Introduction to Optimal Control via LMIs

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Lecture 01: Optimal Control via LMIs

Who Am I?

Website: http://control.asu.edu

Research Interests: Computation, Optimization and Control

Focus Areas:

- Control of Nuclear Fusion
- Immunology
- Thermostats, Renewable Energy, and Power Distribution

Expertise with LMI Methods:

- Optimization of Polynomials
- Parallel Computing for Control
- Control of Delayed Systems
- Control of PDE Systems
- Control of Nonlinear Systems

My Background:

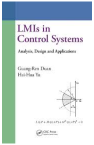
- B.Sc. University of Texas at Austin
- Ph.D. Stanford University
- Postdoc at INRIA Paris
- NSF CAREER Awardee

Office: ERC 253; Lab: GWC 531

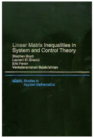
LMI Methods in Optimal and Robust Control

A Toolbox

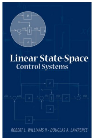
Required: LMIs in Control Systems
by Duan and Yu



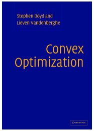
LMIs in Systems and Control Theory
by S. Boyd
Link: Available Online Here



Linear State-Space Control Systems
by Williams and Lawrence



Convex Optimization
by S. Boyd
Link: Available Online Here



Link: Entire Course Online Here

This course is on **RECENT** Developments in Control

- Techniques Developed in the Last 20 years
- Computational Methods
  - No Root Locus
  - No Bode Plots
  - No PID (Proportion-Integral-Differential)

We focus on State-Space Methods

- In the time-domain
- We use large state-space matrices

$$\frac{d}{dt}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\\x_4(t)\end{bmatrix}=\begin{bmatrix}-1 & 1.2 & -1 & .8\\1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\\x_4(t)\end{bmatrix}+\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}u_1(t)\\u_2(t)\end{bmatrix}$$

- We require Matlab
  - Need robust control toolbox.
  - Recommend using YALMIP.
    Link: Installs YALMIP and some other toolboxes

What is Optimization?

An Optimization Problem has 3 parts.

$$\min_{x \in \mathbb{F}} \quad f(x) : \qquad \text{subject to}$$
$$g_i(x) \geq 0 \qquad i = 1, \cdots K_1$$
$$h_i(x) = 0 \qquad i = 1, \cdots K_2$$

Variables: $x \in \mathbb{F}$

- The things you must choose.
- $\mathbb{F}$ represents the set of possible choices for the variables.
- Can be vectors, matrices, functions, systems, locations, colors...
  - However, computers prefer vectors or matrices.

Objective: $f(x)$

- A function which assigns a *scalar* value to any choice of variables.
  - e.g. $[x_1, x_2] \mapsto x_1 - x_2$; red $\mapsto 4$; et c.

Constraints: $g(x) \geq 0;\ h(x) = 0$

- Defines what is a minimally acceptable choice of variables.
- Equality and Inequality constraints are common.
  - x is OK if $g(x) \geq 0$ and $h(x) = 0$.
- Constraints mean variables are not independent. (Constrained optimization is much harder).

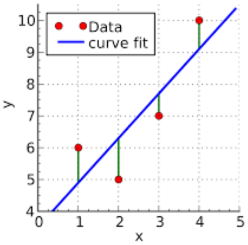
Least Squares

Unconstrained Optimization

Problem: Given a bunch of data in the form

- Inputs: a_i
- Outputs: b_i

Find the function $f(a) = b$ which best fits the data.



For **Least Squares**: Assume $f(a) = z^T a + z_0$ where $z \in \mathbb{R}^n, z_0 \in \mathbb{R}$ are the variables with objective

$$\min \ h(z) := \sum_{i=1}^{K} |f(a_i) - b_i|^2 = \sum_{i=1}^{K} |z^T a_i + z_0 - b_i|^2$$

The **Optimization Problem** is:

$$\min_{z \in \mathbb{R}^n} \|Az - b\|^2$$

where

$$A := \begin{bmatrix} a_1^T & 1 \\ & \vdots \\ a_K^T & 1 \end{bmatrix} \qquad b := \begin{bmatrix} b_1 \\ \vdots \\ b_K \end{bmatrix}$$

In *Classification* we have inputs (data) (x_i), each of which has a binary label $(y_i \in \{-1, +1\})$

- $y_i = +1$ means the output of x_i belongs to group 1
- $y_i = -1$ means the output of x_i belongs to group 2

We want to find a rule (a classifier) which takes the data x and predicts which group it is in.

- Our rule has the form of a function $f(x) = w^T x - b$. Then
  - x is in group 1 if $f(x) = w^T x - b > 0$.
  - x is in group 2 if $f(x) = w^T x - b < 0$.

Question: How to find the best w and b??

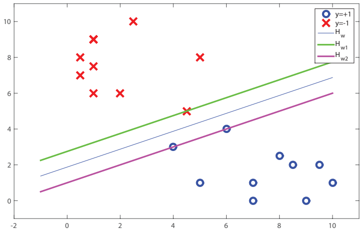
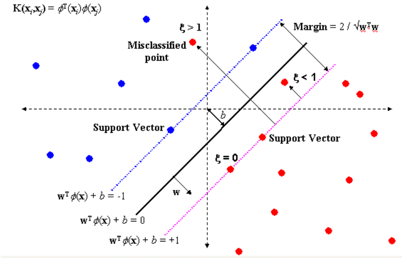


Figure: We want to find a rule which separates two sets of data.



Definition 1.

- A **Hyperplane** is the generalization of the concept of line/plane to multiple dimensions.
$$\{x \in \mathbb{R}^n \ : \ w^T x - b = 0\}$$
- **Half-Spaces** are the parts above and below a Hyperplane.
$$\{x \in \mathbb{R}^n \ : \ w^T x - b \geq 0\} \qquad \text{OR} \qquad \{x \in \mathbb{R}^n \ : \ w^T x - b \leq 0\}$$

We want to separate the data into disjoint half-spaces and maximize the distance between these half-spaces

Variables: $w \in \mathbb{R}^n$ and b define the hyperplane
Constraint: Each existing data point should be correctly labelled.

- $w^T x - b > 1$ when $y_i = +1$ and $w^T x - b < -1$ when $y_i = -1$ (Strict Separation)
- Alternatively: $y_i(w^T x_i - b) \geq 1$.

These two constraints are **Equivalent**.

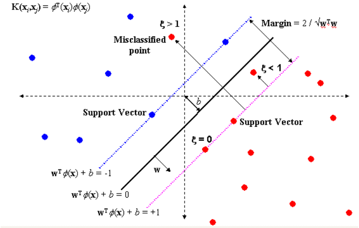


Figure: Maximizing the distance between two sets of Data

Objective: The distance between Hyperplanes $\{x \,:\, w^T x - b = 1\}$ and $\{x \,:\, w^T x - b = -1\}$ is

$$f(w, b) = 2\frac{1}{\sqrt{w^T w}}$$

Machine Learning algorithms solve

$$\min_{w\in\mathbb{R}^p, b\in\mathbb{R}} \frac{1}{2}w^T w, \qquad \text{subject to}$$

$$y_i(w^T x_i - b) \geq 1, \quad \forall i = 1, ..., K.$$

Soft Margin Problems

The hard margin problem can be relaxed to maximize the distance between hyperplanes PLUS the magnitude of classification errors

$$\min_{w\in\mathbb{R}^p, b\in\mathbb{R}} \frac{1}{2}\|w\|^2 + c\sum_{i=1}^{n} \max(0, 1-(w^T x_i - b)y_i).$$

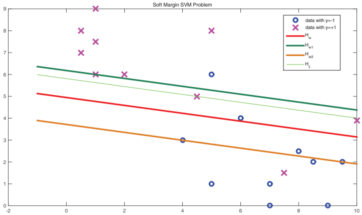


Figure: Data separation using soft-margin metric and distances to associated hyperplanes

Link: Repository of Interesting Machine Learning Data Sets

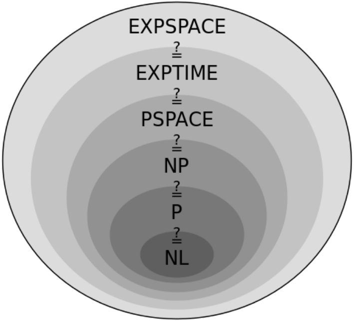
Computational Complexity

In Computer Science, we focus on **Complexity of the PROBLEM**

- NOT complexity of the algorithm.

On an Turing machine, the # of steps is a fn of problem size (number of variables)

- NL: A logarithmic # (SORT)
- P: A polynomial # (LP)
- NP: A polynomial # for verification (TSP)
- NP HARD: at least as hard as NP (TSP)
- NP COMPLETE: A set of Equivalent* NP problems (MAX-CUT, TSP)
- EXPTIME: Solvable in $2^{p(n)}$ steps. p polynomial. (Chess)
- EXPSPACE: Solvable with $2^{p(n)}$ memory.



*Equivalent means there is a polynomial-time reduction from one to the other.

Linear Algebra Review: Matrix Positivity - Definition

Try not to define positivity using eigenvalues. (Eigenvalues don't add)

Definition 2.

A symmetric matrix $P \in \mathbb{S}^n$ is **Positive Semidefinite**, denoted $P \geq 0$ if

$$x^T P x \geq 0 \qquad \text{for all } x \in \mathbb{R}^n$$

Definition 3.

A symmetric matrix $P \in \mathbb{S}^n$ is **Positive Definite**, denoted $P > 0$ if

$$x^T P x > 0 \qquad \text{for all } x \neq 0$$

- P is **Negative Semidefinite** if $-P \geq 0$
- P is **Negative Definite** if $-P > 0$
- A matrix which is neither Positive nor Negative Semidefinite is **Indefinite**

The set of positive or negative matrices is a *convex cone*.

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OK, yes, symmetric matrix has real eigenvalues and the matrix is PSD if and only if its eigenvalues are all PSD and PD iff the eigenvalues are PD.

Pleasant Properties of Positive Matrices

Lemma 4.

$P \in \mathbb{S}^n$ is positive definite if and only if all its eigenvalues are positive.

In this case, the SVD and Unitary (Schur) Diagonalization are the same.

$$\begin{bmatrix}4 & 1 & 2\\ 1 & 5 & 3\\ 2 & 3 & 6\end{bmatrix} = \underbrace{\begin{bmatrix}-.37 & .82 & -.44\\ -.58 & -.58 & -.58\\ -.73 & .04 & .69\end{bmatrix}}_{U} \underbrace{\begin{bmatrix}9.4 & 0 & 0\\ 0 & 3.4 & 0\\ 0 & 0 & 2.2\end{bmatrix}}_{\Lambda=\Sigma} \underbrace{\begin{bmatrix}-.37 & .82 & -.44\\ -.58 & -.58 & -.58\\ -.73 & .04 & .69\end{bmatrix}^T}_{U^T=V^T}$$

Fact: If T is invertible, then $P > 0$ is equivalent to $T^T P T > 0$.

- $P > 0 \to (Tx)^T P (Tx) = x^T T^T P T x > 0$
- $T^T P T > 0 \to (T^{-1}x)^T T^T P T (T^{-1}x) = x^T P x > 0$

Fact: A Positive Definite matrix is invertible: $P^{-1} = U\Sigma^{-1}U^T$.

Fact: The inverse of a positive definite matrix is positive definite: $\Sigma^{-1} > 0$

Fact: For any $P > 0$, there exists a positive square root, $P^{\frac{1}{2}} > 0$ where $P = P^{\frac{1}{2}} P^{\frac{1}{2}}$.

$$P^{\frac{1}{2}} = U\Sigma^{\frac{1}{2}}U^T > 0 \qquad P^{\frac{1}{2}}P^{\frac{1}{2}} = U\Sigma^{\frac{1}{2}}U^T U\Sigma^{\frac{1}{2}}U^T = U\Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}}U^T = U\Sigma U^T = P$$

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Schur Decomposition uses eigenvalues and $V \neq U$

Building Linear Matrix Inequalities

Fact: $\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0$, implies both $X > 0$ and $Z > 0$.

Proof: True since $\begin{bmatrix} 0 \\ z \end{bmatrix}^T \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} 0 \\ z \end{bmatrix} > 0$ and $\begin{bmatrix} x \\ 0 \end{bmatrix}^T \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} > 0$

Fact: $X > 0$ and $Z > 0$ is equivalent to $\begin{bmatrix} X & 0 \\ 0 & Z \end{bmatrix} > 0$.

Proof: True since $x^T X x > 0$ and $z^T Z z > 0$ implies
$\begin{bmatrix} x \\ z \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = x^T X x + z^T Z z > 0$.

Theorem 5 (Schur Complement).

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0 \Leftrightarrow \begin{bmatrix} X & 0 \\ 0 & Z - Y^T X^{-1} Y \end{bmatrix} > 0 \Leftrightarrow \begin{bmatrix} X - Y Z^{-1} Y^T & 0 \\ 0 & Z \end{bmatrix} > 0$$

Diagonal Dominance: If X and Z are big enough, Y doesn't matter.

Leftover Factoids on Positive Matrices

Things which are true:

- $P > 0$ and $Q > 0$ implies $P + Q > 0$.
- $P > 0$ implies $\mu P > 0$ for any positive scalar $\mu > 0$.
- $M^T M \geq 0$ for any matrix, M.
- $P > 0$ implies $M^T P M > 0$ if nullspace of M is empty.

Things which are NOT TRUE (Fallacies):

- $P > 0$ implies $TPT^{-1} > 0$.
- $P > 0$ and $Q > 0$ implies $PQ > 0$.
- $P > 0$ implies $T^T P + PT > 0$
- $P \geq 0$ implies P invertible.
- A has positive eigenvalues implies $A + A^T > 0$.($[1 \; -3; 0 \, 1]$)

Semidefinite Programming - Dual Form

$$\begin{array}{rll} \text{minimize} & \operatorname{trace} CX & \\ \text{subject to} & \operatorname{trace} A_i X = b_i & \text{for all } i \\ & X \succeq 0 & \end{array}$$

- The variable X is a symmetric matrix
- $X \succeq 0$ is another way to say X is positive semidefinite
- The feasible set is the intersection of an affine set with the *positive semidefinite cone*

$$\left\{ X \in \mathbb{S}^n \mid X \succeq 0 \right\}$$

Recall $\operatorname{trace} CX = \sum_{i,j} C_{i,j} X_{j,i}$.

Semidefinite Programming - Dual Form

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Recall $\operatorname{trace} CX = \sum_{i,j} C_{i,j} X_{j,i}$.

SeDuMi uses a Dual Form Input Format

- But we will use YALMIP which is closer to the Primal Form

SDPs with Explicit Variables - Primal Form

We can also explicitly parametrize the affine set to give

$$\begin{array}{rl} \text{minimize} & c^T x \\ \text{subject to} & F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_n F_n \preceq 0 \end{array}$$

where $F_0, F_1, \ldots, F_n$ are symmetric matrices.

The inequality constraint is called a *Linear Matrix Inequality (LMI)*; e.g.,

$$\begin{bmatrix} x_1 - 3 & x_1 + x_2 & -1 \\ x_1 + x_2 & x_2 - 4 & 0 \\ -1 & 0 & x_1 \end{bmatrix} \preceq 0$$

which is equivalent to

$$\begin{bmatrix} -3 & 0 & -1 \\ 0 & -4 & 0 \\ -1 & 0 & 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \preceq 0$$

Linear Matrix Inequalities

Linear Matrix Inequalities are often a *Simpler* way to solve control problems.
Common Form:

$$\text{Find } X :$$
$$\sum_i A_i X B_i + Q > 0$$

There are several very efficient **LMI/SDP Solvers** which interface with YALMIP:

- **SeDuMi**
  - Fast, but somewhat unreliable.
  - Link: http://sedumi.ie.lehigh.edu/
- **LMI Lab** (Part of Matlab's Robust Control Toolbox)
  - Universally disliked, but you already have it.
  - Link: http://www.mathworks.com/help/robust/lmis.html
- **MOSEK** (commercial, but free academic licenses available)
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YALMIP uses a hybrid of the Primal and LMI formats

Using YALMIP to Solve LMIs!

An Example

The system

$$\dot{x} = Ax$$

is stable (eigenvalues have negative real part) if and only if there exists a $P > 0$ such that

$$A^T P + PA < 0$$

YALMIP Code for Stability Analysis:

```
> A = [-1 2 0; -3 -4 1; 0 0 -2];
> P = sdpvar(3,3);
> F = [P >= eye(3)];
> F = [F, A'*P+P*A <= 0];
> optimize(F);
```

If Feasible, YALMIP Code to Retrieve the Solution:

```
> Pfeasible = value(P);
```

Stability of Continuous and Discrete-Time Systems

Definition 6.

A is **Hurwitz** if $\operatorname{Re}\lambda_i(A) < 0$ for all i.

Theorem 7.

$\dot{x}(t) = Ax(t)$ is stable if and only if A is Hurwitz.

For **Discrete-Time Systems:** $x_{k+1} = Ax_k$,

$$x_k = A^k x_0$$

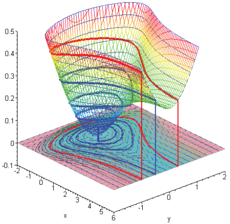
Definition 8.

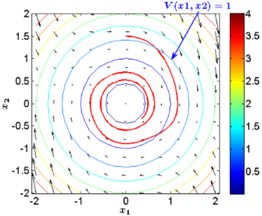
A is **Schur** if $|\lambda_i(A)| < 1$ for all i.

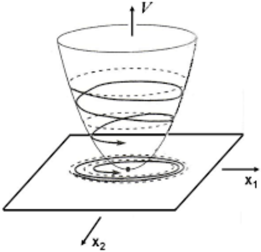
Theorem 9.

$x_{k+1} = Ax_k$ is stable if and only if A is Schur.

Thing to Know: Lyapunov Functions Prove Global Stability







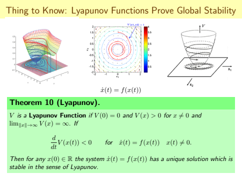
$$\dot{x}(t) = f(x(t))$$

Theorem 10 (Lyapunov).

*V is a **Lyapunov Function** if $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ and $\lim_{\|x\|\to\infty} V(x) = \infty$. If*

$$\frac{d}{dt}V(x(t)) < 0 \qquad \textit{for} \quad \dot{x}(t) = f(x(t)) \quad x(t) \neq 0.$$

Then for any $x(0) \in \mathbb{R}$ the system $\dot{x}(t) = f(x(t))$ has a unique solution which is stable in the sense of Lyapunov.



If $\dot{x}(t) = f(x(t))$, then

$$\dot{V}(x) = \nabla_x V(x)^T f(x)$$

The Lyapunov Inequality (Our First LMI)

Lemma 11 (An LMI for Hurwitz Stability).

A is Hurwitz if and only if there exists a $P > 0$ such that

$$A^T P + PA < 0$$

Proof.

Suppose there exists a $P > 0$ such that $A^T P + PA < 0$.

- Define the Lyapunov function $V(x) = x^T P x$.
- Then $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$.
- Furthermore,

$$\begin{aligned} \dot{V}(x(t)) &= \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) \\ &= x(t)^T A^T P x(t) + x(t)^T P A x(t) \\ &= x(t)^T \left(A^T P + PA\right) x(t) \end{aligned}$$

- Hence $\dot{V}(x(t)) < 0$ for all $x \neq 0$. Thus the system is globally stable.
- Global stability implies A is Hurwitz.

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- Hence $\dot{V}(x(t)) < 0$ for all $x \neq 0$. Thus the system is globally stable.
- Global stability implies A is Hurwitz.

- Although all problems can be posed as optimization problems, not all optimization problems are solvable. In particular, the problem must be convex; the constraints typically should be in the form of a convex cone or inequality. Unfortunately, the constraint that the eigenvalues of a matrix lie in the left half-plane is not easy to enforce directly. The eigenvalues of a matrix are a complex function of the elements of the matrix and, in addition, are quite sensitive to errors in those elements. For this reason, we seek to reformulate the constraint that a matrix be Hurwitz.
- Unlike eigenvalues, the set of Lyapunov functions is a convex cone. This means it is by definition an inequality and therefore well-suited to numerical optimization. Furthermore, the restriction to quadratic Lyapunov functions is likewise a cone constraint and finally, there is a 1-1 relationship between positive matrices and positive Lyapunov functions. This is because the definition of positivity we defined for matrices is identical to the definition of positivity we defined for Lyapunov functions.
- Our first LMI provides the kernel by which we will transmute problems which involve the placement of eigenvalues into problems on the feasibility of certain LMIs. We will use variations of this proof throughout the course, with perhaps the culmination being the KYP Lemma.

The Lyapunov Inequality

Proof.

For the other direction, if A is Hurwitz, for any $Q > 0$, let

$$P = \int_0^\infty e^{A^T s} Q e^{As} ds$$

- Converges because A is Hurwitz.
- Furthermore

$$\begin{aligned} PA &= \int_0^\infty e^{A^T s} Q e^{As} A ds \\ &= \int_0^\infty e^{A^T s} Q A e^{As} ds = \int_0^\infty e^{A^T s} Q \frac{d}{ds}\left(e^{As}\right) ds \\ &= \left[e^{A^T s} Q e^{As}\right]_0^\infty - \int_0^\infty \frac{d}{ds} e^{A^T s} Q e^{As} \\ &= -Q - \int_0^\infty A^T e^{A^T s} Q e^{As} = -Q - A^T P \end{aligned}$$

- Thus $PA + A^T P = -Q < 0$.

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- Thus $PA + A^T P = -Q < 0$.

□

- Sufficiency is easy. Obviously P defines a Lyapunov function in the obvious way. Now simply expand the $\dot{V}$ and we are done.
- Necessity is not as obvious. It comes from the fact that one interpretation of a Lyapunov function is that it represents the magnitude of the forward-time solution starting from a point x. Since $x(t) = e^{At} x_0$, we then have

$$V(x_0) = \int_0^\infty x(s)^T x(s) ds = \int_0^\infty x_0^T e^{A^T s} e^{As} x_0 ds = x_0^T P x_0$$

Discrete-Time Lyapunov Functions

Lemma 12 (An LMI for Schur Stability).

A is Schur if and only if there exists a $P > 0$ such that

$$A^T P A - P < 0$$

Proof.

Suppose there exists a $P > 0$ such that $A^T P A - P < 0$.

- Define the Lyapunov function $V(x) = x^T P x$.
- Then $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$.
- Furthermore,

$$\begin{aligned} V(x_{k+1}) &= x_{k+1}^T P x_{k+1} \\ &= x_k^T A^T P A x_k \\ &< x_k^T P x_k = V(x_k) \end{aligned}$$

- Hence $V(x_{k+1}) < V(x_k)$ for all $k \geq 0$. Thus the system is Stable.
- Stability implies A is Schur.

Proof.

For the other direction, if A is Hurwitz, for any $Q > 0$, let

$$P = \sum_{k=0}^{\infty} (A^T)^k Q A^k$$

$$\begin{aligned} A^T P A - P &= \sum_{k=1}^{\infty} (A^T)^k Q A^k - \sum_{k=0}^{\infty} (A^T)^k Q A^k \\ &= -(A^T)^0 Q A^0 = -Q < 0 \end{aligned}$$

- Thus $A^T P A - P < 0$. □

YALMIP Code:

```
> P = sdpvar(n); eta=.1;
> F=[P>=eta*eye(n)];
> F=[F; A' P A - P<=0];
> optimize(F);
```

Lyapunov Functions

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YALMIP Code:

```
> P = sdpvar(n); eta=.1;
> F=[P>=eta*eye(n)];
> F=[F; A' P A - P<=0];
> optimize(F);
```

- Necessity in the discrete-time case is similar to the continuous-time case. Now, however, the solution is a sequence and its size is defined in the ℓ_2 sense. $x_k = A^k x_0$, we then have

$$V(x_0) = \|x\|_{\ell_2} = \sum_{i=0}^{\infty} x_0^T (A^T)^i A^i x_0$$

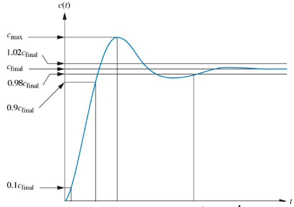
- Note the nonlinearity in A. This will be a problem which will require fixing. For now, however, A is fixed, so the matrix inequality is linear.

Pole Locations AKA D-stability

Some people still care about pole (eigenvalue) locations.

- For these people, we have D-stability.

To begin, you have to define the acceptable region of the complex plane using inequality constraints.



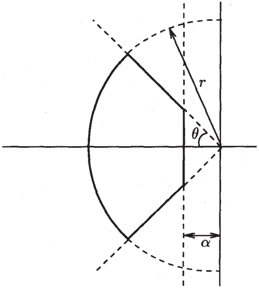
- **Rise Time (t_r):** $\|z\| \le \frac{1.8}{t_r}$
- **Settling Time (t_s):** $\operatorname{Re} z \le -\frac{4.6}{t_s}$
- **Percent Overshoot (M_p):** $\operatorname{Re} z \le -\frac{\ln M_p}{\pi} |\operatorname{Im} z|$

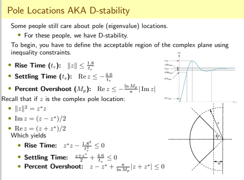
Recall that if z is the complex pole location:

- $\|z\|^2 = z^* z$
- $\operatorname{Im} z = (z - z^*)/2$
- $\operatorname{Re} z = (z + z^*)/2$

Which yields

- **Rise Time:** $z^* z - \frac{1.8^2}{t_r^2} \le 0$
- **Settling Time:** $\frac{z+z^*}{2} + \frac{4.6}{t_s} \le 0$
- **Percent Overshoot:** $z - z^* + \frac{\pi}{\ln M_p} |z + z^*| \le 0$





- Now its time to start introducing lots of LMIs and ways to define them. Up to now, we have been very theory-oriented. This was the minimal amount of theory needed to allow you to understand the next few pages. The next few pages are more practical. Still hard, however.
- The goal, again, is to translate constraints on the eigenvalues of A (which are horribly nonlinear and non-convex functions of the elements of A) to the feasibility of matric inequalities which are linear in A.
- Note that none of these approximations are valid unless the system is composed of only 2 poles and no zeros.
- See, e.g. Franklin, Powell, Enami for derivations.

An LMI for Pole Locations

Gutman proposed a nice LMI for D-stability with a single constraint

Theorem 13 (Gutman).

The pole locations, $z \in \mathbb{C}$ of A satisfy

$$z \in \{z \in \mathbb{C} \, : \, \sum_{k,l} c_{kl} z^k (z^*)^l < 0\}$$

if and only if there exists some $P > 0$ such that

$$\sum_{k,l} c_{kl} A^k P (A^T)^l < 0$$

But this has some disadvantages

- There can only be one constraint.
- The LMI is not linear in A.
  - So controller synthesis is not an LMI.

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But this has some disadvantages

- There can only be one constraint.
- The LMI is not linear in A.
  - So controller synthesis is not an LMI.

- Simply replace z with A and put a P in the middle!
- Region need not be convex!

An LMI for Convex Regions of the Complex Plane

To get around the limitations of Gutman's result, we introduce the concept of LMI regions.

- These are regions which can be represented using LMIs in the z and z^* variables

Definition 14.

An **LMI Region** of the complex plane has the form

$$\{z \in \mathbb{C} \;:\; F_0 + zF_1 + z^*F_2 < 0\}$$

Such regions are hard to visualize (Spectahedron!), but

- Are convex
    - e.g. Minimum rise time is not allowed!
- Can intersect multiple convex regions

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Such regions are hard to visualize (Spectahedron!), but

- Are convex
  - e.g. Minimum rise time is not allowed!
- Can intersect multiple convex regions

- Remember multiple LMI constraints can be concatenated on the diagonal to obtain a single larger LMI constraint.

Rise Time: $z^*z - \underbrace{\frac{1.8^2}{t_r^2}}_{r^2} \le 0$

$$\begin{bmatrix} -r & z \\ z^* & -r \end{bmatrix} = \underbrace{\begin{bmatrix} -r & 0 \\ 0 & -r \end{bmatrix}}_{F_0} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{F_1} z + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{F_2} z^* < 0$$

Which by the Schur complement is equivalent to $r - z^* r^{-1} z > 0$.

Settling Time: $\frac{4.6}{t_s} + \frac{z+z^*}{2} \le 0$

Percent Overshoot: $|z - z^*| + \frac{\pi}{\ln M_p}(z + z^*) \le 0$

$$\begin{bmatrix} \pi(z+z^*) & \ln M_p(z-z^*) \\ \ln M_p(z-z^*)^* & \pi(z+z^*) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{F_0} + \underbrace{\begin{bmatrix} \pi & \ln M_p \\ -\ln M_p & \pi \end{bmatrix}}_{F_1} z + \underbrace{\begin{bmatrix} \pi & -\ln M_p \\ \ln M_p & \pi \end{bmatrix}}_{F_2} z^*$$

Which by the Schur complement is equivalent to $z + z^* < 0$ and $(z - z^*)^2 - (\frac{\pi}{\ln M_p})^2 |z + z^*|^2 > 0$.

An LMI for Convex Regions of the Complex Plane

Examples

Rise Time: $z^*z - \underbrace{\frac{1.8^2}{t_r^2}}_{r^2} \leq 0$

$$\begin{bmatrix} -r & z \\ z^* & -r \end{bmatrix} = \underbrace{\begin{bmatrix} -r & 0 \\ 0 & -r \end{bmatrix}}_{F_0} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{F_1} z + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{F_2} z^* < 0$$

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Which by the Schur complement is equivalent to $z + z^* < 0$ and $(z - z^*)^2 - (\frac{\pi}{\ln M_p})^2|z + z^*|^2 > 0$.

- For rise time $r - z^*r^{-1}z > 0$ is equivalent to $r^2 - z^*z > 0$
- For PO, recall $a > b$ is equivalent to $a^2 > b^2$ if $a, b > 0$ as the square is monotonic increasing

$$\begin{bmatrix} -\pi(z+z^*) & -\ln M_p(z-z^*) \\ -\ln M_p(z-z^*)^* & -\pi(z+z^*) \end{bmatrix} \geq 0$$

$$-\pi(z+z^*) + (\ln M_p)^2(z-z^*)^*(z-z^*)\frac{1}{\pi(z+z^*)} \geq 0$$

$$-(\pi)^2(z+z^*)^2 + (\ln M_p)^2(z-z^*)^*(z-z^*) \leq 0 \qquad (\text{Since } z + z^* < 0)$$

$$-(\frac{\pi}{\ln M_p})^2(z+z^*)^2 + |z-z^*|^2 \leq 0$$

$$|z-z^*|^2 \leq (\frac{\pi}{\ln M_p})^2(|z+z^*|)^2$$

$$(\frac{\pi}{\ln M_p})|z+z^*| \geq |z-z^*|$$

$$(\text{Since } z + z^* < 0)$$

An LMI for Convex Regions of the Complex Plane

Theorem 15 (Chilali + Gahinet).

The pole locations, $z \in \mathbb{C}$ of A satisfy

$$z \in \{z \in \mathbb{C} \;:\; F_0 + zF_1 + z^* F_2 < 0\}$$

if and only if there exists some $P > 0$ such that

$$F_0 \otimes P + F_1 \otimes (AP) + F_2 \otimes (AP)^T < 0$$

The notation $F \otimes P$ is Kronecker notation and means for each element of Fz, replace the scalar z with the matrix P. So, e.g.

$$\underbrace{\begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix}}_{F_0} \otimes P := \begin{bmatrix} f_{11}P & f_{12}P \\ f_{12}P & f_{22}P \end{bmatrix}$$

An LMI for Sector Regions of the Complex Plane

Rise Time:
$$\begin{bmatrix}-r & z\\ z^* & -r\end{bmatrix} = \underbrace{\begin{bmatrix}-r & 0\\ 0 & -r\end{bmatrix}}_{F_0} + \underbrace{\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}}_{F_1} z + \underbrace{\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}}_{F_2} z^* < 0$$

becomes

Lemma 16.

The pole locations, $z \in \mathbb{C}$ of A satisfy $z^* z \le r^2$ if and only if there exists some $P > 0$ such that
$$\begin{bmatrix}-rP & AP\\ (AP)^T & -rP\end{bmatrix} < 0$$

Settling Time: $\frac{4.6}{t_s} + z + z^* \le 0$
becomes

Lemma 17.

The pole locations, $z \in \mathbb{C}$ of A satisfy $2\,\mathrm{Re}\, x \le -\alpha$ if and only if there exists some $P > 0$ such that
$$AP + (AP)^T + \alpha P < 0$$

An LMI for Sector Regions of the Complex Plane

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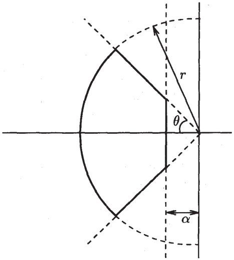
This last LMI is also used to get the exponential decay rate, α

An LMI for Sector Regions of the Complex Plane



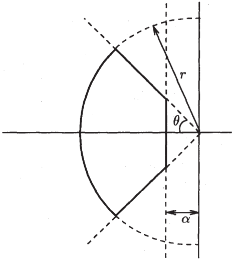
Percent Overshoot: $z + z^* \le -\frac{\ln M_p}{\pi}|z - z^*|$

Lemma 18.

The pole locations, $z \in \mathbb{C}$ of A satisfy $z + z^* \le -\frac{\ln M_p}{\pi}|z - z^*|$ if and only if there exists some $P > 0$ such that

$$\begin{bmatrix} \pi(AP + (AP)^T) & \ln M_p(AP - (AP)^T) \\ \ln M_p(AP - (AP)^T)^T & \pi(AP + (AP)^T) \end{bmatrix} < 0$$

A Combined LMI for D-stability



Theorem 19.

The pole locations, $z \in \mathbb{C}$ of A satisfy $z^* z \le r^2$, $\mathrm{Re}\, x \le -\alpha$ and $z + z^* \le -\frac{\ln M_p}{\pi}|z - z^*|$ if and only if there exists some $P > 0$ such that

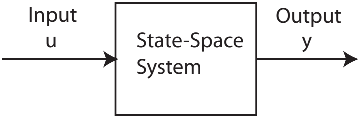
$$\begin{bmatrix} -rP & AP \\ (AP)^T & -rP \end{bmatrix} < 0,$$

$$AP + (AP)^T + 2\alpha P < 0, \quad \textit{and}$$

$$\begin{bmatrix} \pi(AP + (AP)^T) & \ln M_p(AP - (AP)^T) \\ \ln M_p(AP - (AP)^T)^T & \pi(AP + (AP)^T) \end{bmatrix} < 0$$

State-Space System:

$$\dot{x} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \qquad x(0) = 0$$



Input-Output Map:

$$x(t) = \int_0^t e^{A(t-s)} Bu(s)ds$$

$$y(t) = Cx(t) + Du(t) = \int_0^t Ce^{A(t-s)} Bu(s)ds + Du(t)$$

The Static State-Feedback Problem

Lets start with the problem of stabilization.

Definition 20.

The **Static State-Feedback Problem** is to find a feedback matrix K such that

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$u(t) = Kx(t)$$

is stable

- Find K such that $A + BK$ is Hurwitz.

Can also be put in LMI format:

$$\textbf{Find } X > 0,\ K\ :$$
$$X(A+BK) + (A+BK)^T X < 0$$

Problem: Bilinear in K and X.

- The bilinear problem in K and X is a common paradigm.
- Bilinear optimization is not convex.

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- The bilinear problem in K and X is a common paradigm.
- Bilinear optimization is not convex.

State-feedback refers to the fact that $u(t) = Kx(t)$ is a function of all the states, which we assume are all individually measurable. Static refers to the fact that the linear function Kx does not vary in time.

- Resolving this bilinearity is a quintessential step in the controller synthesis process.
- Carries over throughout the course in various generalizations
- The resolution is quite simple and elegant.

An Equivalent LMI for Static State-Feedback

- To convexify the problem, we use a change of variables.

Problem 1:

$$\textbf{Find } X > 0, K \;:$$
$$X(A + BK) + (A + BK)^T X < 0$$

Theorem 21 (Peres).

(A, B) is static-state-feedback stabilizable if and only if there exists some $P > 0$ and Z such that

$$AP + PA^T + BZ + Z^T B^T < 0$$

with $u(t) = ZP^{-1}x(t)$.

Proof.

Suppose $Y > 0$ is a solution to Problem 1. Let $X = Y^{-1} > 0$. Then

$$XA + A^T X = X(AX^{-1} + X^{-1}A^T)X = X(AY + YA^T)X < 0$$

□

Conclusion: If $V(x) = x^T P x$ proves stability of $\dot{x} = Ax$,

- Then $V(x) = x^T P^{-1} x$ proves stability of $\dot{x} = A^T x$.

An Equivalent LMI for Static State-Feedback

- To convexify the problem, we use a change of variables.

Problem 1: **Find** $X > 0, K$:

$$X(A+BK)+(A+BK)^T X < 0$$

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(A,B) is static-state-feedback stabilizable if and only if there exists some $P > 0$ and Z such that

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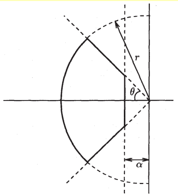
$\square$

Conclusion: If $V(x) = x^T P x$ proves stability of $\dot{x} = Ax$,

- Then $V(x) = x^T P^{-1} x$ proves stability of $\dot{x} = A^T x$.

- A and A^T have the same eigenvalues
- This transformation $A \to A^T$ is sometimes referred to as a duality transformation

Controllers for D-stability



Then we have an LMI which gives us a controller for D-stabilization

Lemma 22 (An LMI for D-Stabilization).

Suppose there exists $X > 0$ *and* Z *such that*

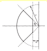
$$\begin{bmatrix} -rP & AP+BZ \\ (AP+BZ)^T & -rP \end{bmatrix} < 0,$$

$$AP + BZ + (AP+BZ)^T + 2\alpha P < 0, \quad \textit{and}$$

$$\begin{bmatrix} AP+BZ+(AP+BZ)^T & c(AP+BZ-(AP+BZ)^T) \\ c((AP+BZ)^T-(AP+BZ)) & AP+BZ+(AP+BZ)^T \end{bmatrix} < 0$$

Then if $K = ZP^{-1}$, *the pole locations,* $z \in \mathbb{C}$ *of* $A + BK$ *satisfy* $|x| \le r$, $\operatorname{Re} x \le -\alpha$ *and* $z + z^* \le -c|z - z^*|$.

Controllers for D-stability



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Lemma 22 (An LMI for D-Stabilization).

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$$AP + BZ + (AP+BZ)^T + 2\alpha P < 0, \quad \textit{and}$$

$$\begin{bmatrix} AP+BZ+(AP+BZ)^T & c(AP+BZ-(AP+BZ)^T) \\ c((AP+BZ)^T-(AP+BZ)) & AP+BZ+(AP+BZ)^T \end{bmatrix} < 0$$

Then if $K = ZP^{-1}$, the pole locations, $z \in \mathbb{C}$ of $A+BK$ satisfy $|x| \le r$, $\mathrm{Re}\, x \le -\alpha$ and $z + z^* \le -c|z - z^*|$.

- Again, we use the fact that A and A^T have the same eigenvalues
- LMIs are particularly useful in that they allow one to directly and sequentially impose constraints on the variables by combining different LMI constraints into a single LMI.
- So we can add closed-loop eigenvalue constraints.
- Or robustness constraints.
- However, this is limited by the variable substitution process $Z = KQ$ and $P = Q^{-1}$.
- Old variables K, P must not appear anywhere in the LMI.

The Discrete-Time Stabilization Problem

Now consider the Schur Stability condition:

$$(A+BK)^T P(A+BK) - P < 0$$

Pre- and Post-multiplying by P^{-1} shows this matrix inequality is equivalent to

$$P^{-1} - P^{-1}(A+BK)^T P(A+BK)P^{-1} > 0$$

Applying the Schur Complement, this matrix inequality is equivalent to

$$\begin{bmatrix} P^{-1} & (A+BK)P^{-1} \\ P^{-1}(A+BK)^T & P^{-1} \end{bmatrix} > 0$$

Using the change of variables $X = P^{-1}$ and $Z = KX$, we get an LMI:

Lemma 23.

Suppose there exists some $X > 0$ and Z such that

$$\begin{bmatrix} X & AX+BZ \\ (AX+BZ)^T & X \end{bmatrix} > 0$$

then if $K = ZX^{-1}$, the closed-loop system matrix $(A+BK)$ is Schur.

Observers

Suppose we have designed a controller

$$u(t) = Fx(t)$$

but we can only measure $y(t) = Cx(t)$!

Question: How to find $x(t)$?

- If (C, A) observable, then we can observe $y(t)$ on $t \in [t, t+T]$.
  - But by then its too late!
  - we need $x(t)$ in *real time*!

Definition 24.

An **Observer**, is an *Artificial Dynamical System* whose output tracks $x(t)$.

Suppose we want to observe the following system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Lets assume the observer is state-space

- What are our inputs and output?
- What is the dimension of the system?

The Luenberger Observer

For now, we consider a special kind of observers, parameterized by the matrix L

$$\begin{aligned}\dot{z}(t) &= (A+LC)z(t) - Ly(t) + (B+LD)u(t)\\ &= \underbrace{Az(t)+Bu(t)}_{\text{Propagate estimate}} + \underbrace{L(\underbrace{Cz(t)+Du(t)}_{\text{Predicted Output}} - \underbrace{y(t)}_{\text{Actual Output}})}_{\text{Correction Term}}\\ \hat{x}(t) &= z(t)\end{aligned}$$

Now define $e(t) = \hat{x}(t) - x(t)$ and the error dynamics simplify to

$$\dot{e}(t) = (A+LC)e(t)$$

Thus the criterion for convergence is $A + LC$ Hurwitz.

Question Can we choose L such that $A + LC$ is Hurwitz?
Similar to choosing $A + BF$.

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Thus the criterion for convergence is $A + LC$ Hurwitz.

Question Can we choose L such that $A + LC$ is Hurwitz?
Similar to choosing $A + BF$.

- The estimator state is itself the estimate of the state.

An LMI for Observer Synthesis

Question: How to compute L?

- The eigenvalues of $A+LC$ and $(A+LC)^T = A^T + C^T L^T$ are the same.
- This is the same problem as controller design!

Theorem 25.

There exists a K such that $A+BK$ is stable if and only if there exists some $P>0$ and Z such that

$$AP + PA^T + BZ + Z^T B^T < 0,$$

where $K = ZP^{-1}$.

Theorem 26.

There exists an L such that $A+LC$ is stable if and only if there exists some $P>0$ and Z such that

$$A^T P + PA + C^T Z + Z^T C < 0,$$

where $L = P^{-1}Z^T$.

So now we know how to design an Luenberger observer.

- Also called an estimator

The error dynamics will be dictated by the eigenvalues of $A+LC$.

- generally a good idea for the observer to converge faster than the plant.

Summary: What do we know?

- How to design a controller which uses the full state.
- How to design an observer which converges to the full state.

Question: Is the combined system stable?

- We know the error dynamics converge.
- Lets look at the coupled dynamics.

Proposition 1.

The system defined by

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t) \\
u(t) &= F\hat{x}(t) \\
\dot{\hat{x}}(t) &= (A + LC + BF + LDF)\,\hat{x}(t) - Ly(t)
\end{aligned}$$

has eigenvalues equal to that of $A + LC$ *and* $A + BF$*.*

Note we have reduced the dependence on $u(t)$.

Observer-Based Controllers

The proof is relatively easy

Proof.

The state dynamics are

$$\dot{x}(t) = Ax(t) + BF\hat{x}(t)$$

Rewrite the estimation dynamics as

$$\begin{aligned}
\dot{\hat{x}}(t) &= (A + LC + BF + LDF)\,\hat{x}(t) - Ly(t) \\
&= (A + LC)\,\hat{x}(t) + (B + LD)\,F\hat{x}(t) - LCx(t) - LDu(t) \\
&= (A + LC)\,\hat{x}(t) + (B + LD)\,u(t) - LCx(t) - LDu(t) \\
&= (A + LC)\,\hat{x}(t) + Bu(t) - LCx(t) \\
&= (A + LC + BF)\,\hat{x}(t) - LCx(t)
\end{aligned}$$

In state-space form, we get

$$\begin{bmatrix}\dot{x}(t)\\ \dot{\hat{x}}(t)\end{bmatrix} = \begin{bmatrix} A & BF \\ -LC & A + LC + BF\end{bmatrix}\begin{bmatrix}x(t)\\ \hat{x}(t)\end{bmatrix}$$

□

Observer-Based Controllers

Proof.

$$\begin{bmatrix}\dot{x}(t)\\ \dot{\hat{x}}(t)\end{bmatrix} = \begin{bmatrix}A & BF\\ -LC & A+LC+BF\end{bmatrix}\begin{bmatrix}x(t)\\ \hat{x}(t)\end{bmatrix}$$

Use the similarity transform $T = T^{-1} = \begin{bmatrix}I & 0\\ I & -I\end{bmatrix}$.

$$\begin{aligned}
T\bar{A}T^{-1} &= \begin{bmatrix}I & 0\\ I & -I\end{bmatrix}\begin{bmatrix}A & BF\\ -LC & A+LC+BF\end{bmatrix}\begin{bmatrix}I & 0\\ I & -I\end{bmatrix}\\
&= \begin{bmatrix}I & 0\\ I & -I\end{bmatrix}\begin{bmatrix}A+BF & -BF\\ A+BF & -(A+LC+BF)\end{bmatrix}\\
&= \begin{bmatrix}A+BF & -BF\\ 0 & A+LC\end{bmatrix}
\end{aligned}$$

which has eigenvalues $A+LC$ and $A+BF$. □

Eigenvalues are invariant under similarity transforms.

Observer-Based Controllers

Proof.

$$\begin{bmatrix}\dot{x}(t)\\ \dot{\hat{x}}(t)\end{bmatrix} = \begin{bmatrix}A & BF\\ -LC & A+LC+BF\end{bmatrix}\begin{bmatrix}x(t)\\ \hat{x}(t)\end{bmatrix}$$

Use the similarity transform $T = T^{-1} = \begin{bmatrix}I & 0\\ I & -I\end{bmatrix}$.

$$\begin{aligned} T\tilde{A}T^{-1} &= \begin{bmatrix}I & 0\\ I & -I\end{bmatrix}\begin{bmatrix}A & BF\\ -LC & A+LC+BF\end{bmatrix}\begin{bmatrix}I & 0\\ I & -I\end{bmatrix}\\ &= \begin{bmatrix}I & 0\\ I & -I\end{bmatrix}\begin{bmatrix}A+BF & -BF\\ A+BF & -(A+LC+BF)\end{bmatrix}\\ &= \begin{bmatrix}A+BF & -BF\\ 0 & A+LC\end{bmatrix}\end{aligned}$$

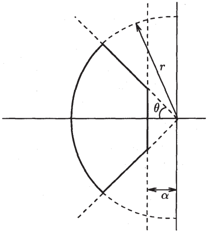
which has eigenvalues $A+LC$ and $A+BF$. □

Eigenvalues are invariant under similarity transforms.

Basically, we just change the states to x and e.

An LMI for Observer D-Stability

- Use the Controller Synthesis LMI to choose K.
- Then use the following LMI to choose L.
- If both $A + LC$ and $A + BK$ satisfy the D-stability condition, then the eigenvalues of the close-loop system will as well.



Lemma 27 (An LMI for D-Observer Design).

Suppose there exists $X > 0$ *and* Z *such that*

$$\begin{bmatrix} -rP & (PA+ZC)^T \\ PA+ZC & -rP \end{bmatrix} < 0,$$

$$(PA+ZC)^T + PA + ZC + 2\alpha P < 0, \quad \textit{and}$$

$$\begin{bmatrix} ((PA+ZC)^T + PA + ZC) & c((PA+ZC)^T - (PA+ZC)) \\ c(PA+ZC-(PA+ZC)^T) & ((PA+ZC)^T + PA + ZC) \end{bmatrix} < 0$$

Then if $L = P^{-1}Z$*, the pole locations,* $z \in \mathbb{C}$ *of* $A + LC$ *satisfy* $|x| \le r$*,* $\operatorname{Re} x \le -\alpha$ *and* $z + z^* \le -c|z - z^*|$*.*

One and Two-Step Discrete-Time Observers

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(C\hat{x}_k + Du_k - y_k)$$

This gives error ($e_k = x_k - \hat{x}_k$) dynamics

$$e_{k+1} = (A + LC)e_k$$

So the Problem is exactly the same as for the continuous-time case.

New Problem: Feedback at step k doesn't include the latest measurements y_k. Instead take the output from the previous estimator and propagate it forward

$$\bar{x}_k = A\hat{x}_{k-1} + Bu_{k-1}, \text{(Current State Estimate w/o update)}$$
$$\hat{x}_k = \bar{x}_k + L(C\bar{x}_k + Du_k - y_k)$$

Eliminating $\hat{x}$, we get the **Current State Estimator!**

$$\bar{x}_{k+1} = A\bar{x}_k + Bu_k + AL(C\bar{x}_k + Du_k - y_k)$$

The error dynamics then become

$$e_{k+1} = (A + LCA)e_k$$

This is not a more difficult problem to solve (replace C with CA)

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The error dynamics then become

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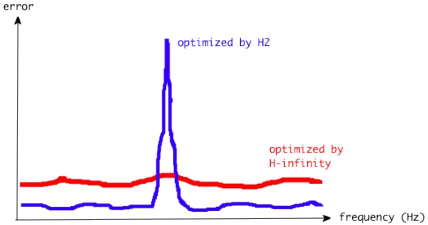
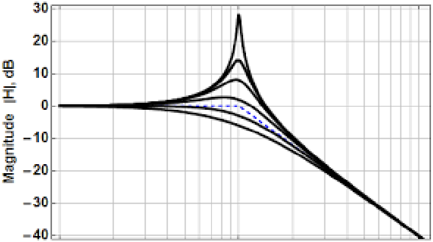
This is not a more difficult problem to solve (replace C with CA)

- The big difference is that in the 2-step case, we first propagate the estimate, then do the correction.
- This means the propagation step uses the previous input and state, but the correction term uses the current input and updated state estimate.





Definition 28.

A function $\hat{G} : \bar{\mathbb{C}}^+ \to \mathbb{C}^{n \times m}$ is in H_∞ if

1. $\hat{G}(s)$ is analytic on the CRHP, $\mathbb{C}^+$.
2. $\lim_{\sigma \to 0^+} \hat{G}(\sigma + \imath\omega) = \hat{G}(\imath\omega)$
3. $\sup_{s \in \mathbb{C}^+} \bar{\sigma}(\hat{G}(s)) < \infty$

- H_∞ is a **Banach Space** with norm

$$\|\hat{G}\|_{H_\infty} = \text{ess} \sup_{\omega \in \mathbb{R}} \bar{\sigma}(\hat{G}(\imath\omega))$$

H_∞ - The space of "Transfer Functions"

From Paley-Wiener, if $G = \Lambda^{-1} M_{\hat{G}} \Lambda$

Theorem 29.

$$\|G\|_{\mathcal{L}(L_2)} = \|M_{\hat{G}}\|_{\mathcal{L}(H_2)} = \|\hat{G}\|_{H_\infty}$$

The **Gain** of the system G can be calculated as $\|\hat{G}\|_{H_\infty}$

- This is the motivation for H_∞ control
- minimize $\sup_u \frac{\|Gu\|_{L_2}}{\|u\|_{L_2}}$.
  - minimize maximum energy of the output.

Conclusion: H_∞ provides a complete parametrization of the space of causal bounded linear time-invariant operators.

H_∞ - The space of "Transfer Functions"

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- This is the motivation for H_∞ control
- minimize $\sup_u \frac{\|Gu\|_{L_2}}{\|u\|_{L_2}}$.
  - minimize maximum energy of the output.

Conclusion: H_∞ provides a complete parametrization of the space of causal bounded linear time-invariant operators.

- Λ is the Laplace Transform
- G is the I/O system on L_2
- $\hat{G} \in H_\infty$ is the transfer function
- $M_{\hat{G}}$ is the multiplication operator on $\hat{L}_2$

The KYP Lemma (AKA: The Bounded Real Lemma)

The most important theorem in this class.

Lemma 30.

Suppose

$$\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then the following are equivalent.

- $\|G\|_{H_\infty} \le \gamma$.
- *There exists a* $X > 0$ *such that*

$$\begin{bmatrix} A^T X + XA & XB \\ B^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0$$

Can be used to calculate the H_∞-norm of a system

- Originally used to solve LMI's using graphs. (Before Computers)
- Now used directly instead of graphical methods like Bode.

The feasibility constraints are linear

- Can be combined with other methods.

The KYP Lemma

Proof.

We will only show that ii) implies i). The other direction requires the Hamiltonian, which we have not discussed.

- We will show that if $y = Gu$, then $\|y\|_{L_2} \le \gamma \|u\|_{L_2}$.
- From the 1 x 1 block of the LMI, we know that $A^TX + XA < 0$, which means A is Hurwitz.
- Because the inequality is strict, there exists some $\epsilon > 0$ such that

$$\begin{aligned}
&\begin{bmatrix} A^TX + XA & XB \\ B^TX & -(\gamma - \epsilon)I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} C^T \\ D^T \end{bmatrix}\begin{bmatrix} C & D \end{bmatrix} \\
&= \begin{bmatrix} A^TX + XA & XB \\ B^TX & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} C^T \\ D^T \end{bmatrix}\begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \epsilon I \end{bmatrix} < 0
\end{aligned}$$

- Let $y = Gu$. Then the state-space representation is

$$\begin{aligned}
y(t) &= Cx(t) + Du(t) \\
\dot{x}(t) &= Ax(t) + Bu(t) \qquad\qquad x(0) = 0
\end{aligned}$$

The KYP Lemma

Proof.

- Let $V(x) = x^T X x$. Then the LMI implies

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \left[\begin{bmatrix} A^T X + XA & XB \\ B^T X & -(\gamma - \epsilon) I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \right] \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

$$= \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^T X + XA & XB \\ B^T X & -(\gamma - \epsilon) I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

$$= \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^T X + XA & XB \\ B^T X & -(\gamma - \epsilon) I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \frac{1}{\gamma} y^T y$$

$$= x^T (A^T X + XA) x + x^T X B u + u^T B^T X x - (\gamma - \epsilon) u^T u + \frac{1}{\gamma} y^T y$$

$$= (Ax + Bu)^T X x + x^T X (Ax + Bu) - (\gamma - \epsilon) u^T u + \frac{1}{\gamma} y^T y$$

$$= \dot{x}(t)^T X x(t) + x(t)^T X \dot{x}(t) - (\gamma - \epsilon) \|u(t)\|^2 + \frac{1}{\gamma} \|y(t)\|^2$$

$$= \dot{V}(x(t)) - (\gamma - \epsilon) \|u(t)\|^2 + \frac{1}{\gamma} \|y(t)\|^2 < 0$$

Proof.

- Now we have $\dot V(x(t)) - (\gamma - \epsilon)\|u(t)\|^2 + \frac{1}{\gamma}\|y(t)\|^2 < 0$
- Integrating in time, we get

$$\begin{aligned}&\int_0^T \left(\dot V(x(t)) - (\gamma - \epsilon)\|u(t)\|^2 + \frac{1}{\gamma}\|y(t)\|^2\right) dt \\ &= V(x(T)) - V(x(0)) - (\gamma - \epsilon)\int_0^T \|u(t)\|^2 dt + \frac{1}{\gamma}\int_0^T \|y(t)\|^2 \Big) dt < 0\end{aligned}$$

- Because A is Hurwitz, $\lim_{t\to\infty} x(t) = 0$.
- Hence $\lim_{t\to\infty} V(x(t)) = 0$.
- Likewise, because $x(0) = 0$, we have $V(x(0)) = 0$.

□

Proof.

- Since $V(x(0)) = V(x(\infty)) = 0$,

$$\lim_{T\to\infty}\left[\dot V(x(T)) - \dot V(x(0)) - (\gamma-\epsilon)\int_0^T \|u(t)\|^2 dt + \frac{1}{\gamma}\int_0^T \|y(t)\|^2 dt\right]$$
$$= 0 - 0 - (\gamma-\epsilon)\int_0^\infty \|u(t)\|^2 dt + \frac{1}{\gamma}\int_0^\infty \|y(t)\|^2 dt$$
$$= -(\gamma-\epsilon)\|u\|_{L_2}^2 + \frac{1}{\gamma}\|y\|_{L_2}^2 dt < 0$$

- Thus

$$\|y\|_{L_2}^2 dt < (\gamma^2 - \epsilon\gamma)\|u\|_{L_2}^2$$

- By definition, this means $\|G\|_{H_\infty}^2 \le (\gamma^2 - \epsilon\gamma) < \gamma^2$ or

$$\|G\|_{H_\infty} < \gamma$$

□

Schur Complement

The KYP condition is

$$\begin{bmatrix} A^TX + XA & XB \\ B^TX & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} C^T \\ D^T \end{bmatrix}\begin{bmatrix} C & D \end{bmatrix} < 0$$

Recall the Schur Complement

Theorem 31 (Schur Complement).

For any $S \in \mathbb{S}^n$, $Q \in \mathbb{S}^m$ and $R \in \mathbb{R}^{n\times m}$, the following are equivalent.

1. $\begin{bmatrix} M & R \\ R^T & Q \end{bmatrix} < 0$
2. $Q < 0$ *and* $M - RQ^{-1}R^T < 0$

In this case, let $Q = -\frac{1}{\gamma}I < 0$,

$$M = \begin{bmatrix} A^TX + XA & XB \\ B^TX & -\gamma I \end{bmatrix} \qquad R = \begin{bmatrix} C & D \end{bmatrix}^T$$

Note we are making the LMI Larger.

H_∞-Optimal Control via Schur Complement?

The Schur Complement says that

$$\begin{bmatrix} A^TX + XA & XB \\ B^TX & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} C^T \\ D^T \end{bmatrix}\begin{bmatrix} C & D \end{bmatrix} < 0$$

if and only if

$$\begin{bmatrix} A^TX + XA & XB & C^T \\ B^TX & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0$$

This leads to the
Full-State Feedback Condition

$$\begin{bmatrix} (A+B_2F)^TX + X(A+B_2F) & XB_1 & (C_1+D_{12}F)^T \\ B_1^TX & -\gamma I & D_{11}^T \\ (C_1+D_{12}F) & D_{11} & -\gamma I \end{bmatrix} < 0$$

which is now bilinear in X and F.

A Variation on the KYP lemma is the positive-real lemma

Lemma 32.

Suppose

$$\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then the following are equivalent.

- G *is passive. i.e.* $(\langle u, Gu\rangle_{L_2} \geq 0)$*.*
- *There exists a* $P > 0$ *such that*

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -D^T - D \end{bmatrix} \leq 0$$

An LMI for the H_2-norm of a System

Theorem 33.

Suppose $\hat{P}(s) = C(sI - A)^{-1}B$. Then the following are equivalent.

1. *A is Hurwitz and $\|\hat{P}\|_{H_2}^2 < \gamma$.*
2. *There exists some $X > 0$ such that*

$$\operatorname{trace} CXC^T < \gamma$$
$$AX + XA^T + BB^T < 0$$

An LMI for the H_2-norm of a System

Theorem 33.

Suppose $\hat{P}(s) = C(sI - A)^{-1}B$. Then the following are equivalent.

1. *A is Hurwitz and $\|\hat{P}\|_{H_2}^2 < \gamma$.*
2. *There exists some $X > 0$ such that*

$$\operatorname{trace} CXC^T < \gamma$$
$$AX + XA^T + BB^T < 0$$

- The H_2 norm of a system is conceptually identical to the Frobenius norm on a matrix.
- Minimizing the H_2-norm using full-state feedback is the LQR problem.
- However, minimizing the H_2 norm reflects a view of the system based on representation and not operation. That is, the controller minimizes the size of the *representation* of the system as opposed to the *performance* of the system.
- Like judging a book based on how many words it has.

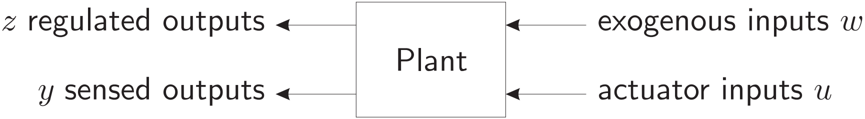
2-input 2-output Framework

We introduce the control framework by separating internal signals from external signals.

Output Signals:

- **z:** Output to be controlled/minimized
  - Regulated output
- **y:** Output used by the controller
  - Must be measured in real-time by sensor
  - May replicate signals from regulated output

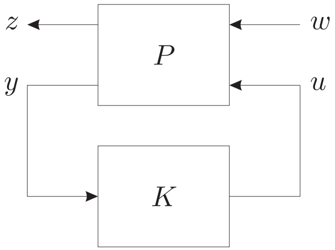
2-input 2-output Framework



Input Signals:

- **w:** Disturbance, Tracking Signal, etc.
  - exogenous input
- **u:** Output from controller
  - Input to actuator
  - Not related to external input

The controller closes the loop from y to u.



For a linear system P, we have 4 subsystems.

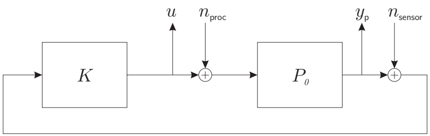
$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

All P_{ij} are MIMO

$$P_{11} : w \mapsto z \qquad P_{12} : u \mapsto z$$

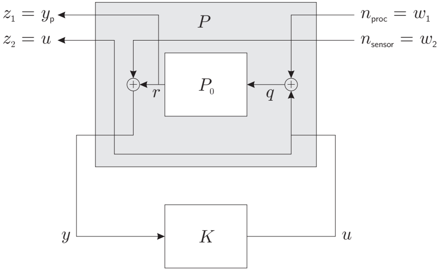
$$P_{21} : w \mapsto y \qquad P_{22} : u \mapsto y$$

The Regulator

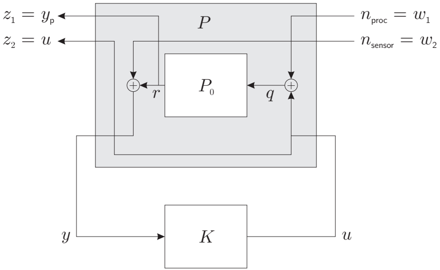


If we define

$$z_2 = u \qquad\qquad q = w_1 + u$$
$$z_1 = y_p \qquad\qquad y = r + w_2$$



The Regulator



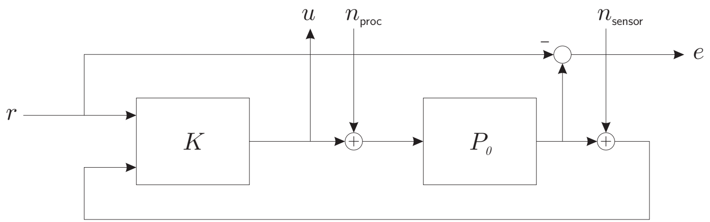
The reconfigured plant P is given by

$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} P_0 & 0 & P_0 \\ 0 & 0 & I \\ P_0 & I & P_0 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \\ u(t) \end{bmatrix}$$

If $P_0 = (A, B, C, D)$, then

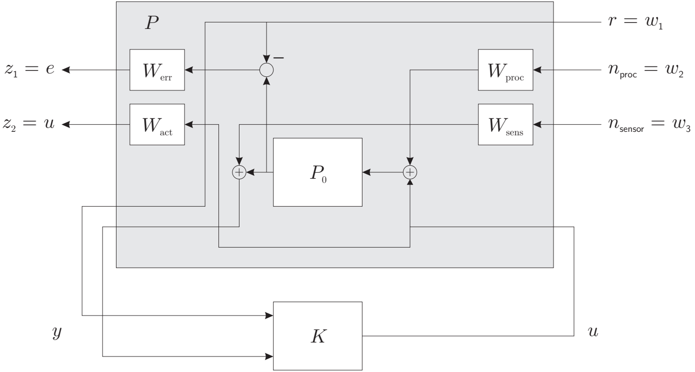
$$P = \left[\begin{array}{c|ccc} A & B & 0 & B \\ \hline C & D & 0 & D \\ 0 & 0 & 0 & I \\ C & D & I & D \end{array}\right]$$

Tracking Control



$r =$ tracking input	$w_2 = n_{proc}$	$w_1 = r$
$e =$ tracking error	$w_3 = n_{sensor}$	$u = u$
$n_{proc} =$ process noise	$z_1 = e$	$y_1 = r$
$n_{sensor} =$ sensor noise	$z_2 = u$	$y_2 = y_p$

Tracking Control

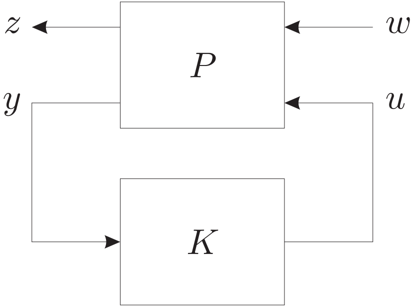


$$P = \begin{bmatrix} I & -P_0 & 0 & -P_0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & P_0 & I & P_0 \end{bmatrix}$$

$$z_1 = r - P_0(n_{proc} + u)$$
$$z_2 = u$$
$$y_1 = r$$
$$y_2 = w_3 + P_0(n_{proc} + u)$$

Linear Fractional Transformation

Close the loop



Plant:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \qquad \text{where} \qquad P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

Controller:

$$u = Ky \qquad \text{where} \qquad K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

Linear Fractional Transformation

$$
\begin{aligned}
z &= P_{11}w + P_{12}u \\
y &= P_{21}w + P_{22}u \\
u &= Ky
\end{aligned}
$$

Solving for u,

$$u = KP_{21}w + KP_{22}u$$

Thus

$$
\begin{aligned}
(I - KP_{22})u &= KP_{21}w \\
u &= (I - KP_{22})^{-1}KP_{21}w
\end{aligned}
$$

Now we solve for z:

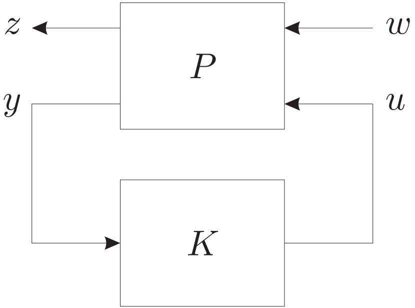
$$z = \left[P_{11} + P_{12}(I - KP_{22})^{-1}KP_{21}\right]w$$

Linear Fractional Transformation

This expression is called the Linear Fractional Transformation of (P, K), denoted

$$\underline{\mathsf{S}}(P, K) := P_{11} + P_{12}(I - KP_{22})^{-1}KP_{21}$$

AKA: Lower Star Product



Closed-Loop Dynamics

In state-space format:

$$\begin{bmatrix}\dot{x}(t)\\ \dot{x}_K(t)\end{bmatrix} = \begin{bmatrix}A & 0\\ 0 & A_K\end{bmatrix}\begin{bmatrix}x(t)\\ x_K(t)\end{bmatrix} + \begin{bmatrix}B_2 & 0\\ 0 & B_K\end{bmatrix}\begin{bmatrix}u(t)\\ y(t)\end{bmatrix} + \begin{bmatrix}B_1\\ 0\end{bmatrix}w(t)$$
$$z(t) = \begin{bmatrix}C_1 & 0\end{bmatrix}\begin{bmatrix}x(t)\\ x_K(t)\end{bmatrix} + \begin{bmatrix}D_{12} & 0\end{bmatrix}\begin{bmatrix}u(t)\\ y(t)\end{bmatrix} + D_{11}w(t)$$

From

$$u(t) = D_K y(t) + C_K x_K(t)$$
$$y(t) = D_{22}u(t) + C_2 x(t) + D_{21}w(t)$$

We have

$$\begin{bmatrix}I & -D_K\\ -D_{22} & I\end{bmatrix}\begin{bmatrix}u(t)\\ y(t)\end{bmatrix} = \begin{bmatrix}0 & C_K\\ C_2 & 0\end{bmatrix}\begin{bmatrix}x(t)\\ x_K(t)\end{bmatrix} + \begin{bmatrix}0\\ D_{21}\end{bmatrix}w(t)$$

Because the rest is state-space, the interconnection is well-posed if and only if the matrix $\begin{bmatrix}I & -D_K\\ -D_{22} & I\end{bmatrix}$ is invertible.

Optimal Control

Choose K to minimize

$$\|P_{11} + P_{12}(I - KP_{22})^{-1}KP_{21}\|_{H_\infty}$$

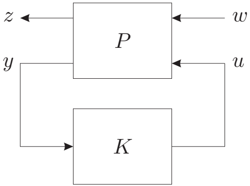
Equivalently choose $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ to minimize

$$\left\|\left[\begin{array}{c|c} \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} & \begin{matrix} B_1 + B_2 D_K Q D_{21} \\ B_K Q D_{21} \end{matrix} \\ \hline \begin{bmatrix} C_1 & 0 \end{bmatrix} + \begin{bmatrix} D_{12} & 0 \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} & D_{11} + D_{12} D_K Q D_{21} \end{array}\right]\right\|_{H_\infty}$$

where $Q = (I - D_{22}D_K)^{-1}$.

In either case, the problem is Nonlinear.

Optimal Full-State Feedback Control



For the full-state feedback case, we consider a controller of the form

$$u(t) = Fx(t)$$

Controller:

$$u = Ky \qquad \text{where} \qquad K = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F \end{array}\right]$$

Plant:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \qquad \text{where} \qquad P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ I & 0 & 0 \end{array}\right]$$

Optimal Full-State Feedback Control

Thus the closed-loop state-space representation is

$$\underline{S}(\hat{P},\hat{K}) = \left[\begin{array}{c|c} A + B_2F & B_1 \\ \hline C_1 + D_{12}F & D_{11} \end{array}\right]$$

By the KYP lemma, $\|\underline{S}(\hat{P},\hat{K})\|_{H_\infty} < \gamma$ if and only if there exists some $X > 0$ such that

$$\begin{bmatrix} (A+B_2F)^TX + X(A+B_2F) & XB_1 \\ B_1^TX & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} (C_1+D_{12}F)^T \\ D_{11}^T \end{bmatrix}\begin{bmatrix} (C_1+D_{12}F) & D_{11} \end{bmatrix} < 0$$

This is a matrix inequality, but is nonlinear

- Quadratic (Not Bilinear)
- May NOT apply variable substitution trick.

Dual KYP Lemma

To apply the variable substitution trick, we must also construct the dual form of this LMI.

Lemma 34 (KYP Dual).

Suppose

$$\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then the following are equivalent.

- $\|G\|_{H_\infty} \le \gamma$.
- *There exists a* $Y > 0$ *such that*

$$\begin{bmatrix} YA^T + AY & B & YC^T \\ B^T & -\gamma I & D^T \\ CY & D & -\gamma I \end{bmatrix} < 0$$

Dual KYP Lemma

To apply the variable substitution trick, we must also construct the dual form of this LMI.

Lemma 34 (KYP Dual).

Suppose

$$\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then the following are equivalent.

- $\|G\|_{H_\infty} \le \gamma$.
- *There exists a* $Y > 0$ *such that*

$$\begin{bmatrix} YA^T + AY & B & YC^T \\ B^T & -\gamma I & D^T \\ CY & D & -\gamma I \end{bmatrix} < 0$$

Simply multiply the Primal KYP on the left and right by

$$\begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and let $Y = P^{-1}$

An LMI for H_∞-Optimal Full-State Feedback Control

We can now apply this result to the state-feedback problem.

Theorem 35.

The following are equivalent:

- *There exists an F such that $\|\underline{S}(P, K(0,0,0,F))\|_{H_\infty} \le \gamma$.*
- *There exist $Y > 0$ and Z such that*

$$\begin{bmatrix} YA^T + AY + Z^T B_2^T + B_2 Z & B_1 & YC_1^T + Z^T D_{12}^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} < 0$$

Then $F = ZY^{-1}$.

An LMI for H_∞-Optimal Output Feedback Control

Theorem 36.

The following are equivalent.

- *There exists a* $\hat{K} = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ *such that* $\|S(K,P)\|_{H_\infty} < \gamma$.
- *There exist* $X_1, Y_1, A_n, B_n, C_n, D_n$ *such that* $\begin{bmatrix} X_1 & I \\ I & Y_1 \end{bmatrix} > 0$

$$\begin{bmatrix} AY_1+Y_1A^T+B_2C_n+C_n^TB_2^T & *^T & *^T & *^T \\ A^T+A_n+[B_2D_nC_2]^T & X_1A+A^TX_1+B_nC_2+C_2^TB_n^T & *^T & *^T \\ [B_1+B_2D_nD_{21}]^T & [X_1B_1+B_nD_{21}]^T & -\gamma I & \\ C_1Y_1+D_{12}C_n & C_1+D_{12}D_nC_2 & D_{11}+D_{12}D_nD_{21} & -\gamma I \end{bmatrix} < 0$$

Moreover,

$$\left[\begin{array}{c|c} A_{K2} & B_{K2} \\ \hline C_{K2} & D_{K2} \end{array}\right] = \begin{bmatrix} X_2 & X_1B_2 \\ 0 & I \end{bmatrix}^{-1} \left[\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} - \begin{bmatrix} X_1AY_1 & 0 \\ 0 & 0 \end{bmatrix}\right] \begin{bmatrix} Y_2^T & 0 \\ C_2Y_1 & I \end{bmatrix}^{-1}$$

for any full-rank X_2 *and* Y_2 *such that*

$$\begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}^{-1}$$

An LMI for H_∞-Optimal Output Feedback Control

Theorem 36.

The following are equivalent.

- *There exists a* $\hat{K} = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ *such that* $\|S(K,P)\|_{H_\infty} < \gamma$.
- *There exist* $X_1, Y_1, A_n, B_n, C_n, D_n$ *such that* $\begin{bmatrix} X_1 & I \\ I & Y_1 \end{bmatrix} > 0$

$$\begin{bmatrix} AY_1 + Y_1 A^T + B_2 C_n + C_n^T B_2^T & *^T & *^T & *^T \\ A^T + A_n + [B_2 D_n C_2]^T & X_1 A + A^T X_1 + B_n C_2 + C_2^T B_n^T & *^T & *^T \\ [B_1 + B_2 D_n D_{21}]^T & [X_1 B_1 + B_n D_{21}]^T & -\gamma I & \\ C_1 Y_1 + D_{12} C_n & C_1 + D_{12} D_n C_2 & D_{11} + D_{12} D_n D_{21} & -\gamma I \end{bmatrix} < 0$$

Moreover,

$$\left[\begin{array}{c|c} A_{K2} & B_{K2} \\ \hline C_{K2} & D_{K2} \end{array}\right] = \begin{bmatrix} X_2 & X_1 B_2 \\ 0 & I \end{bmatrix}^{-1} \left[\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} - \begin{bmatrix} X_1 A Y_1 & 0 \\ 0 & 0 \end{bmatrix} \right] \begin{bmatrix} Y_2^T & 0 \\ C_2 Y_1 & I \end{bmatrix}^{-1}$$

for any full-rank X_2 *and* Y_2 *such that*

$$\begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}^{-1}$$

This LMI requires

- Two changes of variables
- A Half-Dual Transformation
- See slides online for complete proof.

Conclusion

Then, we construct our controller using

$$
\begin{aligned}
D_K &= (I + D_{K2}D_{22})^{-1}D_{K2}\\
B_K &= B_{K2}(I - D_{22}D_K)\\
C_K &= (I - D_K D_{22})C_{K2}\\
A_K &= A_{K2} - B_K(I - D_{22}D_K)^{-1}D_{22}C_K.
\end{aligned}
$$

where

$$
\left[\begin{array}{c|c} A_{K2} & B_{K2} \\ \hline C_{K2} & D_{K2}\end{array}\right] = \begin{bmatrix} X_2 & X_1B_2 \\ 0 & I\end{bmatrix}^{-1}\left[\begin{bmatrix} A_n & B_n \\ C_n & D_n\end{bmatrix} - \begin{bmatrix} X_1AY_1 & 0 \\ 0 & 0\end{bmatrix}\right]\begin{bmatrix} Y_2^T & 0 \\ C_2Y_1 & I\end{bmatrix}^{-1}.
$$

and where X_2 and Y_2 are any matrices which satisfy $X_2Y_2^T = I - X_1Y_1$.

- e.g. Let $Y_2 = I$ and $X_2 = I - X_1Y_1$.
- The optimal controller is NOT uniquely defined.
- Don't forget to check invertibility of $I - D_{22}D_K$

Conclusion

The H_∞-optimal controller is a dynamic system.

- Transfer Function $\hat{K}(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$

Minimizes the effect of external input (w) on external output (z).

$$\|z\|_{L_2} \le \|\underline{\mathsf{S}}(P,K)\|_{H_\infty} \|w\|_{L_2}$$

- Minimum Energy Gain

Lets consider the full-state feedback problem

$$\hat{G}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ I & 0 & 0 \end{array}\right]$$

- D_{12} is the weight on control effort.
- $D_{11} = 0$ is a feed-through term and must be 0.
- $C_2 = I$ as this is state-feedback.

$$\hat{K}(s) = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & K \end{array}\right]$$

H_2-optimal control

Full-State Feedback

Lets consider the full-state feedback problem

$$\hat{G}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ I & 0 & 0 \end{array}\right]$$

- D_{12} is the weight on control effort.
- $D_{11} = 0$ is a feed-through term and must be 0.
- $C_2 = I$ as this is state-feedback.

$$\hat{K}(s) = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & K \end{array}\right]$$

The H_2-norm of a constant (D_{11}) is ∞!!!

H_2-optimal control

Applying the Schur Complement gives the alternative formulation convenient for control.

Theorem 37.

Suppose $\hat{P}(s) = C(sI - A)^{-1}B$. Then the following are equivalent.

1. *A is Hurwitz and $\|\hat{P}\|_{H_2} < \gamma$.*
2. *There exists some $X, W > 0$ such that*

$$\begin{bmatrix} A^T X + XA & XB \\ B^T X & -\gamma I \end{bmatrix} < 0, \qquad \begin{bmatrix} X & C^T \\ C & W \end{bmatrix} > 0, \qquad \mathit{Trace} W < \gamma^2$$

Theorem 38.

The following are equivalent.

1. $\|S(K,P)\|_{H_2} < \gamma$.
2. $K = ZX^{-1}$ *for some* Z *and* $X > 0$ *where*

$$\begin{bmatrix} A & B_2 \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} X & Z^T \end{bmatrix} \begin{bmatrix} A^T \\ B_2^T \end{bmatrix} + B_1 B_1^T < 0$$

$$\begin{bmatrix} X & (C_1 X + D_{12} Z)^T \\ C_1 X + D_{12} Z & W \end{bmatrix} > 0$$

$$\textit{Trace} W < \gamma^2$$

Thus we can solve the H_2-optimal static full-state feedback problem.

H_2-optimal control

Relationship to LQR

To solve the LQR problem using H_2 optimal state-feedback control, let

- $C_1 = \begin{bmatrix} Q^{\frac{1}{2}} \\ 0 \end{bmatrix}$
- $D_{12} = \begin{bmatrix} 0 \\ R^{\frac{1}{2}} \end{bmatrix}$
- $B_2 = B$ and $B_1 = I$.

So that

$$\underline{S}(\hat{P}, \hat{K}) = \left[\begin{array}{c|c} A + B_2 K & B_1 \\ \hline C_1 + D_{12} K & D_{11} \end{array}\right] = \left[\begin{array}{c|c} A + BK & I \\ \hline \begin{matrix} Q^{\frac{1}{2}} \\ R^{\frac{1}{2}} K \end{matrix} & 0 \end{array}\right]$$

And solve the H_2 full-state feedback problem. Then if

$$\begin{aligned} \dot{x}(t) &= A_{CL} x(t) = (A + BK)x(t) = Ax(t) + Bu(t) \\ u(t) &= Kx(t), \qquad x(0) = x_0 \end{aligned}$$

Then

$$\int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) dt = \|x_0\|^2 \|S(K, P)\|_{H_2}^2$$

H_2-optimal output feedback control

Theorem 39 (Scherer, Gahinet).

The following are equivalent.

- *There exists a* $\hat{K} = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ *such that* $\|S(K,P)\|_{H_2} < \gamma$.
- *There exist* $X_1, Y_1, Z, A_n, B_n, C_n, D_n$ *such that*

$$\begin{bmatrix} AY_1 + Y_1 A^T + B_2 C_n + C_n^T B_2^T & *^T & *^T \\ A^T + A_n + [B_2 D_n C_2]^T & X_1 A + A^T X_1 + B_n C_2 + C_2^T B_n^T & *^T \\ [B_1 + B_2 D_n D_{21}]^T & [X_1 B_1 + B_n D_{21}]^T & -I \end{bmatrix} < 0,$$

$$\begin{bmatrix} Y_1 & I & *^T \\ I & X_1 & *^T \\ C_1 Y_1 + D_{12} C_n & C_1 + D_{12} D_n C_2 & Z \end{bmatrix} > 0,$$

$$D_{11} + D_{12} D_n D_{21} = 0, \qquad \mathrm{trace}(Z) < \gamma^2$$

H_2-optimal output feedback control

As before, the controller can be recovered as

$$\left[\begin{array}{c|c} A_{K2} & B_{K2} \\ \hline C_{K2} & D_{K2} \end{array}\right] = \begin{bmatrix} X_2 & X_1 B_2 \\ 0 & I \end{bmatrix}^{-1} \left[\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} - \begin{bmatrix} X_1 A Y_1 & 0 \\ 0 & 0 \end{bmatrix} \right] \begin{bmatrix} Y_2^T & 0 \\ C_2 Y_1 & I \end{bmatrix}^{-1}$$

for any full-rank X_2 and Y_2 such that

$$\begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}^{-1}$$

To find the actual controller, we use the identities:

$$\begin{aligned} D_K &= (I + D_{K2} D_{22})^{-1} D_{K2} \\ B_K &= B_{K2}(I - D_{22} D_K) \\ C_K &= (I - D_K D_{22}) C_{K2} \\ A_K &= A_{K2} - B_K (I - D_{22} D_K)^{-1} D_{22} C_K \end{aligned}$$

An LMI for Mixed H_2-H_∞ optimal output feedback control

Theorem 40.

The following are equivalent.

- *There exists a* $K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ *such that* $\|S(K,P)\|_{H_2} < \gamma_1$ *and* $\|S(K,P)\|_{H_\infty} < \gamma_2$.
- *There exist* $X_1, Y_1, Z, A_n, B_n, C_n, D_n$ *such that*

$$\begin{bmatrix} AY_1+Y_1A^T+B_2C_n+C_n^TB_2^T & *^T & *^T \\ A^T+A_n+[B_2D_nC_2]^T & X_1A+A^TX_1+B_nC_2+C_2^TB_n^T & *^T \\ [B_1+B_2D_nD_{21}]^T & [X_1B_1+B_nD_{21}]^T & -I \end{bmatrix} < 0,$$

$$\begin{bmatrix} Y_1 & I & *^T \\ I & X_1 & *^T \\ C_1Y_1+D_{12}C_n & C_1+D_{12}D_nC_2 & Z \end{bmatrix} > 0,$$

$$D_{11}+D_{12}D_nD_{21} = 0, \qquad \operatorname{trace}(Z) < \gamma_1^2$$

$$\begin{bmatrix} AY_1+Y_1A^T+B_2C_n+C_n^TB_2^T & *^T & *^T & *^T \\ A^T+A_n+[B_2D_nC_2]^T & X_1A+A^TX_1+B_nC_2+C_2^TB_n^T & *^T & *^T \\ [B_1+B_2D_nD_{21}]^T & [X_1B_1+B_nD_{21}]^T & -\gamma_2 I & *^T \\ C_1Y_1+D_{12}C_n & C_1+D_{12}D_nC_2 & D_{11}+D_{12}D_nD_{21} & -\gamma_2 I \end{bmatrix} < 0$$

An LMI for the Kalman Filter! - Continuous Time

System:

$$\dot{x} = Ax + Bu + w$$
$$y = Cx + v$$

Filter:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$
$$\hat{y} = C\hat{x}$$

Error: $$\dot{e}(t) = (A + LC)e(t) + w(t) + Lv(t)$$

The Kalman Filter chooses L to minimize the cost $J = \mathbf{E}[e^T e]$.

$$L = \Sigma C^T V_2^{-1}$$

where $V_1 = \mathbf{E}[\mathbf{w(t)w(t)^T}]$ and $V_2 = \mathbf{E}[\mathbf{v(t)v(t)^T}]$ and Σ satisfies

$$A\Sigma + \Sigma A^T + V_1 = \Sigma C^T V_2^{-1} C\Sigma$$

If we choose $u = K\hat{x}$ where $A + BK$ is stable,

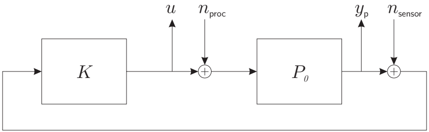
- $A + LC$ is stable if system is observable (not detectable).
- Closed-Loop is stable by the separation principle (has Luenberger form).
- A Dual to the LQR problem. Replace (A, B, Q, R, K) with $(A^T, C^T, V_1, V_2, L^T)$

What is Uncertainty?

The Known Unknowns

CASE 1: External Disturbances

- The most benign source of uncertainty.
- Finite Energy (L_2-norm bounded).
- H_∞ optimal control minimizes the effect of these uncertainties.



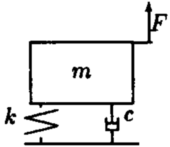
Benign Sources:

- Vibrations, Wind, 60 Hz noise
- Initial Conditions
- Sensor Noise
- Changes in Reference Signal

Not-So-Benign Sources:

- Higher-Order Dynamics
- Nonlinearity (Saturation)
- Delay
- Modeling Errors (Parametric vs. Structural)
- Model Reduction
- Logical Switching

There are **Three Main Types of Parametric Uncertainty**



$$\ddot{y}(t) = \frac{c}{m}\dot{y}(t) + \frac{k}{m}y(t) = \frac{F(t)}{m}$$

- **Uncertainty in Parameters** c, k, m

Multiplicative Uncertainty

- $m = m_0(1 + \eta_m \delta_m)$
- $c = c_0(1 + \eta_c \delta_c)$
- $k = k_0(1 + \eta_k \delta_k)$

Where $\delta_m, \delta_c, \delta_k$ are bounded.

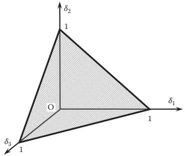
Additive Uncertainty

- $m = m_0 + \eta_m \delta_m$
- $c = c_0 + \eta_c \delta_c$
- $k = k_0 + \eta_k \delta_k$

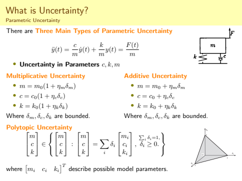
Where $\delta_m, \delta_c, \delta_k$ are bounded.

Polytopic Uncertainty

$$\begin{bmatrix} m \\ c \\ k \end{bmatrix} \in \left\{ \begin{bmatrix} m \\ c \\ k \end{bmatrix} : \begin{bmatrix} m \\ c \\ k \end{bmatrix} = \sum_i \delta_i \begin{bmatrix} m_i \\ c_i \\ k_i \end{bmatrix}, \begin{matrix} \sum_i \delta_i = 1, \\ \delta_i \geq 0. \end{matrix} \right\}$$



where $\begin{bmatrix} m_i & c_i & k_i \end{bmatrix}^T$ describe possible model parameters.

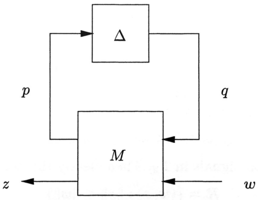


- Additive or multiplicative uncertainty arises from guessing and error tolerances.
- Polytopic Uncertainty arises from multiple conflicting measurements or operating points

Linear-Fractional Representation

The **Nominal System**, M:

$$\begin{bmatrix} p \\ z \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} q \\ w \end{bmatrix}$$



Nominal System Representations:

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} A & B_2 & B_1 \\ C_2 & D_{22} & D_{21} \\ C_1 & D_{12} & D_{11} \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ q(t) \end{bmatrix}$$

$$M_{11} = \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right], M_{12} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_1 & D_{12} \end{array}\right], M_{21} = \left[\begin{array}{c|c} A & B_1 \\ \hline C_2 & D_{21} \end{array}\right], M_{22} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right].$$

Closed-Loop: Can be expressed using only matrices

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \bar{S}(P,\Delta) \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} = (P_{22} + P_{21}(I - \Delta P_{11})^{-1}\Delta P_{12}) \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}$$

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = \bar{S}(P,\Delta) = \begin{bmatrix} A & B_2 \\ C_2 & D_{22} \end{bmatrix} + \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} (I - \Delta D_{11})^{-1}\Delta \begin{bmatrix} C_1 & D_{12} \end{bmatrix}$$

$$\dot{x}(t) = (A + B_1(I - \Delta D_{11})^{-1}\Delta C_1)x(t) + (B_2 + B_1(I - \Delta D_{11})^{-1}\Delta D_{12})w(t)$$

$$z(t) = (C_2 + D_{21}(I - \Delta D_{11})^{-1}\Delta C_1)x(t) + (D_{22} + D_{21}(I - \Delta D_{11})^{-1}\Delta D_{12})w(t)$$

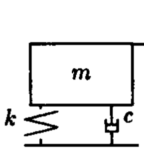
Example of Parametric Uncertainty

Nominal System: P

$$\begin{bmatrix}\dot x_1\\ \dot x_2\end{bmatrix} = \begin{bmatrix}0 & m_0^{-1}\\ -k_0 & -c_0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}F(t) + \begin{bmatrix}-\eta & 0 & 0\\ 0 & \eta_k & \eta_c\end{bmatrix}q(t)$$

$$p(t) = \begin{bmatrix}0 & m_0^{-1}\\ -k_0 & 0\\ 0 & -c_0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}-\eta_m & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}q(t)$$

$$z(t) = x_1(t)$$



Uncertain System: Δ

$$q = \Delta p = \begin{bmatrix}\delta_m & 0 & 0\\ 0 & \delta_k & 0\\ 0 & 0 & \delta_c\end{bmatrix}p$$

Closed-Loop:

$$\begin{bmatrix}\dot x_1\\ \dot x_2\\ z(t)\end{bmatrix} = (P_{22}+P_{21}(I-\Delta P_{11})^{-1}\Delta P_{12})\begin{bmatrix}x_1(t)\\ x_2(t)\\ F(t)\end{bmatrix}$$

where

$$P_{22} = \begin{bmatrix}A & B_2\\ C_2 & D_{22}\end{bmatrix}, P_{21} = \begin{bmatrix}B_1\\ D_{21}\end{bmatrix}, P_{12} = \begin{bmatrix}C_1 & D_{12}\end{bmatrix}, P_{11} = D_{11},$$

Questions:

- How to formulate the uncertainty matrix?
- What if the uncertainty is time-varying?

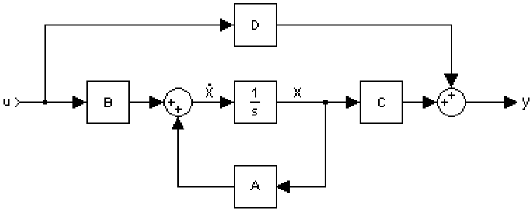
Formulating the LFT

Consider the Example From Gu, Petkoz, Konstantinov

Recall:

State-Space Systems can be represented in Block-Diagram Form. e.g.

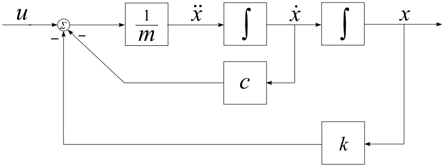
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$



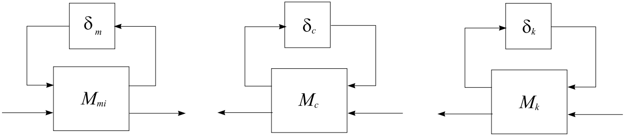
$$m\ddot{x} + c\dot{x} + kx = F \qquad x(s) = \frac{1}{ms^2 + cs + k} u(s)$$

Lets consider how to do this problem in General with Block Diagrams.

Step 1: Isolate all the uncertain parameters:



Step 2: Rewrite all the uncertain blocks as LFTs



For the $\frac{1}{m_0(1+\eta_m\delta_m)}$ Term:

$$\frac{1}{m} = \frac{1}{m_0(1+\eta_m\delta_m)} = \frac{1}{m_0} - \frac{\eta_m}{m_0}(1+\eta_m\delta_m)^{-1} = \bar{S}(M_m, \delta_m)$$

where $M_m = \begin{bmatrix} -\eta_m & \frac{1}{m_0} \\ -\eta_m & \frac{1}{m_0} \end{bmatrix}$.

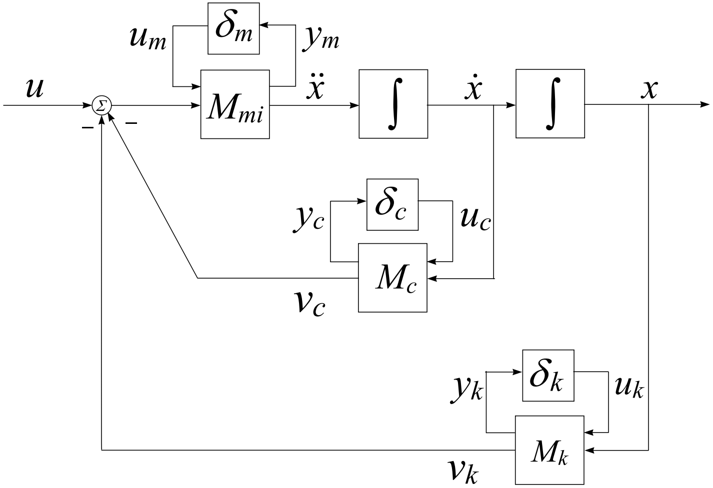
For the $c_0(1+\eta_c\delta_c)$ and $k_0(1+\eta_k\delta_k)$ Terms:

$$c = c_0(1+\eta_c\delta_c) = \bar{S}(M_c, \delta_c) \qquad M_c = \begin{bmatrix} 0 & c_0 \\ \eta_c & c_0 \end{bmatrix}$$

$$k = k_0(1+\eta_k\delta_k) = \bar{S}(M_k, \delta_c) \qquad M_k = \begin{bmatrix} 0 & k_0 \\ \eta_k & k_0 \end{bmatrix}$$

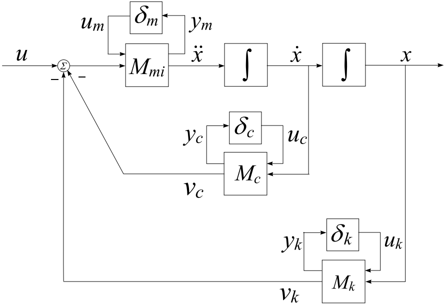
Formulating the LFT

Step 3: Write down all your equations!



Set $x_1 = x$, $x_2 = \dot{x}$, $z = x_1$ so $\ddot{x} = \dot{x}_2$.

Formulating the LFT



$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\eta_m u_m + \frac{1}{m_0}(w - v_c - v_k)$$
$$y_m = -\eta_m u_m + \frac{1}{m_0}(w - v_c - v_k)$$
$$y_c = c_0 x_2$$
$$y_k = k_0 x_1$$
$$v_c = \eta_c u_c + c_0 x_2, \qquad v_k = \eta_k u_k + k_0 x_1$$
$$z = x_1$$
$$u_m = \delta_m y_m, \quad u_c = \delta_c y_c, \quad u_k = \delta_k y_k$$

Eliminating v_c and v_k, we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ y_m \\ y_c \\ y_k \\ z \end{bmatrix} = \left[\begin{array}{cc|ccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_0}{m_0} & -\frac{c_0}{m_0} & -\eta_m & -\frac{\eta_c}{m_0} & -\frac{\eta_k}{m_0} & \frac{1}{m_0} \\ \hline -\frac{k_0}{m_0} & -\frac{c_0}{m_0} & -\eta_m & -\frac{\eta_c}{m_0} & -\frac{\eta_k}{m_0} & \frac{1}{m_0} \\ 0 & c_0 & 0 & 0 & 0 & 0 \\ k_0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ u_m \\ u_c \\ u_k \\ w \end{bmatrix} \qquad u = \begin{bmatrix} \delta_m & 0 & 0 \\ 0 & \delta_k & 0 \\ 0 & 0 & \delta_c \end{bmatrix} y$$

Alternatives to the LFT

Additive Affine Time-Varying Interval and Polytopic Uncertainty

- Time-Varying Uncertainty can cause problems
- Because dealing with *Structured Uncertainty* is difficult, we often look for alternative representations.

Consider the following form of time-varying uncertainty

$$\dot{x}(t) = (A_0 + \Delta A(t))x(t)$$

where

$$\Delta A(t) = A_1\delta_1(t) + \cdots + A_k\delta_k(t)$$

where $\delta(t)$ lies in either the intervals

$$\delta_i(t) \in [\delta_i^-, \delta_i^+]$$

or the simplex

$$\delta(t) \in \{\alpha \,:\, \sum_i \alpha_i = 1,\, \alpha_i \geq 0\}$$

For convenience, we denote this *Convex Hull* as

$$Co(A_1, \cdots, A_k) := \left\{ \sum_i A_i\alpha_i \;:\; \alpha_i \geq 0, \sum_i \alpha_i = 1 \right\}$$

For example,

$$m\ddot{x} + c\dot{x} + kx = F \qquad x(s) = \frac{1}{ms^2 + cs + k} u(s)$$

Define $x_1 = y$ and $x_2 = m\dot{y}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & m^{-1} \\ -k & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F$$

Then if $m \in [m^-, m^+]$, $c \in [c^-, c^+]$, $k \in [k^-, k^+]$, then

$$m^{-1} \in [\frac{1}{m^+}, \frac{1}{m^-}]$$

$$\frac{c}{m} \in [\frac{c^-}{m^+}, \frac{c^+}{m^-}]$$

Note: This doesn't always work!

- e.g. if in addition there were a c coefficient (appearing w/o $1/m$).
- Need a change of parameters which becomes affine in the parameters.
- Then you are stuck with the LFT.

Discrete-Time Case

All frameworks are readily adapted to the Discrete-Time Case:

LFT Framework:

$$\begin{bmatrix} x_{k+1} \\ z_k \end{bmatrix} = \bar{S}(P, \Delta) \begin{bmatrix} x_k \\ w_k \end{bmatrix}$$

Alternative Framework:

$$x_{k+1} = (A_0 + \Delta A_k)x_k + (B_0 + \Delta B_k)u_k$$

where

$$\Delta A_k = A_1\delta_{1,k} + \cdots + A_k\delta_{K,k}$$

where δ_k lies in either the intervals

$$\delta_{i,k} \in [\delta_i^-, \delta_i^+]$$

or the simplex

$$\delta_k \in \{\alpha : \sum_i \alpha_i = 1,\ \alpha_i \geq 0\}$$

Types of Uncertainty

To Summarize, we have many choices for our uncertainty Set, $\mathbf{\Delta}$

- **Unstructured, Dynamic, norm-bounded:**
$$\mathbf{\Delta} := \{\Delta \in \mathcal{L}(L_2) \,:\, \|\Delta\|_{H_\infty} < 1\}$$
- **Structured, Static, norm-bounded:**
$$\mathbf{\Delta} := \{\mathrm{diag}(\delta_1, \cdots, \delta_K, \Delta_1, \cdots \Delta_N) \,:\, |\delta_i| < 1,\ \bar{\sigma}(\Delta_i) < 1\}$$
- **Structured, Dynamic, norm-bounded:**
$$\mathbf{\Delta} := \{\Delta_1, \Delta_2, \cdots \in \mathcal{L}(L_2) \,:\, \|\Delta_i\|_{H_\infty} < 1\}$$
- **Unstructured, Parametric, norm-bounded:**
$$\mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n\times n} \,:\, \|\Delta\| \le 1\}$$
- **Parametric, Polytopic:**
$$\mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n\times n} \,:\, \Delta = \sum_i \alpha_i H_i,\ \alpha_i \ge 0,\ \sum_i \alpha_i = 1\}$$
- **Parametric, Interval:**
$$\mathbf{\Delta} := \left\{ \sum_i \Delta_i \delta_i \,:\, \delta_i \in [\delta_i^-, \delta_i^+] \right\}$$

Each of these can be Time-Varying or Time-Invariant!

Definitions: Use Robust Stability for Static Uncertainty

Definition 41.

The system
$$\dot{x}(t) = (A_0 + \Delta(t))x(t)$$
is **Robustly Stable** over $\boldsymbol{\Delta}$ if $A_0 + \Delta$ is Hurwitz for all $\Delta \in \boldsymbol{\Delta}$.

Note that Robust Stability DOES NOT imply stability if $\Delta(t)$ is time-varying.

- It implies that for any $\Delta \in \boldsymbol{\Delta}$, there exists a $P(\Delta) > 0$ such that
$$(A + \Delta)^T P(\Delta) + P(\Delta)(A + \Delta) < 0 \quad \text{for all } \Delta \in \boldsymbol{\Delta}$$
- For a fixed Δ, this implies stability using Lyapunov function $V(x) = x^T P(\Delta) x$.
- Does not imply stability for TV Δ because if $V(x,t) = x^T P(\Delta(t))x$,

$$\begin{aligned}\frac{d}{dt}V(x(t),t) &= x(t)^T \Big((A + \Delta(t))^T P(\Delta(t)) + P(\Delta(t))(A + \Delta(t))\Big) x(t) \\ &\quad + x(t)^T \left(\frac{d}{dt}P(\Delta(t))\right) x(t) \\ &\le x(t)^T \left(\frac{d}{dt}P(\Delta(t))\right) x(t)\end{aligned}$$

Definitions: Use Quadratic Stability for Static Uncertainty

Definition 42.

The system
$$\dot{x}(t) = (A_0 + \Delta(t))x(t)$$
is **Quadratically Stable** over $\mathbf{\Delta}$ if there exists a $P > 0$ such that
$$(A + \Delta)^T P + P(A + \Delta) < 0 \quad \text{for all } \Delta \in \mathbf{\Delta}.$$

Quadratic Stability **Implies Stability** of trajectories for any $\Delta(t)$ with $\Delta(t) \in \mathbf{\Delta}$ for all $t \geq 0$.

- Use the Lyapunov function $V(x) = x^T P x$.

$$\frac{d}{dt}V(x(t)) = x(t)^T((A + \Delta(t))^T P + P(A + \Delta(t))x(t) < 0$$

Counterintuitive:

- Robust Stability does not imply stability!
- Stability does not imply quadratic stability!

Definitions: Use Quadratic Stability for Static Uncertainty

Definition 42.

The system
$$\dot{x}(t) = (A_0 + \Delta(t))x(t)$$
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Quadratic Stability **Implies Stability** of trajectories for any $\Delta(t)$ with $\Delta(t) \in \mathbf{\Delta}$ for all $t \geq 0$.

- Use the Lyapunov function $V(x) = x^T P x$.

$$\frac{d}{dt}V(x(t)) = x(t)^T((A + \Delta(t))^T P + P(A + \Delta(t))x(t) < 0$$

Counterintuitive:

- Robust Stability does not imply stability!
- Stability does not imply quadratic stability!

Quadratic Stability refers to the METHOD, rather than the system property

An LMI for Polytopic Quadratic Stability

Definition 43.

The pair $(A + \Delta, \mathbf{\Delta})$ is **Quadratically Stable** over $\mathbf{\Delta}$ if there exists a $P > 0$ such that

$$(A + \Delta)^T P + P(A + \Delta) < 0 \quad \text{for all } \Delta \in \mathbf{\Delta}.$$

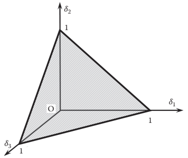
Theorem 44.

$(A + \Delta, \mathbf{\Delta})$ is quadratically stable over $\mathbf{\Delta} := Co(A_1, \cdots, A_k)$ if and only if there exists a $P > 0$ such that

$$(A + A_i)^T P + P(A + A_i) < 0 \quad \textit{for } i = 1, \cdots, k$$

The theorem says the LMI only needs to hold at the EXTREMAL POINTS or VERTICES of the polytope.

- In Fact, Quadratic Stability MUST be expressed as an LMI
- There is NO Ricatti Eqn. Equivalent.



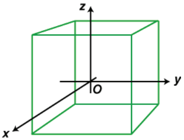
An LMI for Interval Quadratic Stability

Recall the system with Affine Time-Varying uncertainty.

$$\dot{x}(t) = (A_0 + \Delta(t))x(t)$$

where

$$\Delta(t) = A_1\delta_1(t) + \cdots + A_k\delta_k(t)$$



where $\delta_i(t) \in [\delta_i^-, \delta_i^+]$. Note: $\delta(t)$ lies in a *hypercube*.
Interval Stability is a Kind of Polytopic Uncertainty.

The vertices of the hypercube define the vertices of the uncertainty set

$$V := \left\{ \sum_i A_i \left(\frac{1-(-1)^{d_i}}{2}\delta_i^- + \frac{1+(-1)^{d_i}}{2}\delta_i^+ \right),\ d_i \in \{0,1\} \right\}$$

Define the corner values: $I_{i,j} := \left(\frac{1-(-1)^j}{2}\delta_i^- + \frac{1+(-1)^j}{2}\delta_i^+ \right)$

Theorem 45 (Quadratic Stability using 2^k LMI constraints!).

$(A+\Delta, \boldsymbol{\Delta})$ *is quadratically stable over* $\boldsymbol{\Delta} := Co(V)$ *if and only if there exists a* $P > 0$ *such that*

$$\left(A + \sum_i A_i I_{i,v_i} \right)^T P + P \left(A + \sum_i A_i I_{i,v_i} \right) < 0 \quad \textit{for every } v \in \{0,1\}^k$$

An LMI for Quadratic Polytopic Stabilization

Controller Synthesis is a simple application of the previous theorem:

Theorem 46.

There exists a K *such that*

$$\dot{x}(t) = (A + \Delta_A + (B + \Delta_B)K)x(t)$$

is **quadratically stable** *for* $(\Delta_A, \Delta_B) \in Co((A_1, B_2), \cdots, (A_k, B_k))$ *if and only if there exists some* $P > 0$ *and* Z *such that*

$$(A + A_i)P + P(A + A_i)^T + (B + B_i)Z + Z^T(B + B_i)^T < 0 \qquad \text{for } i = 1, \cdots k.$$

with $K = ZP^{-1}$.

Note that here the controller doesn't depend on Δ!

- If you want K to depend on Δ, the problem is harder.
- But this would require sensing Δ in real-time.

An LMI for Quadratic D-Stabilization

Lemma 47 (An LMI for Quadratic D-Stabilization).

Suppose there exists $X > 0$ *and* Z *such that*

$$\begin{bmatrix} -rP & AP+BZ \\ (AP+BZ)^T & -rP \end{bmatrix} + \begin{bmatrix} 0 & A_iP+B_iZ \\ (A_iP+B_iZ)^T & 0 \end{bmatrix} < 0,$$

$$AP+BZ+(AP+BZ)^T + A_iP+B_iZ+(A_iP+B_iZ)^T + 2\alpha P < 0, \quad \textit{and}$$

$$\begin{bmatrix} AP+BZ+(AP+BZ)^T & c(AP+BZ-(AP+BZ)^T) \\ c((AP+BZ)^T-(AP+BZ)) & AP+BZ+(AP+BZ)^T \end{bmatrix}$$
$$+ \begin{bmatrix} A_iP+B_iZ+(A_iP+B_iZ)^T & c(A_iP+B_iZ-(A_iP+B_iZ)^T) \\ c((A_iP+B_iZ)^T-(A_iP+B_iZ)) & A_iP+B_iZ+(A_iP+B_iZ)^T \end{bmatrix} < 0$$

for $i = 1, \cdots, k$. *Then if* $K = ZP^{-1}$, *the pole locations,* $z \in \mathbb{C}$ *of* $A(\Delta) + B(\Delta)K$ *satisfy* $|x| \leq r$, $\operatorname{Re} x \leq -\alpha$ *and* $z + z^* \leq -c|z - z^*|$ *for all* $\Delta \in Co(\Delta_1, \cdots, \Delta_k)$.

An LMI for Quadratic D-Stabilization

Lemma 47 (An LMI for Quadratic D-Stabilization).

Suppose there exists $X > 0$ and Z such that

$$\begin{bmatrix} -rP & AP+BZ \\ (AP+BZ)^T & -rP \end{bmatrix} + \begin{bmatrix} 0 & A_iP+B_iZ \\ (A_iP+B_iZ)^T & 0 \end{bmatrix} < 0,$$

$$AP+BZ+(AP+BZ)^T + A_iP+B_iZ+(A_iP+B_iZ)^T + 2\alpha P < 0, \quad \textit{and}$$

$$\begin{bmatrix} AP+BZ+(AP+BZ)^T & c(AP+BZ-(AP+BZ)^T) \\ c((AP+BZ)^T-(AP+BZ)) & AP+BZ+(AP+BZ)^T \end{bmatrix} + \begin{bmatrix} A_iP+B_iZ+(A_iP+B_iZ)^T & c(A_iP+B_iZ-(A_iP+B_iZ)^T) \\ c((A_iP+B_iZ)^T-(A_iP+B_iZ)) & A_iP+B_iZ+(A_iP+B_iZ)^T \end{bmatrix} < 0$$

for $i = 1, \cdots, k$. Then if $K = ZP^{-1}$, the pole locations, $z \in \mathbb{C}$ of $A(\Delta) + B(\Delta)K$ satisfy $|x| \le r$, $\mathrm{Re}\, x \le -\alpha$ and $z + z^* \le -c|z - z^*|$ for all $\Delta \in Co(\Delta_1, \cdots, \Delta_k)$.

This is ridiculous

- Eigenvalues are undefined for TV systems
- The behaviour of the CL would be completely unpredictable

An LMI for Quadratic Polytopic H_∞-Optimal State-Feedback Control

Recall the closed-loop in state feedback is:

$$\underline{S}(P,K) = \left[\begin{array}{c|c} A + B_2F & B_1 \\ \hline C_1 + D_{12}F & D_{11} \end{array}\right]$$

Now add uncertainty to system matrices A, B_1, B_2, C_1, D_{12} and D_{11}.

Theorem 48.

There exists an F such that $\|\underline{S}(P(\Delta), K(0,0,0,F))\|_{H_\infty} \le \gamma$ for all $\Delta \in Co(\Delta_1, \cdots \Delta_k)$ if there exist $Y > 0$ and Z such that

$$\begin{bmatrix} Y(A+A_i)^T + (A+A_i)Y + Z^T(B_2+B_{2,i})^T + (B_2+B_{2,i})Z & *^T & *^T \\ (B_1+B_{1,i})^T & -\gamma I & *^T \\ (C_1+C_{1,i})Y + (D_{12}+D_{12,i})Z & D_{11}+D_{11,i} & -\gamma I \end{bmatrix} < 0 \quad i = 1, \cdots, k$$

Then $F = ZY^{-1}$.

$$\underline{S}(P(\Delta),K) = \left[\begin{array}{c|c} A + B_2F & B_1 \\ \hline C_1 + D_{12}F & D_{11} \end{array}\right] + \Delta \qquad \Delta \in Co(\Delta_1, \cdots \Delta_k)$$

$$\Delta_i = \left[\begin{array}{c|c} A_1 + B_{2,i}F & B_{1,i} \\ \hline C_{1,i} + D_{12,i}F & D_{11,i} \end{array}\right]$$

An LMI for Quadratic Polytopic H_2-Optimal State-Feedback Control

Similarly

Theorem 49.

There exists an F such that $\|\underline{S}(P(\Delta), K(0,0,0,F))\|_{H_2}^2 \le \gamma$ for all $\Delta \in Co(\Delta_1, \cdots \Delta_k)$ if there exist $X > 0$ and Z such that

$$\begin{bmatrix} AX + B_2Z + XA^T + Z^TB_2^T & B_1 \\ B_1^T & -I \end{bmatrix} + \begin{bmatrix} A_iX + B_{2,i}Z + XA_i^T + Z^TB_{2,i}^T & B_{1,i} \\ B_{1,i}^T & 0 \end{bmatrix} < 0 \qquad i = 1, \cdots, k$$

$$\begin{bmatrix} X & (C_1X + D_{12}Z)^T \\ C_1X + D_{12}Z & W \end{bmatrix} + \begin{bmatrix} 0 & (C_{1,i}X + D_{12,i}Z)^T \\ C_{1,i}X + D_{12,i}Z & 0 \end{bmatrix} > 0 \qquad i = 1, \cdots, k$$

$$\textit{Trace} W < \gamma$$

Then $F = ZY^{-1}$.

Similar Steps can be taken for robust estimator design, using the LMIs in Duan.

- However, I am not aware of a robust version of the general optimal output feedback LMI for polytopic uncertainty.

An LMI for Quadratic Schur Stabilization

State Equations: $u_k = Fx_k$

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ &= Ax_k + BFx_k \\ &= (A+BF)x_k \end{aligned}$$

Lemma 50.

Suppose there exists some $X > 0$ and Z such that

$$\begin{bmatrix} X & AX+BZ \\ (AX+BZ)^T & X \end{bmatrix} + \begin{bmatrix} 0 & A_iX+B_iZ \\ (A_iX+B_iZ)^T & 0 \end{bmatrix} > 0$$

then if $F = ZX^{-1}$, then trajectories of the closed-loop system $(A+BK)$ are stable for any $\Delta \in Co(\Delta_1, \cdots \Delta_k)$.

Types of Uncertainty

In this Lecture, we will cover

- **Unstructured, Dynamic, norm-bounded:**

$$\mathbf{\Delta} := \{\Delta \in \mathcal{L}(L_2) \,:\, \|\Delta\|_{H_\infty} < 1\}$$

- **Structured, Static, norm-bounded:**

$$\mathbf{\Delta} := \{\text{diag}(\delta_1, \cdots, \delta_K, \Delta_1, \cdots \Delta_N) \,:\, |\delta_i| < 1,\ \bar{\sigma}(\Delta_i) < 1\}$$

- **Structured, Dynamic, norm-bounded:**

$$\mathbf{\Delta} := \{\Delta_1, \Delta_2, \cdots \in \mathcal{L}(L_2) \,:\, \|\Delta_i\|_{H_\infty} < 1\}$$

- **Unstructured, Static, norm-bounded:**

$$\mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n \times n} \,:\, \|\Delta\| \le 1\}$$

- **Parametric, Polytopic:**

$$\mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n \times n} \,:\, \Delta = \sum_i \alpha_i H_i,\ \alpha_i \ge 0,\ \sum_i \alpha_i = 1\}$$

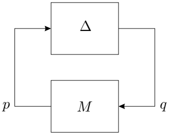
- **Parametric, Interval:**

$$\mathbf{\Delta} := \{\sum_i \Delta_i \delta_i \,:\, \delta_i \in [\delta_i^-, \delta_i^+]\}$$

Each of these can be Time-Varying or Time-Invariant!

Back to the Linear Fractional Transformation

The interval and polytopic cases rely on **Linearity** of the uncertain parameters.



$$\dot{x}(t) = (A_0 + \Delta(t))x(t)$$

The Linear-Fractional Transformation, however

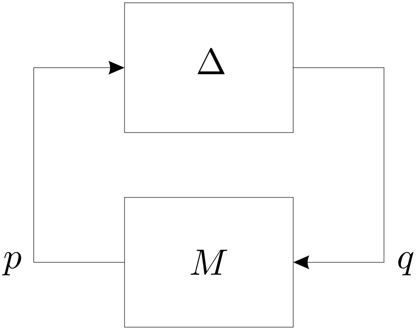
$$\begin{bmatrix}\dot{x}_1\\ z(t)\end{bmatrix} = \bar{S}(P,\Delta)\begin{bmatrix}x_1(t)\\ F(t)\end{bmatrix} = (P_{22}+P_{21}(I-\Delta P_{11})^{-1}\Delta P_{12})\begin{bmatrix}x(t)\\ F(t)\end{bmatrix}$$

is an arbitrary rational function.

We focus on two results:

- The S-Procedure for Unstructured Uncertainty Sets
- The Structured Singular Value for Structured Uncertainty Sets.

Robust Stability



Questions:

- Is $\underline{\mathsf{S}}(\Delta, M)$ stable for all $\Delta \in \mathbf{\Delta}$?
- Is $I - \Delta M_{11}$ invertible for all $\Delta \in \mathbf{\Delta}$?

Redefine Robust and Quadratic Stability

Suppose we have the system

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

Definition 51.

The pair $(M, \mathbf{\Delta})$ is **Robustly Stable** if $(I - M_{22}\Delta)$ is invertible for all $\Delta \in \mathbf{\Delta}$.

Alternatively, if

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \bar{S}(M, \Delta) \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}$$

Definition 52 (Continuous-Time).

The pair $(M, \mathbf{\Delta})$ is **Robustly Stable** if for some $\beta > 0$,
$M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12} + \beta I$ is Hurwitz for all $\Delta \in \mathbf{\Delta}$.

Alternatively, if

$$\begin{bmatrix} x_{k+1} \\ z_k \end{bmatrix} = \bar{S}(M, \Delta) \begin{bmatrix} x_k \\ w_k \end{bmatrix}$$

Definition 53 (Discrete-Time).

The pair $(M, \mathbf{\Delta})$ is **Robustly Stable** if
$\rho(M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}) = \beta < 1$ for all $\Delta \in \mathbf{\Delta}$.

Quadratic Stability - Parametric Uncertainty

Focus on the 1,1 block of $\bar{S}(M,\Delta)$:

If
$$\dot{x}(t) = \bar{S}(M,\Delta)x(t),$$

Definition 54 (Continuous Time).

The pair $(M, \mathbf{\Delta})$ is **Quadratically Stable** if there exists a $P > 0$ such that

$$\bar{S}(M,\Delta)^T P + P\bar{S}(M,\Delta) < -\beta I \qquad \text{for all } \Delta \in \mathbf{\Delta}$$

Alternatively, if
$$x_{k+1} = \bar{S}(M,\Delta)x_k,$$

Definition 55 (Discrete Time).

The pair $(M, \mathbf{\Delta})$ is **Quadratically Stable** if there exists a $P > 0$ such that

$$\bar{S}(M,\Delta)^T P\bar{S}(M,\Delta) - P < -\beta I \qquad \text{for all } \Delta \in \mathbf{\Delta}$$

for all $\Delta \in \mathbf{\Delta}$.

Parametric, Norm-Bounded Time-Varying Uncertainty

Consider the state-space representation:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Mp(t), & p(t) &= \Delta(t)q(t), \\ q(t) &= Nx(t) + Qp(t), & \Delta(t) &\in \mathbf{\Delta} \end{aligned}$$

- **Parametric, Norm-Bounded Uncertainty:**

$$\mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n \times n} \,:\, \|\Delta\| \leq 1\}$$

Parametric, Norm-Bounded Uncertainty

Quadratic Stability: There exists a $P > 0$ such that

$$P(Ax(t)+Mp)+(Ax(t)+Mp)^T P < 0 \text{ for all } p \in \left\{ p \,:\, p = \Delta q, \begin{matrix} q = Nx + Qp, \\ \Delta \in \mathbf{\Delta} \end{matrix} \right\}$$

Theorem 56.

The system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Mp(t), \qquad & p(t) &= \Delta(t) q(t), \\ q(t) &= Nx(t) + Qp(t), \qquad & \Delta &\in \mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n \times n} \,:\, \|\Delta\| \le 1\} \end{aligned}$$

is quadratically stable if and only if there exists some $P > 0$ such that

$$\begin{bmatrix} x \\ p \end{bmatrix} \begin{bmatrix} A^T P + PA & PM \\ M^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} < 0$$

$$\text{for all } \begin{bmatrix} x \\ p \end{bmatrix} \in \left\{ \begin{bmatrix} x \\ p \end{bmatrix} \,:\, \begin{bmatrix} x \\ p \end{bmatrix} \begin{bmatrix} -N^T N & -N^T Q \\ -Q^T N & I - Q^T Q \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \le 0 \right\}$$

Parametric, Norm-Bounded Uncertainty

If.

If

$$\begin{bmatrix} x \\ p \end{bmatrix} \begin{bmatrix} A^T P + PA & PM \\ M^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} < 0$$

$$\text{for all } \begin{bmatrix} x \\ p \end{bmatrix} \in \left\{ \begin{bmatrix} x \\ p \end{bmatrix} \,:\, \begin{bmatrix} x \\ p \end{bmatrix} \begin{bmatrix} -N^T N & -N^T Q \\ -Q^T N & I - Q^T Q \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \le 0 \right\}$$

then

$$x^T P(Ax + Mp) + (Ax + Mp)^T Px < 0$$

for all x, p such that

$$\|p\|^2 \le \|Nx + Qp\|^2$$

Therefore, since $p = \Delta q$ implies $\|p\| \le \|q\|$, we have quadratic stability.
The *only if* direction is similar.

□

Quadratic stability here requires positivity of a matrix on a *subset*.

- This is Generally a very hard problem
- NP-hard to determine if $x^T F x \geq 0$ for all $x \geq 0$. (Matrix Copositivity)

S-procedure to the rescue!

The S-procedure asks the question:

- Is $z^T F z \geq 0$ for all $z \in \{x : x^T G x \geq 0\}$?

Corollary 57 (S-Procedure).

$z^T F z \geq 0$ for all $z \in \{x : x^T G x \geq 0\}$ if there exists a scalar $\tau \geq 0$ such that $F - \tau G \succeq 0$.

The S-procedure is **Necessary** if $\{x : x^T G x > 0\}$ has an interior point.

Theorem 58 (Dual Version).

The system

$$\dot{x}(t) = Ax(t) + Mp(t), \qquad p(t) = \Delta(t)q(t),$$

$$q(t) = Nx(t) + Qp(t), \qquad \Delta \in \mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n \times n} \,:\, \|\Delta\| \leq 1\}$$

is quadratically stable if and only if there exists some $\mu \geq 0$ *and* $P > 0$ *such that*

$$\begin{bmatrix} AP + PA^T & PN^T \\ NP & 0 \end{bmatrix} + \mu \begin{bmatrix} MM^T & MQ^T \\ QM^T & QQ^T - I \end{bmatrix} < 0\}$$

Noting that the LMI can be written as

$$\begin{bmatrix} AP + PA^T & PN^T \\ NP & -\mu I \end{bmatrix} + \mu \begin{bmatrix} M \\ Q \end{bmatrix} \begin{bmatrix} M \\ Q \end{bmatrix}^T < 0$$

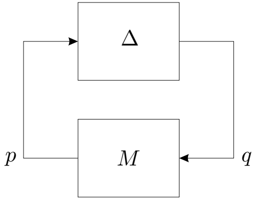
or

$$\begin{bmatrix} AP + PA^T & PN^T & M^T \\ NP & -\mu I & Q^T \\ M & Q & -\frac{1}{\mu} I \end{bmatrix} < 0$$

we see that this condition is simply an H_∞ gain condition on the nominal system $\|\cdot\|_{H_\infty} < 1$.

Necessity of the Small-Gain Condition

This leads to the interesting result:



If $\mathbf{\Delta} := \{\Delta \in \mathcal{L}(L_2) \, : \, \|\Delta\| \le 1\}$, then

- $\bar{S}(P, \Delta) \in H_\infty$ if **and only if** $\|P_{11}\|_{H_\infty} < 1$
- The small gain condition is necessary and sufficient for stability.
- Quadratic Stability is equivalent to stability.
- Holds for Dynamic and Parametric Uncertainty
  - Does this mean Quadratic and Robust Stability are Equivalent?

Quadratic Stability and Equivalence to Robust Stability

Consider Quadratic Stability in Discrete-Time: $x_{k+1} = S_l(M,\Delta)x_k$.

Definition 59.

$(S_l, \mathbf{\Delta})$ is QS if

$$S_l(M,\Delta)^T P S_l(M,\Delta) - P < 0 \qquad \text{for all } \Delta \in \mathbf{\Delta}$$

Theorem 60 (Packard and Doyle).

Let $M \in \mathbb{R}^{(n+m)\times(n+m)}$ be given with $\rho(M_{11}) \le 1$ and $\sigma(M_{22}) < 1$. Then the following are equivalent.

1. *The pair $(M, \Delta = \mathbb{R}^{m\times m})$ is quadratically stable.*
2. *The pair $(M, \Delta = \mathbb{C}^{m\times m})$ is quadratically stable.*
3. *The pair $(M, \Delta = \mathbb{C}^{m\times m})$ is robustly stable.*

Quadratically Stabilizing Controllers with Parametric Norm-Bounded Uncertainty

However, we can add controllers:

Theorem 61.

The system with $u(t) = Kx(t)$ *and*

$$\dot{x}(t) = A_0 x(t) + Bu(t) + Mp(t), \qquad p(t) = \Delta(t)q(t),$$
$$q(t) = Nx(t) + Qp(t) + D_{12}u(t), \qquad \Delta \in \mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n\times n} \,:\, \|\Delta\| \le 1\}$$

is quadratically stable if and only if there exists some $\mu \ge 0$ *and* $P > 0$ *such that*

$$\begin{bmatrix} (A+BK)P + P(A+BK)^T & P(N+D_{12}K)^T \\ (N+D_{12}K)P & 0 \end{bmatrix} + \mu \begin{bmatrix} MM^T & MQ^T \\ QM^T & QQ^T - I \end{bmatrix} < 0\}$$

Of course, this is bilinear in P and K, so we make the change of variables $Z = KP$.

An LMI for Quadratically Stabilizing Controllers with Parametric Norm-Bounded Uncertainty

Theorem 62.

There exists a K *such that the system with* $u(t) = Kx(t)$

$$\dot{x}(t) = Ax(t) + Bu(t) + Mp(t), \qquad p(t) = \Delta(t)q(t),$$
$$q(t) = Nx(t) + Qp(t) + D_{12}u(t), \qquad \Delta \in \boldsymbol{\Delta} := \{\Delta \in \mathbb{R}^{n \times n} \,:\, \|\Delta\| \leq 1\}$$

is quadratically stable if and only if there exists some $\mu \geq 0$, Z *and* $P > 0$ *such that*

$$\begin{bmatrix} AP + BZ + PA^T + Z^T B^T & PN^T + Z^T D_{12}^T \\ NP + D_{12}Z & 0 \end{bmatrix} + \mu \begin{bmatrix} MM^T & MQ^T \\ QM^T & QQ^T - I \end{bmatrix} < 0\}.$$

Then $K = ZP^{-1}$ *is a quadratically stabilizing controller.*

We can also extend this result to optimal control in the H_∞ norm.

An LMI for Quadratically Stabilizing Controllers with Parametric Norm-Bounded Uncertainty

Theorem 62.

There exists a K such that the system with $u(t) = Kx(t)$

$$\dot{x}(t) = Ax(t) + Bu(t) + Mp(t), \qquad p(t) = \Delta(t)q(t),$$
$$q(t) = Nx(t) + Qp(t) + D_{12}u(t), \qquad \Delta \in \boldsymbol{\Delta} := \{\Delta \in \mathbb{R}^{n \times n} : \|\Delta\| \le 1\}$$

is quadratically stable if and only if there exists some $\mu \ge 0$, Z and $P > 0$ such that

$$\begin{bmatrix} AP + BZ + PA^T + Z^T B^T & PN^T + Z^T D_{12}^T \\ NP + D_{12}Z & 0 \end{bmatrix} + \mu \begin{bmatrix} MM^T & MQ^T \\ QM^T & QQ^T - I \end{bmatrix} < 0\}.$$

Then $K = ZP^{-1}$ is a quadratically stabilizing controller.

We can also extend this result to optimal control in the H_∞ norm.

This is from Boyd page 101

An LMI for H_∞-Optimal Quadratically Stabilizing Controllers with Parametric Norm-Bounded Uncertainty

In this case, we set $Q = 0$.

Theorem 63.

There exists a K such that the system with $u(t) = Kx(t)$

$$\dot{x}(t) = Ax(t) + Bu(t) + Mp(t) + B_2 w(t), \qquad p(t) = \Delta(t) q(t),$$
$$q(t) = Nx(t) + D_{12} u(t), \qquad \Delta \in \mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n \times n} \, : \, \|\Delta\| \le 1\}$$
$$y(t) = Cx(t) + D_{22} u(t)$$

satisfies $\|y\|_{L_2} \le \gamma \|u\|_{L_2}$ if there exists some $\mu \ge 0$, Z and $P > 0$ such that

$$\begin{bmatrix} AP + BZ + PA^T + Z^T B^T + B_2 B_2^T + \mu M M^T & (CP + D_{22} Z)^T & PN^T + Z^T D_{12}^T \\ CP + D_{22} Z & -\gamma^2 I & 0 \\ NP + D_{12} Z & 0 & -\mu I \end{bmatrix} < 0.$$

Then $K = ZP^{-1}$ is the corresponding controller.

An LMI for H_∞-Optimal Quadratically Stabilizing Controllers with Parametric Norm-Bounded Uncertainty

In this case, we set $Q = 0$.

Theorem 63.

There exists a K such that the system with $u(t) = Kx(t)$

$$\dot{x}(t) = Ax(t) + Bu(t) + Mp(t) + B_2 w(t), \qquad p(t) = \Delta(t) q(t),$$
$$q(t) = Nx(t) + D_{12} u(t), \qquad \Delta \in \boldsymbol{\Delta} := \{\Delta \in \mathbb{R}^{n \times n} : \|\Delta\| \le 1\}$$
$$y(t) = Cx(t) + D_{22} u(t)$$

satisfies $\|y\|_{L_2} \le \gamma \|u\|_{L_2}$ if there exists some $\mu \ge 0$, Z and $P > 0$ such that

$$\begin{bmatrix} AP + BZ + PA^T + Z^T B^T + B_2 B_2^T + \mu M M^T & (CP + D_{22} Z)^T & PN^T + Z^T D_{12}^T \\ CP + D_{22} Z & -\gamma^2 I & 0 \\ NP + D_{12} Z & 0 & -\mu I \end{bmatrix} < 0.$$

Then $K = ZP^{-1}$ is the corresponding controller.

This is from Boyd page 110.

I believe it relies on the following alternative to the S-procedure [Xie, 1992], which is similar to Finsler's Lemma

Theorem 64.

The following are equivalent

1. $$Q + F\Delta E + E^T \Delta F^T > 0 \qquad \textit{for all } \|\Delta\| < 1$$
2. *There exists some $\epsilon > 0$ such that*
$$Q + \epsilon F F^T + \epsilon^{-1} E^T E > 0$$

Unfortunately, to put the LMI in the form of 1 requires us to eliminate the pass-through term Q.

Structured, Norm-Bounded Uncertainty

For the case of structured parametric uncertainty, we define the structured set

$$\mathbf{\Delta} = \{\Delta = \text{diag}(\delta_1 I_{n1}, \cdots, \delta_s I_{ns}, \Delta_{s+1}, \cdots, \Delta_{s+f}) \,:\, \delta_i \in \mathbb{R},\, \Delta \in \mathbb{R}^{n_k \times n_k}\}$$

$$\Delta = \begin{bmatrix} \delta_1 I_{n1} & & & & & \\ & \cdots & & & & \\ & & \delta_s I_{ns} & & & \\ & & & \Delta_{s+1} & & \\ & & & & \cdots & \\ & & & & & \Delta_{s+f} \end{bmatrix}$$

- δ and Δ represent unknown parameters.
- s is the number of scalar parameters.
- f is the number of matrix parameters.

The Structured Singular Value

For the case of structured parametric uncertainty, we define the structured singular value.

Definition 65.

Given system $M \in \mathcal{L}(L_2)$ and set $\mathbf{\Delta}$ as above, we define the **Structured Singular Value** of $(M, \mathbf{\Delta})$ as

$$\mu(M, \mathbf{\Delta}) = \frac{1}{\inf_{\substack{\Delta \in \mathbf{\Delta} \\ I - M\Delta \text{ is singular}}} \|\Delta\|}$$

Of course, $\bar{S}(M, \mathbf{\Delta})$ is stable if and only if $\mu(M_{11}, \mathbf{\Delta}) < 1$.

- Obviously, $\mu(M, \mathbf{\Delta}) < \|M\|$
- For $\mathbf{\Delta} := \{\Delta \in \mathcal{L}(L_2) \,:\, \|\Delta\| \le 1\}$, $\mu(M, \mathbf{\Delta}) = \|M\|$
- $\mu(\alpha M, \mathbf{\Delta}) = |\alpha| \mu(M, \mathbf{\Delta})$
- Can increase M by a factor $\frac{1}{\mu(M, \mathbf{\Delta})}$ before losing stability.
- In general, computing μ is NP-hard unless uncertainty is unstructured.

Scalings and The Structured Singular Value

Suppose $\mathbf{\Theta} = \{\Theta : \Theta\Delta = \Delta\Theta \text{ for all } \Delta \in \mathbf{\Delta}\}$

- Then $\mu(M, \mathbf{\Delta}) = \inf_{\Theta \in \mathbf{\Theta}} \|\Theta M \Theta^{-1}\|$.
- Θ is the set of *scalings*.

$$\boldsymbol{\Delta} = \{\Delta = \text{diag}(\delta_1 I_{n1}, \cdots, \delta_s I_{ns}, \Delta_{s+1}, \cdots, \Delta_{s+f}) \,:\, \delta_i \in \mathbb{R},\, \Delta \in \mathbb{R}^{n_k \times n_k}\}$$

Define the set of scalings

$$\mathbf{P\Theta} := \{\text{diag}(\Theta_1, \cdots, \Theta_s, \theta_{s+1} I, \cdots, \theta_{s+f} I \,:\, \Theta_i > 0,\, \theta_j > 0\}$$

Theorem 66.

Suppose system M has transfer function $\hat{M}(s) = C(sI - A)^{-1}B + D$ with $\hat{M} \in H_\infty$. The following are equivalent

- *There exists $\Theta \in \boldsymbol{\Theta}$ such that $\|\Theta M \Theta^{-1}\|^2 < \gamma$.*
- *There exists $\Theta \in \mathbf{P\Theta}$ and $X > 0$ such that*

$$\begin{bmatrix} A^T X + XA & XB \\ B^T X & -\Theta \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \Theta \begin{bmatrix} C & D \end{bmatrix} < 0$$

Note: To minimize γ, you must use bisection.

An LMI for Stability of Structured, Norm-Bounded Uncertainty

This allows us to generalize the S-procedure to structured uncertainty

Theorem 67.

The system

$$\dot{x}(t) = Ax(t) + Mp(t), \qquad p(t) = \Delta(t)q(t),$$
$$q(t) = Nx(t) + Qp(t), \qquad \Delta \in \mathbf{\Delta},\ \ \|\Delta\| \le 1$$

is quadratically stable if and only if there exists some $\Theta \in \mathbf{P\Theta}$ *and* $P > 0$ *such that*

$$\begin{bmatrix} AP + PA^T & PN^T \\ NP & 0 \end{bmatrix} + \begin{bmatrix} M\Theta M^T & M\Theta Q^T \\ Q\Theta M^T & Q\Theta Q^T - \Theta \end{bmatrix} < 0\}$$

This is an LMI in Θ and P.

- The constraint $\Theta \in \mathbf{P\Theta}$ is linear

$$\mathbf{P\Theta} := \{\text{diag}(\Theta_1, \cdots, \Theta_s, \theta_{s+1} I, \cdots, \theta_{s+f} I) \,:\, \Theta_i > 0,\, \theta_j > 0\}$$

An LMI for Stability with Structured, Norm-Bounded Uncertainty

To prove the theorem, we can take a closer look at the scalings:

Since $T\Delta = \Delta T$ for $T \in \mathbf{P\Theta}$, the system can equivalently be written as

$$\dot{x}(t) = Ax(t) + MT^{-1}p(t), \qquad p(t) = \Delta(t)q(t),$$
$$q(t) = TNx(t) + TQT^{-1}p(t), \qquad \Delta \in \mathbf{\Delta},\ \|\Delta\| \le 1$$

for any $T \in \mathbf{P\Theta}$. Then

$$\begin{bmatrix} AP + PA^T & PN^T \\ NP & 0 \end{bmatrix} + \begin{bmatrix} MM^T & MQ^T \\ QM^T & QQ^T - I \end{bmatrix} < 0$$

becomes

$$\begin{bmatrix} AP + PA^T & PN^T T^T \\ TNP & 0 \end{bmatrix} + \begin{bmatrix} MT^{-2}M^T & MT^{-2}Q^T T^T \\ TQT^{-2}M^T & TQT^{-2}Q^T T^T - I \end{bmatrix} < 0\}$$

Pre- and Post-multiplying by $\begin{bmatrix} I & 0 \\ 0 & T^{-1} \end{bmatrix}$, and using $\Theta = T^{-2} \in \mathbf{P\Theta}$, we recover the LMI condition.

An LMI for Stabilizing State-Feedback Controllers with Structured Norm-Bounded Uncertainty

Theorem 68.

There exists a K *such that the system with* $u(t) = Kx(t)$

$$\dot{x}(t) = Ax(t) + Bu(t) + Mp(t), \qquad p(t) = \Delta(t)q(t),$$
$$q(t) = Nx(t) + Qp(t) + D_{12}u(t), \qquad \Delta \in \mathbf{\Delta},\ \|\Delta\| \le 1$$

is quadratically stable if and only if there exists some $\Theta \in \mathbf{P\Theta}$, $P > 0$ *and* Z *such that*

$$\begin{bmatrix} AP + BZ + PA^T + Z^T B^T & PN^T + Z^T D_{12}^T \\ NP + D_{12}Z & 0 \end{bmatrix} + \begin{bmatrix} M\Theta M^T & M\Theta Q^T \\ Q\Theta M^T & Q\Theta Q^T - \Theta \end{bmatrix} < 0.$$

Then $K = ZP^{-1}$ *is a quadratically stabilizing controller.*

We can also extend this result to optimal control in the H_∞ norm.

An LMI for Stabilizing State-Feedback Controllers with Structured Norm-Bounded Uncertainty

Theorem 68.
There exists a K such that the system with $u(t) = Kx(t)$

$$\dot{x}(t) = Ax(t) + Bu(t) + Mp(t), \qquad p(t) = \Delta(t)q(t),$$
$$q(t) = Nx(t) + Qp(t) + D_{12}u(t), \qquad \Delta \in \mathbf{\Delta}, \; \|\Delta\| \le 1$$

is quadratically stable if and only if there exists some $\Theta \in \mathbf{P\Theta}$, $P > 0$ and Z such that

$$\begin{bmatrix} AP + BZ + PA^T + Z^TB^T & PN^T + Z^TD_{12}^T \\ NP + D_{12}Z & 0 \end{bmatrix} + \begin{bmatrix} M\Theta M^T & M\Theta Q^T \\ Q\Theta M^T & Q\Theta Q^T - \Theta \end{bmatrix} < 0.$$

Then $K = ZP^{-1}$ is a quadratically stabilizing controller.

We can also extend this result to optimal control in the H_∞ norm.

This is from Boyd, page 102

An LMI for Optimal State-Feedback Controllers with Structured Norm-Bounded Uncertainty

In this case, we set $Q = 0$.

Theorem 69.

There exists a K such that the system with $u(t) = Kx(t)$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Mp(t) + B_2 w(t), \qquad p(t) = \Delta(t) q(t), \\ q(t) &= Nx(t) + D_{12} u(t), \qquad \Delta \in \mathbf{\Delta}, \; \|\Delta\| \le 1 \\ y(t) &= Cx(t) + D_{22} u(t) \end{aligned}$$

satisfies $\|y\|_{L_2} \le \gamma \|u\|_{L_2}$ if there exists some $\Theta \in \mathbf{P\Theta}$, Z and $P > 0$ such that

$$\begin{bmatrix} AP + BZ + PA^T + Z^T B^T + B_2 B_2^T + M\Theta M^T & (CP + D_{22} Z)^T & PN^T + Z^T D_{12}^T \\ CP + D_{22} Z & -\gamma^2 I & 0 \\ NP + D_{12} Z & 0 & -\Theta \end{bmatrix} < 0.$$

Then $K = ZP^{-1}$ is the corresponding controller.

An LMI for Optimal State-Feedback Controllers with Structured Norm-Bounded Uncertainty

Using the equivalent scaled system

$$\dot{x}(t) = Ax(t) + Bu(t) + MT^{-1}p(t) + B_2 w(t), \qquad p(t) = \Delta(t)q(t),$$
$$q(t) = TNx(t) + TD_{12}u(t), \qquad \Delta \in \mathbf{\Delta},\ \|\Delta\| \le 1$$
$$y(t) = Cx(t) + D_{22}u(t)$$

we get

$$\begin{bmatrix} AP + BZ + PA^T + Z^T B^T + B_2 B_2^T + MT^{-2}M^T & (CP + D_{22}Z)^T & PN^T T^T + Z^T D_{12}^T T^T \\ CP + D_{22}Z & -\gamma^2 I & 0 \\ TNP + TD_{12}Z & 0 & -I \end{bmatrix} < 0.$$

Pre- and Post-multiplying by $\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & T^{-1} \end{bmatrix}$, and using $\Theta = T^{-2} \in \mathbf{P\Theta}$, we recover the LMI condition.

An LMI for Optimal State-Feedback Controllers with Structured Norm-Bounded Uncertainty

Using the equivalent scaled system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + MT^{-1}p(t) + B_2w(t), \qquad p(t) = \Delta(t)q(t),\\ q(t) &= TNx(t) + TD_{12}u(t), \qquad \Delta \in \boldsymbol{\Delta},\ \|\Delta\| \le 1\\ y(t) &= Cx(t) + D_{22}u(t)\end{aligned}$$

we get

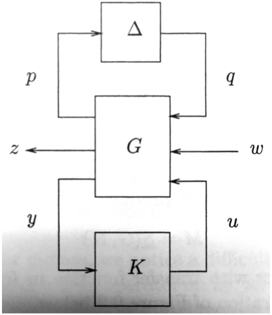
$$\begin{bmatrix} AP + BZ + PA^T + Z^TB^T + B_2B_2^T + MT^{-2}M^T & (CP + D_{22}Z)^T & PN^TT^T + Z^TD_{12}^TT^T \\ CP + D_{22}Z & -\gamma^2 I & 0 \\ TNP + TD_{12}Z & 0 & -I \end{bmatrix} < 0.$$

Pre- and Post-multiplying by $\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & T^{-1} \end{bmatrix}$, and using $\Theta = T^{-2} \in \mathbf{P\Theta}$, we recover the LMI condition.

This is not from Boyd, but should be

Output-Feedback Robust Controller Synthesis

How to Solve the Output Feedback Case???



$$\inf_{K} \sup_{\Delta \in \mathbf{\Delta}} \|\underline{S}(\bar{S}(G,\Delta),K)\|_{H_\infty}$$

Finally, we mention a Heuristic for Output-Feedback Controller synthesis.

Initialize: $\Theta = I$.
Define:

$$\hat{G}_\Theta(s) = \left[\begin{array}{c|cc} A & B_1\Theta^{-\frac{1}{2}} & B_2 \\ \hline \Theta^{\frac{1}{2}}C_1 & \Theta^{\frac{1}{2}}D_{11}\Theta^{-\frac{1}{2}} & \Theta^{\frac{1}{2}}D_{12} \\ C_2 & D_{21}\Theta^{-\frac{1}{2}} & 0 \end{array}\right]$$

Step 1: Fix Θ and solve

$$\inf_K \|\underline{S}(G_\Theta, K)\|_{H_\infty}$$

Step 2: Fix K and minimize γ such that there exists $\Theta \in \mathbf{P\Theta}$ (or $\Theta \in \mathbf{P\Theta} \times I$ if you include the regulated output channel.) and $X > 0$ such that

$$\begin{bmatrix} A_{cl}^T X + X A_{cl} & X B_{cl} \\ B_{cl}^T X & -\Theta \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} C_{cl}^T \\ D_{cl}^T \end{bmatrix} \Theta \begin{bmatrix} C_{cl} & D_{cl} \end{bmatrix} < 0$$

where $A_{cl}, B_{cl}, C_{cl}, D_{cl}$ define $\underline{S}(G_I, K)$. (Requires Bisection).

Step 3: `GOTO` Step 1

A Word on D-K Iteration with Static Uncertainty

A Heuristic for Dynamic Output Feedback Synthesis

The D-K iteration outlined in this lecture is only valid for *Dynamic Uncertainty*: $\Delta(t)$.

- Our Scalings Θ are time-invariant.

For Static uncertainties, we should search for *Dynamic Scaling Factors*

- $\Theta(s)$ is a *Transfer Function*
- This is much harder to represent as an LMI (Or by any other method!).
- Matlab has built-in functionality, but it is hard to use.

We will return to μ analysis for static uncertainties when we consider more advanced forms of optimization.