M \geq_1 0$ and $N \geq_2 0$, then $V(\phi) \geq 0$. However, the derivative of V is given by

$$\dot{V}(\phi) = \int_{-\tau_K}^{0} \begin{bmatrix} \phi(-\tau_0) \\ \vdots \\ \phi(-\tau_K) \\ \phi \end{bmatrix}^T D(\theta) \begin{bmatrix} \phi(-\tau_0) \\ \vdots \\ \phi(-\tau_K) \\ \phi \end{bmatrix} d\theta + \int_{-\tau_K}^{0} \int_{-\tau_K}^{0} \phi(\theta) L(\theta, \omega) \phi(\omega) d\theta d\omega \leq 0$$

The derivative is **negative** if

- $-L \geq_2 0$
- $-D \geq_3 0$

Definition 8.

$D \geq_3 0$ if for all $\phi \in \mathcal{C}_\tau$

$$\int_{-\tau_K}^{0} \begin{bmatrix} \phi(-\tau_0) \\ \vdots \\ \phi(-\tau_K) \\ \phi \end{bmatrix}^T D(\theta) \begin{bmatrix} \phi(-\tau_0) \\ \vdots \\ \phi(-\tau_K) \\ \phi \end{bmatrix} d\theta \geq 0$$

Result: We can use a generalization of Theorem 5 for $D \geq_3 0$

A Lyapunov Inequality

Theorem 9.

The linear time-delay system is asymptotically stable if there exist polynomials M and R and constant $\eta > 0$ such that

Positive Functional:

- $M \geq_{\mathbf{1}} \eta I$
- $R \geq_{\mathbf{2}} 0$

Negative Derivative:

- $D \leq_{\mathbf{3}} -\eta I$
- $L \leq_{\mathbf{2}} 0$

Where for a single delay,

$$D(\theta) = \begin{bmatrix} D_{11} & PB - Q(-\tau) & \tau(A^T Q(\theta) - \dot{Q}(\theta) + R(0,\theta)) \\ *^T & -S(-\tau) & \tau(B^T Q(\theta) - R(-\tau,\theta)) \\ *^T & *^T & -\tau \dot{S}(\theta) \end{bmatrix}$$

$$L(\theta,\omega) = \frac{d}{d\theta} R(\theta,\omega) + \frac{d}{d\omega} R(\theta,\omega)$$

$$D_{11} = PA + A^T P + Q(0) + Q(0)^T + S(0)$$

where we represent M as $\quad M(\theta) = \begin{bmatrix} P & \tau Q(\theta) \\ \tau Q(\theta)^T & \tau S(\theta) \end{bmatrix}$

We now consider a system with multiple delays.

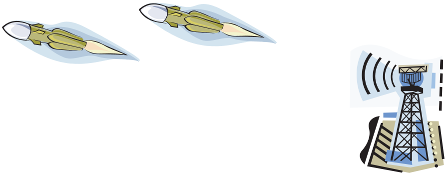
$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -\frac{9}{10} \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \left[\frac{1}{20} x(t - \frac{\tau}{2}) + \frac{19}{20} x(t - \tau) \right]$$

A bisection method was used and results are listed below.

SOS Approach			Piecewise Functional		
d	$\tau_{\min}$	$\tau_{\max}$	N_2	$\tau_{\min}$	$\tau_{\max}$
1	.20247	1.354	1	.204	1.35
2	.20247	1.3722	2	.203	1.372
Analytic	.20246	1.3723			

Table : τ_{max} and τ_{min} using a piecewise-linear functional and our approach and compared to the analytical limit.

Example: Remote Control



A Simple Inertial System: Suppose we are given a specific type of PD controller that we want to implement.

$$\ddot{x}(t) = -ax(t) - \frac{a}{2}\dot{x}(t)$$

The controller is stable for all positive a. Now suppose we want to maintain control from a remote location. When we include the **communication delay**, the equation becomes.

$$\ddot{x}(t) = -ax(t-\tau) - \frac{a}{2}\dot{x}(t-\tau)$$

Question: For what range of a and τ will the controller be stable. The model is linear, but contains a parameter and an uncertain delay.

Delay and Parametric Uncertainty

We can make the Lyapunov functional robust by adding the uncertain parameters $a \in [a_{\min}, a_{\max}]$ and $\tau \in [h_{\min}, h_{\max}]$. We represent this uncertainty as

$$(a,\tau) \in \{(a,\tau) \,:\, g_1(a) \geq 0,\, g_2(\tau) \geq 0\}$$

where $g_1(\tau) = (\tau - \tau_{\min})(\tau_{\max} - \tau)$ and $g_2(a) = (a - a_{\min})(a_{\max} - a)$.

Then the Lyapunov functional is

$$\begin{aligned} V(\phi,a,\tau) = &\int_{-\tau_K}^{0} \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix}^T M(\theta,a,\tau) \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix} d\theta \\ &+ \int_{-\tau_K}^{0} \int_{-\tau_K}^{0} \phi(\theta) N(\theta,\omega,a,\tau) \phi(\omega) d\theta d\omega \end{aligned}$$

Then for positivity, we require

- There exists $R_i \in \Sigma_s$, polynomials $T(a,\tau,s)$ and $\epsilon' > 0$ such that

$$\begin{aligned}&\int_{-\tau}^{0} T(a,\tau,s)ds = 0,\\ &M(a,\tau,s) + \begin{bmatrix} T(a,\tau,s) & 0 \\ 0 & 0 \end{bmatrix} - \epsilon' I \\ &\qquad\qquad = R_0(a,\tau) + g_1(a)R_1(a,\tau) + g_2(\tau)R_2(a,\tau)\end{aligned}$$

- There exists $S_i \in \Sigma_s$ such that

$$\begin{aligned}N(s,t,a,\tau) + N(t,s,a,\tau)^T &= Z_d(s)^T Q(a,\tau) Z_d(t)\\ Q(a,\tau) &= S_0(a,\tau) + g_1(a)S_1(a,\tau) + g_2(\tau)S_2(a,\tau)\end{aligned}$$

Negativity of the derivative is enforced in a similar manner.

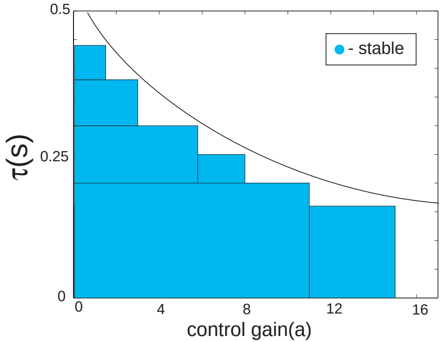
Example: Remote Control

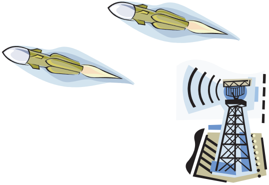
Recall that we considered an inertial system controlled remotely using PD control

$$\ddot{x}(t) = -ax(t-\tau) - \frac{a}{2}\dot{x}(t-\tau)$$

Question: For what range of a and τ will the controller be stable?

- We use parameter-dependent functionals.





Is M polynomial?

Does a continuous solution imply a polynomial solution?

Neglecting N, stability is a Feasibility Problem: Find $M, T, Q \in \mathcal{C}[-h, 0]$:

$$M(s) + \begin{bmatrix} T(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \text{for all } s \in [-h, 0] \quad \text{and} \quad \int_{-h}^{0} T(s) ds = 0$$

$$-A(M, \dot{M})(s) + \begin{bmatrix} Q(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \text{for all } s \in [-h, 0] \quad \text{and} \quad \int_{-h}^{0} Q(s) ds = 0$$

Here $A : \mathcal{C}[-h, 0] \times \mathcal{C}[-h, 0] \to \mathcal{C}[-h, 0]$ is a linear operator given by

$$A(M, \dot{M})(s) = \begin{bmatrix} A_0^T M_{11} + M_{11} A_0 & M_{11} A_1 & 0 \\ A_1^T M_{11} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & A_0^T M_{12}(s) \\ 0 & 0 & A_1^T M_{12}(s) \\ M_{21}(s) A_0 & M_{21}(s) A_1 & 0 \end{bmatrix}$$
$$+ \frac{1}{h} \begin{bmatrix} M_{12}(0) + M_{21}(0) + M_{22}(0) & -M_{12}(-h) & 0 \\ -M_{21}(-h) & -M_{22}(-h) & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -\dot{M}_{12}(s) \\ 0 & 0 & 0 \\ -\dot{M}_{21}(s) & 0 & -\dot{M}_{22}(s) \end{bmatrix}$$

Question:

Can we assume that M, T, and Q are polynomials?

Corollary 10.

Let $L : \mathcal{C}([0,1], \mathbb{R}^{p\times q}) \to \mathbb{R}^{n\times m}$ *be a bounded linear operator and* $f \in \mathcal{C}([0,1], \mathbb{R}^{p\times q})$.

Then for any $\delta > 0$*, there exists a matrix of polynomials* r *such that*

$$\begin{aligned}\|f - r\|_\infty &\le \delta,\\ Lr &= Lf.\end{aligned}$$

Lyapunov-Krasovskii Functionals Can Partly Polynomial

At least for the first term.

Theorem 11.

Suppose there exist continuous functions M, T, Q such that

$$M(s) + \begin{bmatrix} T(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \textit{for all } s \in [-h, 0] \quad \textit{and} \quad \int_{-h}^{0} T(s) ds = 0$$

$$-A(M, \dot{M})(s) + \begin{bmatrix} Q(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \textit{for all } s \in [-h, 0] \quad \textit{and} \quad \int_{-h}^{0} Q(s) ds = 0$$

Then there exist polynomials B, C, D

$$B(s) + \begin{bmatrix} C(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \textit{for all } s \in [-h, 0] \quad \textit{and} \quad \int_{-h}^{0} C(s) ds = 0$$

$$-A(B, \dot{B})(s) + \begin{bmatrix} D(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \textit{for all } s \in [-h, 0] \quad \textit{and} \quad \int_{-h}^{0} D(s) ds = 0$$

What about the Second Term?

Maybe Not

Properties of the term are determined by the kernel.

Definition 12.

A kernel, $N(s,t) \in \mathbb{R}^{n \times n}$ is **separable** if there exists $N_1(s), N_2(s) \in \mathbb{R}^{m \times n}$ such that

$$N(s,t) = N_1(s)^T N_2(t).$$

Theorem 13.

If

$$\int_{-h}^{0} \int_{-h}^{0} x(s) N(s,t) x(t) ds dt > 0$$

for all $x \neq 0$, then N is **NOT** *separable.*

- A functional defined by a separable will Not Be Strictly Positive.

Polynomial Kernels are Separable!

Full-Rank Integral Operators

Semi-Separable Kernels

Semi-separable kernels have been considered for defining the solution map of linear time-vary systems.

Definition 14.

A kernel $N(s,t)$ is **semi-separable** if there exist functions $M_1(s), M_2(s)$ and $N_1(s), N_2(s)$ such that

$$P(s,t) = \begin{cases} N_1(s)M_1(t) & s < t \\ N_2(s)M_2(t) & s \ge t \end{cases}.$$

Some interesting examples of semi-separable kernels include

$$e^{|s-t|} = \begin{cases} e^{-s}e^{t} & s < t \\ e^{s}e^{-t} & s \ge t \end{cases} \qquad \text{and} \qquad I(s-t) = \begin{cases} 0 & s < t \\ 1 & s \ge t \end{cases}$$

We use a sum-of-squares approach to positivity of the operator.

Definition 15.

We say that a polynomial semi-separable kernel, $N : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n \times n}$, is **sum-of-squares** if it can be represented as

$$N(s,t) = \int_{-h}^{0} k(s,u)k(u,t)du$$

where $k(u,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n \times n}$ is a semi-separable kernel.

Polynomial Semi-Separable Kernels

Positivity

Any sum-of-squares semi-separable kernel is positive since

$$(Ax)(s) = \int N(s,t)x(t)dt = \int \int k(s,u)k(u,t)dudt$$

$$= \int k(s,u) \int k(u,t)x(t)dtdu = (B^*Bx)(s)$$

where

$$Bx(s) = \int k(s,t)x(t)dt$$

A sum-of-squares semi-separable kernel

- Defines a positive operator.
- Is semi-separable.

Unlike polynomial kernels, it may be that not all positive semi-separable kernels are sum-of-squares.

- The square root of a compact operator may not be compact.

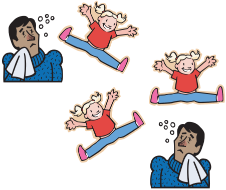
Positivity of Semi-Separable Kernels

Theorem 16.

A polynomial semi-separable kernel, N is sum-of-squares if and only if there exists a $d \geq 0$ and $Q \geq 0$ such that

$$
\begin{aligned}
N(\omega,t) &= \begin{cases} N_1(\omega,t) & \omega \le t \\ N_2(\omega,t) & \omega > t, \end{cases} \\
N_1(\omega,t) &= \int_{-h}^{\omega} R_{11}(t,s,\omega)\,ds + \int_{\omega}^{t} R_{21}(t,s,\omega)\,ds + \int_{t}^{0} R_{22}(t,s,\omega)\,ds, \\
N_2(\omega,t) &= \int_{-h}^{t} R_{11}(t,s,\omega)\,ds + \int_{t}^{\omega} R_{12}(t,s,\omega)\,ds + \int_{\omega}^{0} R_{22}(t,s,\omega)\,ds. \\
R(t,s,\omega) &= \begin{bmatrix} R_{11}(t,s,\omega) & R_{12}(t,s,\omega) \\ R_{12}(t,s,\omega)^T & R_{22}(t,s,\omega) \end{bmatrix} \\
&= Z_{2n,d}(t,s)^T Q Z_{2n,d}(\omega,s) \\
Z_{n,d}(x) &= I_n \otimes Z_d(x)
\end{aligned}
$$

Example: Epidemiological Model of Infection



Consider a human population subject to non-lethal infection by a cold virus. The disease has **incubation period** (τ). Cooke(1978) models the percentage of infected humans(y) using the following equation.

$$\dot{y}(t) = -ay(t) + by(t-\tau)\left[1 - y(t)\right]$$

Where

- a is the rate of recovery for infected persons
- b is the rate of infection for exposed people

The model is nonlinear and contains delay. Equilibria exist at $y^* = 0$ and $y^* = (b-a)/b$.

Example: Epidemiological Model

Recall the dynamics of infection are given by

$$\dot{y}(t) = -ay(t) + by(t-\tau)\left[1 - y(t)\right]$$

Cooke used the following Lyapunov functional to prove delay-independent stability of the 0 equilibrium for $a > b > 0$.

$$V(\phi) = \frac{1}{2}\phi(0)^2 + \frac{1}{2}\int_{-\tau}^{0} a\phi(\theta)^2 d\theta$$

Using semidefinite programming, we were also able to prove delay-independent stability for $a > b > 0$ using the following functional.

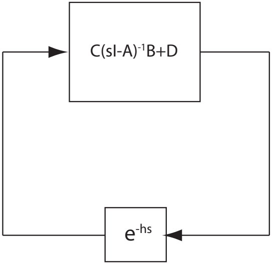
$$V(\phi) = 1.75\phi(0)^2 + \int_{-\tau}^{0} (1.47a + .28b)\phi(\theta)^2 d\theta$$

Conclusion: When the rate of recovery is greater than the rate of infection, the epidemic will die out.

Complexity Reductions: A Reformulation

Couple ODE and Difference Equations

Separate into an ODE and a static difference equation.



$$\dot{x}(t) = Ax(t) + By(t)$$
$$y(t) = Cx(t-\tau)$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$

A new "Complete Quadratic" Functional

Couple ODE and Difference Equations

We use the general form

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{K} B_j y_j(t - r_j),$$

$$y_i(t) = C_i x(t) + \sum_{j=1}^{K} D_{ij} y_j(t - r_j), \qquad i = 1, 2, \ldots K,$$

Which as shown in Gu et al., 2009 has a converse Lyapunov functional of form

$$\begin{aligned} V(x(t), y_t) =& \sum_{i=1}^{K} \int_{-\tau_i}^{0} \begin{bmatrix} x(t) \\ y_i(t+s) \end{bmatrix}^T M_i(s) \begin{bmatrix} x(t) \\ y_i(t+s) \end{bmatrix} ds \\ &+ \sum_{i=1}^{K} \sum_{j=1}^{K} \int_{-\tau_i}^{0} \int_{-\tau_j}^{0} y_i(t+s)^T N_{i,j}(s,\theta) y_j(t+\theta) ds\, d\theta \end{aligned}$$

Apply our SOS positivity conditions to this functional.

Numerical Results

$$\dot{x}(t) = \begin{bmatrix} 0 & .5 & 0 & 0 & 0 & 0 \\ -.5 & -.5 & 0 & 0 & 0 & 0 \\ 0 & 1 & .1 & 1 & 0 & 0 \\ 0 & 0 & -2 & .2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -.9 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ -.5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_1(t - \frac{\tau}{\sqrt{2}}) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -2 & 0 \\ -1 & -1.45 \end{bmatrix} u_2(t-\tau)$$

$$u_1(t) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} x(t), \qquad u_2(t) = \begin{bmatrix} -.2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} x(t)$$

- The system has 6 states, 2 delays and 3 delay channels

Numerical Testing

Comparision of Asymptotic Algorithms for Delay Stability Testing

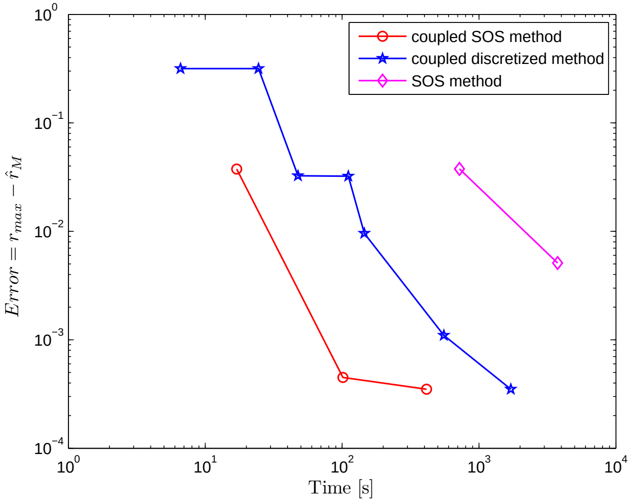


Figure : Log-Log plot of accuracy vs. computation time using SeDuMi

PDE Systems

Now lets see if our approach to analysis of time-delay systems can be expanded.

- Expand to PDE systems
- Expand to the problem of synthesis

Problems:

- PDE systems are all different
  - State-space Theory is different for every system.
  - No universal converse Lyapunov functional.

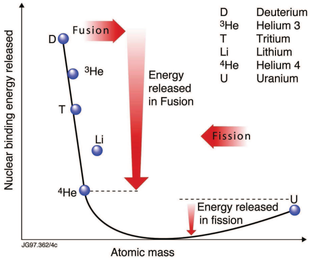
Nuclear Fusion

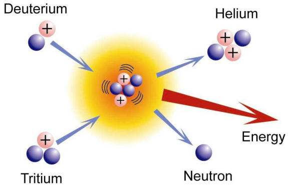
A Renewable Energy Source

Fusion energy is the potential energy difference between particles in free state and particles bound together by the strong nuclear force.

The ^{2}H + ^{3}H to ^{4}He + ^{1}n reaction:

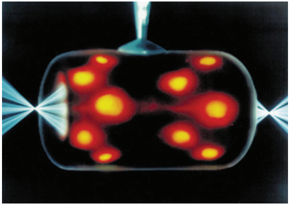
- Strong Nuclear Force: $\Delta E = -3.5$ **MeV/nucleon** decrease in potential energy.
- electrostatic repulsion: $\Delta E = +0.01$ **MeV/nucleon** increase in potential energy (Coulomb Barrier).
- Nuclear Fission of U^{235} only releases $-.85$ MeV/nucleon
- Unfortunately $.01$ MeV/nucleon mean kinetic energy implies a temperature of $120 \cdot 10^6$ K.
  - Temperature at center of sun is $15.7 \cdot 10^6$K.
  - From Maxwell-Boltzmann distribution, we only need $\cong 10^6$K for a statistically significant reaction rate





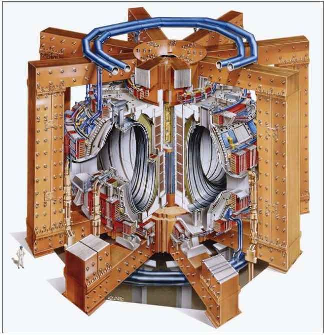
Inertial Confinement

- Compress the fuel quickly
- Plasma does not have time to expand spatially before creating additional reactions.
  - Similar to a hydrogen bomb.



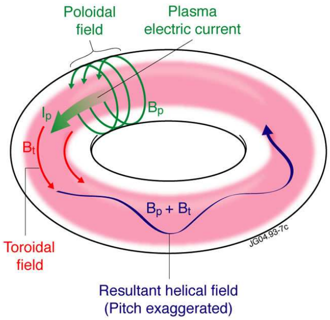
Magnetic Confinement

- **Plasma:** At high temperature, atoms ionize.
  - Hydrogen $\to {}^2\mathrm{H}^+$ ion $+\ \mathrm{e}^+$ electron .
- Charged particles oscillate in a uniform magnetic field.
  - But a uniform field must eventually end.
  - Particles will eventually escape.
- Tokamaks loop the field back on itself.
  - Particles rotate indefinitely.



Magnetic Confinement of Plasma in Tokamaks

Poloidal and Toroidal Fields



The plasma is contained through the combined action of toroidal ϕ and poloidal ψ fields.

- torroidal field is generated from fixed electromagnets
- poloidal field is generated by the motion of the plasma

We need to control the gradient of the poloidal field, ψ_x.

The Dynamics of the Poloidal Flux Gradient

State: $\psi_r(r,t) = \frac{\partial}{\partial r}\psi(r,t)$ - poloidal flux gradient.
Input: j_{ni}, non-inductive current (from ECCD and LH antennae)

$$\frac{\partial \psi_r(r,t)}{\partial t} = \frac{1}{\mu_0 a^2}\frac{\partial}{\partial r}\left(\frac{\eta_\|(r,t)}{r}\frac{\partial}{\partial r}\left(r\psi_r(r,t)\right)\right) + R_0\frac{\partial}{\partial r}\left(\eta_\|(r,t) j_{ni}(r,t)\right).$$

where

$$\begin{aligned} a &= \text{mean radius} \\ R_0 &= \text{magnetic center location} \\ \mu_0 &= \text{permeability of free space} \\ \eta_\|(r,t) &= \text{ plasma resistivity} \\ j_{ni}(r,t) &= \text{ non-inductive current density} \end{aligned}$$

with the boundary conditions

$$\psi_r(0,t) = 0 \text{ and } \psi_r(1,t) = 0.$$

The dynamics are coupled to electron temperature via **Plasma Resistivity**, $\eta_\|$.

- Depends on dynamics of temperature, density, etc.
- Treat as time-varying parameter

Dynamical System Representation

Generalize the problem: A PDE in ODE form

$$\dot{\psi}(t) = A\psi(t) + Bu(t)$$

where A and B are the operators

$$\left(A\psi\right)(r) := \frac{1}{\mu_0 a^2}\frac{\partial}{\partial r}\left(\eta_{\|}(r)\frac{\partial}{\partial r}\left(r\psi(r)\right)\right)$$
$$\left(Bj_{ni}\right)(r) := \frac{\partial}{\partial r}\left(\eta_{\|}(r)j_{eni}(r)\right)$$

Define

$$D_A = \{y \in L_2[0,1] \,:\, y, y_r, y_{rr} \in L_2[0,1],\, y(0) = y(1) = 0\}.$$

For any $\psi(0) \in D_A$, this system has a solution such that $\psi(t) \in D_A$ for all $t \geq 0$ and is associated with Hilbert space $X = L_2[0,1]$.

How to prove stability?

- Find a Lyapunov function, e.g.

$$V = \int \psi(r) M(r) \psi(r) dr + \int \int \psi(r) N(r,\theta) \psi(\theta) dr\, d\theta$$

- A Convex optimization problem.

Theorem 17.

Suppose the operator A generates a strongly continuous semigroup on Hilbert space X with domain D_A. Then

$$\dot{x}(t) = Ax(t)$$

is stable if and only if there exist a positive operator $P \in \mathcal{L}(X \to X)$ such that

$$\langle x, (A^* P + PA)x \rangle_X < \|x\|_X^2$$

for all $x \in D_A$.

- **Optimization** with **Variable:** P
- Same as the Lyapunov question with $V(x) = \langle x, Px \rangle_{L_2}$.
- How to parameterize P and enforce positivity?

Question: How to parameterize a set of operators?

A Class of Operators: $x \in L_2[0,1]$

$$(Px)(s) = M(s)x(s) + \int_0^1 N(s,t)x(t)dt$$

- $M(s)$ is the multiplier of a **Multiplier Operator**.
- $N(s,t)$ is the kernel of an **Integral Operator**.

Question: How to parameterize multiplier and integral operators

- We consider *polynomial* multipliers, $M(s)$ and kernels, $N(s,t)$
- For a finite monomial basis, the set of operators is finite-dimensional

Operator Positivity

Now, how do we enforce positivity on D_A?

- Consider the simplest case where $D_A = L_2[0,1]$. Then

Theorem 18 (Multiplier Operator).

Let $P_M : L_2 \to L_2$ be defined as.

$$(P_M x)(s) = M(s)x(s)$$

Then $P_M \succeq 0$ if and only if $M(s) \geq 0$ for all $s \in [0,1]$.

Theorem 19 (Integral Operator).

Suppose N is polynomial, and let $P_N : L_2 \to L_2$ be defined as.

$$(P_N x)(s) = \int_0^1 N(s,t)x(t)dt$$

Then $P_N \succeq 0$ if and only if $N(s,\theta) = Z(s)^T Q Z(\theta)$ for some $Q \geq 0$ where Z is the vector of monomials.

Full-State Feedback: A Dual LOI

Let $u = Kx$. Then

$$\dot{x}(t) = (A + BK)x(t)$$

Synthesis Problem: Find Lyapunov operator $P \succ 0$ and Control operator K where

$$A^*P + PA + K^*B^*P + PBK \prec 0$$

Unfortunately, this is bilinear in P and K.

Theorem 20 (Dual Stability condition).

Suppose the operator A generates a strongly continuous semigroup on Hilbert space X with domain D_A. The system

$$\dot{x}(t) = Ax(t)$$

is stable if there exist a positive, self-adjoint operator $P \in \mathcal{L}(D_A \to D_A)$ such that

$$\langle x, (PA^* + AP)x \rangle_X < \|x\|_X^2$$

for all $x \in D_A$.

An LOI for synthesis of full-state feedback controllers

For the ∞-dimensional System $\dot{x}(t) = (A + BK)x(t)$,

- Use dual stability condition: $PA^* + AP + (KP)^*B^* + BKP \prec 0$
- Define new variable $Z = KP$.

Theorem 21 (Variable Substitution Trick).

The system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is stabilizable via full-state feedback if there exist operators $P : D_A \to D_A$ *and* Z *such that* $P \succ 0$ *and*

$$PA^* + AP + BZ + Z^*B \prec 0.$$

Furthermore, $K = ZP^{-1}$ *is a stabilizing controller.*

A Convex Optimization Problem

Lyapunov Operator: For simplicity, choose

$$(Px)(r) = M(r)x(r).$$

Control Operator: Choose a relatively simple structure: $K : D_A \to X$

$$(K\psi)(r) = K_1(r)\psi(r) + \frac{d}{dr}\left(K_2(r)\psi(r)\right)$$

New Variable: But K is not the operator we are looking for! We need $Z = KP$.

- The structure of K and P gives the structure of $Z = KP$:

$$(Z\psi)(r) = (KP\psi)(r) = Z_1(r)\psi(r) + \frac{d}{dr}\left(Z_2(r)\psi(r)\right)$$

Recover : Given a *solution*, P, Z, we recover the controller from $K = ZP^{-1}$:

$$K_1(r) = Z_1(r)M(r)^{-1}$$
$$K_2(r) = Z_2(r)M(r)^{-1}.$$

Recall the form for A and B:

$$(A\psi)(r) := \frac{1}{\mu_0 a^2}\frac{\partial}{\partial r}\left(\eta_{\|}(r)\frac{\partial}{\partial r}(r\psi(r))\right)$$

$$(Bj_{ni})(r) := \frac{\partial}{\partial r}\left(\eta_{\|}(r)j_{eni}(r)\right)$$

Positivity Constraint: First we must enforce positivity of P

$$\langle \psi, P\psi\rangle = \int_0^1 \psi(r)M(r)\psi(r)dr \geq 0$$

which holds if and only if

$$M(r) \geq 0 \qquad \text{for all } r \in [0,1]$$

Negativity Constraint: Expanding the synthesis LOI $PA^* + AP + BZ + Z^*B \prec 0$, we get two terms

$$\langle \psi, (PA^*+AP+BZ+Z^*B)\psi\rangle = \int_0^1 \psi(r)R_1(r)\psi(r)ds + \int_0^1 \dot{\psi}(r)R_2(r)\dot{\psi}(r)ds \leq 0$$

where the polynomials R_1 and R_2 are on the following slide.

For negativity, we require both

$$R_1(r) < 0 \qquad \text{and} \qquad R_2(r) < 0 \qquad \text{for all } r \in [0,1].$$

Enforcing Positivity

As promised:

$$R_1(s) := \frac{1}{\mu_0 a^2} b_1\left(r, \frac{d}{dr}\right) M(r) + b_2\left(r, \frac{d}{dr}\right) Z_1(r) + b_3\left(r, \frac{d}{dr}\right) Z_2(r)$$
$$R_2(s) := \frac{1}{\mu_0 a^2} c_1(r) M(r) + c_2(r) Z_2(r).$$

where

$$\begin{aligned}
b_1\left(r, \frac{d}{dr}\right) &= f(r)\left(\frac{\eta_{\|,r}}{r} - \frac{\eta_\|}{r^2}\right) + f'(r)\left(-\frac{\eta_\|}{r} + \eta_{\|,r}\right) \\
&\quad + f''(r)\eta_\| + \frac{f(r)\eta_\|}{r}\frac{d}{dr} + \left(f(r)\eta_\| + f(r)\eta_{\|,r}\right)\frac{d^2}{dr^2}, \\
b_2\left(r, \frac{d}{dr}\right) &= -f'(r) + f(r)\frac{d}{dr}, \\
b_3\left(r, \frac{d}{dr}\right) &= \eta_{\|,r} f'(r) + \eta_\| f''(r) + \eta_{\|,r} f(r)\frac{d}{dr} + \eta_\| f(r)\frac{d^2}{dr^2},
\end{aligned}$$

$c_1(r) = -\eta_\| f(r), c_2(r) = -2\eta_\| f(r)$ and $f(r) = r^2(1-r)$.

Simulation

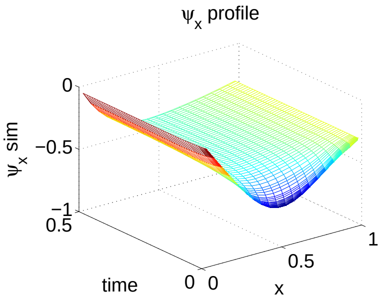


Figure : Time evolution of ψ_x-profile.

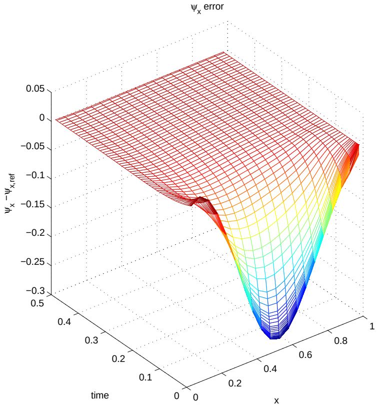


Figure : ψ_x-profile error, $\psi_x - \psi_{x,ref}$. Here $\psi_{x,ref}$ is obtained from the reference q-profile, q_{ref}.

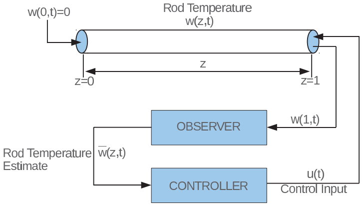
Observing PDE systems: Heat Equation Example

Problem: Feedback requires a knowledge of the heat distribution.

- Sensors can only measure heat at a single point.

Consider the dynamics of heat flux.

$$w_t(x,t) = w_{xx}(x,t)$$



Point Observation: $y(t) = w(1,t)$
Point Actuation: $w_z(1,t) = u(t)$

Design a Feedback Controller:

$$u(t) = K\hat{w}(t)$$

where $\hat{w}$ is the state estimate.
Coupled with a Luenberger Observer:

$$\dot{\hat{w}}(t) = (A + LC + BF)\hat{w}(t) - Ly(t)$$

Using a state-separation argument, the closed loop is stable if $A + LC$ and $A + BF$ are stable.

Convex LOI Problem: Find $P \succ 0, Q \succ 0, Z, R$ such that

$$PA^* + AP + BZ + Z^*B \prec 0 \qquad \text{and} \qquad A^*Q + QA + C^*R^* + RC \prec 0.$$

Observer-Based Controller

The Heat Equation

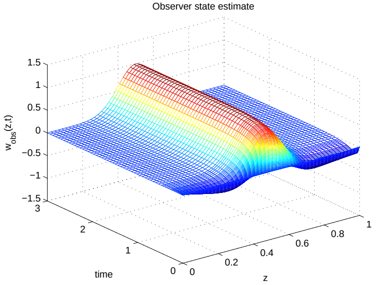


Figure : Estimate of the State

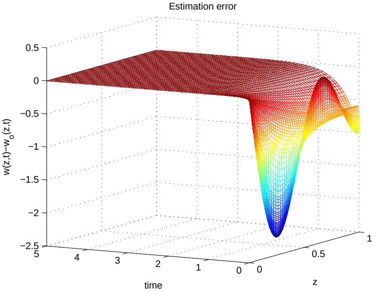


Figure : Error in the Observed State

Observer-Based Controller

The Heat Equation

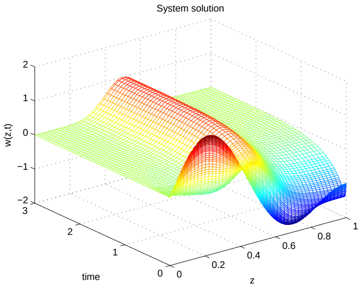


Figure : Effect of Observer-Based Boundary Control

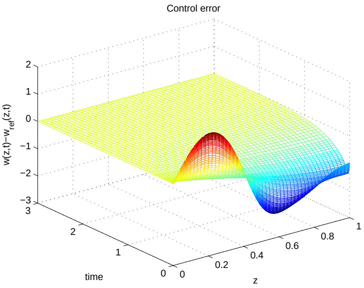


Figure : Error in Observer-Based Boundary Control