A Converse Sum-of-Squares Lyapunov Result with Degree Bound

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Complex Problems

- PDEs.
- Nonlinearity.
- Delay.
- Coupled Systems.
- Computation creates a metric for complexity.

Successes for Linear Systems:

- LMI's for solving linear finite-dimensional systems.
- polynomial-time complexity.

Challenges for NP-hard Problems:

- How to solve nonlinear and infinite-dimensional problems.
- How to use parallel/distributed computation to solve large-scale problems?

Asymptotic Algorithms

- A sequence of Algorithms.
- Guaranteed Error Bounds.
- Error decreases as complexity increases.

A New Metric

- Non-polynomial.
- Fixed error complexity.

Consider: A System of Nonlinear Ordinary Differential Equations

$$\dot{x}(t) = f(x(t))$$

Problem: Stability
Given a specific polynomial $f : \mathbb{R}^n \to \mathbb{R}^n$,
find the largest $X \subset \mathbb{R}^n$
such that for any $x(0) \in X$,
$\lim_{t\to\infty} x(t) = 0$.

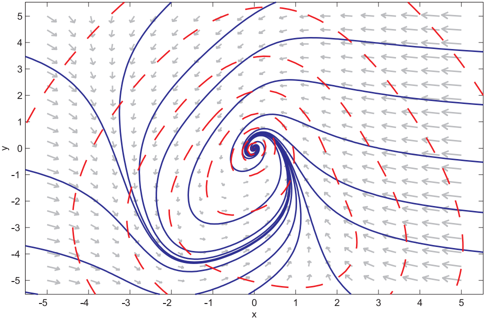
Lyapunov Functions

Necessary and Sufficient for Stability

Consider

$$\dot{x}(t) = f(x(t))$$

with $x(0) \in \mathbb{R}^n$.



Theorem 1 (Lyapunov Stability).

Suppose there exists a continuous V and $\alpha, \beta, \gamma > 0$ where

$$\beta\|x\|^2 \le V(x) \le \alpha\|x\|^2$$
$$-\nabla V(x)^T f(x) \ge \gamma\|x\|^2$$

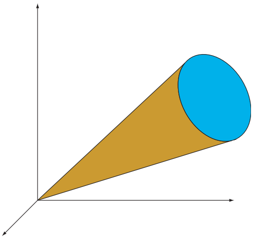
for all $x \in X$. Then any sub-level set of V in X is a **Domain of Attraction**.

Tractable or Intractable?

Convex Optimization

Problem:

$$\max \ bx$$
$$\text{subject to} \quad Ax \in C$$



The problem is *convex optimization* if

- C is a convex cone.
- b, A and B are affine.

Computational Tractability: Convex Optimization over C is, in general, tractable if

- The set membership test for $y \in C$ is in P.
- x is finite dimensional.

The Stability Problem is Convex

Convex Optimization of Functions: Variables $V \in \mathcal{C}[\mathbb{R}^n]$ and $\gamma \in \mathbb{R}$

$$\max\ \gamma$$
subject to
$$V(x) - x^T x \geq 0 \qquad \forall x$$
$$\nabla V(x)^T f(x) + \gamma x^T x \leq 0 \qquad \forall x$$

The problem is *finite-dimensional* if $V(x)$ is *polynomial* of bounded degree.

Convex Optimization of Polynomials: Variables $c \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$

$$\max\ \gamma$$
subject to
$$c^T Z(x) - x^T x \geq 0 \qquad \forall x$$
$$c^T \nabla Z(x) f(x) + \gamma x^T x \leq 0 \qquad \forall x$$

- $Z(x)$ is a fixed vector of monomial bases.

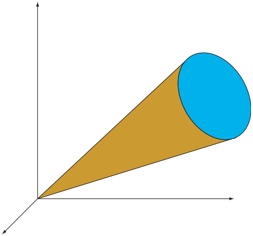
A Unified Framework for NP-hard Problems

Optimization of Polynomial Variables

Problem:

$$\max\ b^T x$$

$$\text{subject to } A_0(y) + \sum_i^n x_i A_i(y) \succeq 0 \quad \forall y$$



The A_i are matrices of polynomials in y. e.g. Using multi-index notation,

$$A_i(y) = \sum_\alpha A_{i,\alpha}\ y^\alpha$$

Computationally Intractable

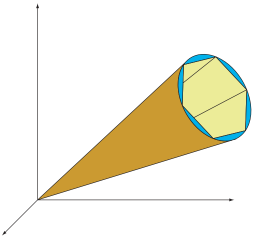
The problem: "Is $p(x) \geq 0$ for all $x \in \mathbb{R}^n$?" (i.e. "$p \in \mathbb{R}^+[x]$?") is NP-hard.

A Popular Approach to Optimizing Polynomials
Sum-of-Squares (SOS) Programming

Problem:

$$\max \ b^T x$$

subject to $\quad A_0(y) + \sum_i^n x_i A_i(y) \in \Sigma_s$



Definition 2.

$\Sigma_s \subset \mathbb{R}^+[x]$ is the cone of *sum-of-squares* matrices. If $S \in \Sigma_s$, then for some $G_i \in \mathbb{R}[x]$,

$$S(y) = \sum_{i=1}^r G_i(y)^T G_i(y)$$

Computationally Tractable: $S \in \Sigma_s$ is an SDP constraint.

SOS Programming:

Why is $M \in \Sigma_s$ an SDP?

Define $Z_d^n(x)$ to be the vector of monomial bases in dimension n of degree d or less.
e.g., if $x \in \mathbb{R}^2$, then

$$Z_2^1(x)^T = \begin{bmatrix} 1 & x_1 & x_2 & x_1 x_2 & x_1^2 & x_2^2 \end{bmatrix}$$

and

$$Z_1^2(x)^T = \begin{bmatrix} 1 & x_1 & x_2 & & & \\ & & & 1 & x_1 & x_2 \end{bmatrix} = \begin{bmatrix} Z_1^1(x) & \\ & Z_1^1(x) \end{bmatrix}$$

Feasibility Test:

Lemma 3.

Suppose M is polynomial of degree $2d$. $M \in \Sigma_s$ iff there exists some $Q \succeq 0$ such that

$$M(x) = Z_d(x)^T Q Z_d(x).$$

Consider the system

$$\dot{x}(t) = f(x(t))$$

Theorem 4 (Peet, TAC 2009).

*If f is **sufficiently smooth** and the system is exponentially stable on a compact set X. Then there exists a **polynomial** $V : \mathbb{R}^n \to \mathbb{R}$ and constants $\alpha, \beta, \gamma > 0$ such that*

$$\alpha \|x\|_2^2 \le V(x) \le \beta \|x\|_2^2$$
$$\nabla V(x)^T f(x) \le -\gamma \|x\|_2^2$$

for all $x \in X$.

Note: f is **sufficiently smooth** if it is 3 times continuously differentiable.

Picard Iteration

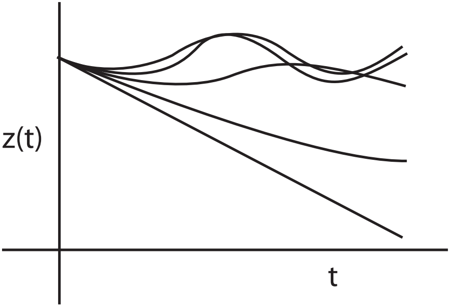
Picard Iteration:

- A method for constructing solutions to ordinary differential equations

$$\dot{x}(t) = f(x(t)) \qquad x(0) = x_0$$

Make an initial guess at the solution

$$y_1(t) = x_0 + f(x_0)t$$



1. Define a new guess as

$$y_{i+1} = x_0 + \int_0^t f(y_i(s))ds$$

- The map $y_i \to y_{i+1}$ is a contraction in the $\sup$ norm on $[0,T]$.
- The solution map will converge on some interval, $[0,T]$.

Picard Iteration for the Solution Map

Consider

$$\dot{x}(t) = f(x(t)) \qquad\qquad x(0) = x_0$$

If solutions exist, we can define the **Solution Map**, $z(x_0, t)$.

z is the solution map if

$$\frac{d}{dt} z(x,t) = f(z(x,t)) \qquad \text{and} \qquad z(x,0) = x,$$

for all $t \in Y$, $x_0 \in X$

Picard iteration can be applied to the solution map

$$z_{i+1}(x,s) = x + \int_0^t f(z_i(s,x))ds$$

The Picard iteration converges uniformly on $x \in X$ for some interval $t \in [0, T]$.

Polynomial Approximation of the Solution Map with Properties of the Solution

Recall the Picard iteration

$$z_{i+1}(x,s) = x + \int_0^t f(z_i(s,x))ds$$

If

- z_i is a polynomial of degree h in x
- f is a polynomial of degree d

Then z_{i+1} is a polynomial of degree hd in x.

Conclusion: We can construct successive polynomial approximations of the solution map with a simple degree bound.

These approximations preserve properties of the solution!

$$\frac{d}{dt} z_{i+1}(x,t) = f(z_i(t,x))$$

Converse Lyapunov Form

The Solution Map is often used to define converse Lyapunov functions.

$$V(x) = \int_0^\delta z(x,s)^T z(x,s) ds$$

Theorem 5.

Suppose

$$\dot{x}(t) = f(x(t))$$

is exponentially stable. Then

$$\dot{V}(x(t)) = \frac{d}{dt}\int_0^\delta z(x(t),s)^T z(x(t),s) ds < -\alpha \|x(t)\|^2$$

for δ large enough and some $\alpha > 0$.

Converse Lyapunov Form using Picard Iteration

We propose the new converse Lyapunov form

$$V_i(x) = \int_0^\delta z_i(x,s)^T z_i(x,s) ds$$

- V_i is polynomial for any i
- V_i is SOS for any i
- V_i is degree d^{2i} in x.

Theorem 6.

Suppose

$$\dot{x}(t) = f(x(t))$$

is exponentially stable. Then for δ, i large enough,

$$\dot{V}_i(x(t)) < -\alpha \|x\|^2$$

for some $\alpha > 0$.

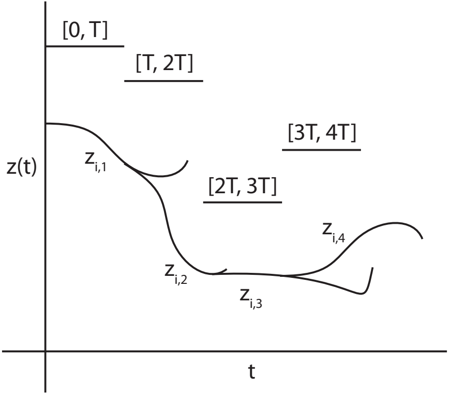
Extensions of the Picard Iteration

Unfortunately, even for well-behaved systems, the Picard iteration only converges on sufficiently short intervals, $[0, T]$.

We extend the Picard iteration indefinitely by defining a new iteration at each interval:

$$z_{i+1,j}(x,t)$$
$$= z_{i+1,j-1}(x, jT) + \int_{jT}^{t} f(z_{i,j}(s,x))ds$$

for $t \in [jT, jT+T]$.



By induction

- Each $z_{i+1,j}$ is polynomial
- We have a different polynomial on each interval $[jT, jT+T]$.
- Approximation is smooth at the points jT.

Extensions of the Picard Iteration

Use the new functional:

$$V_i(x) = \sum_j \int_{jT}^{jT+T} z_{i,j}(x,s)^T z_{i,j}(x,s) ds$$

- Each $\int_{jT}^{jT+T} z_{i,j}(x,s)^T z_{i,j}(x,s) ds$ is a squared polynomial.
- V_i is a sum of squared polynomials (SOS).

Degree Bound: The degree bound, $2d^{(Nk-1)}$, depends on both

- The number of iterations, k.
- The number of Picard extensions, N.

These factors depend on

- Lipschitz factor, L, for f.
- Size of the region of convergence, r.
- Stability factors K, λ

$$\|x(t)\| \leq Ke^{-\lambda t}\|x_0\|$$

Polynomials Systems have SOS Lyapunov Functions

Consider

$$\dot{x}(t) = f(x(t))$$

Theorem 7 (Peet and Papachristodoulou).

If $\|x(t)\| \le K e^{-\lambda t}$ *and the polynomial* f *satisfies* $\|\nabla f(x)\| \le L$ *on* $\|x\| \le r$*, then there exists a sum-of-squares polynomial* $V \in \Sigma_s$ *such that*

$$c_1 \|x\|^2 \le V(x) \le c_2 \|x\|^2$$
$$\nabla V(x)^T f(x) \le -c_3 \|x\|^2$$

Furthermore, the degree of V *can be bounded by*

$$\textit{degree}(V) \le d(L, \gamma, K, r)$$

$d(L, \gamma, K, r)$ can only be found via algorithm.

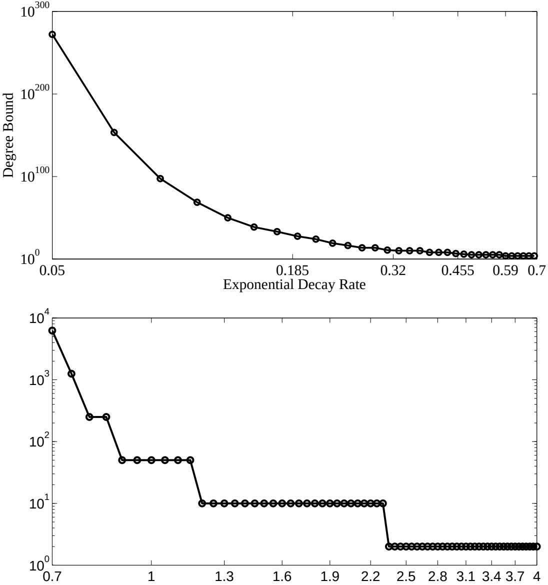


Figure: Degree bound vs. Convergence Rate for $K = 1.2$, $r = L = 1$, and $q = 5$

Concluding Remarks: Two Problems

Optimization of Polynomials:

- Problems for Polynomial Computing
  - Nonlinear Stability
  - Time-Delay Controller Synthesis
  - PDE Controller Synthesis
- Complexity bounds for Asymptotic Algorithms.
  - Do polynomial solutions exist?
  - Can we find degree bounds?

Solving Polynomial Computing Problems:

- Sum-of-Squares
  - Positivstellensatz Degree bounds
  - Bounds on Error
- Parallel Computing
  - Polya's Algorithm
  - Computable bounds on Polya's Exponent
  - PS for Polya?

Some algorithms are available for download at:

`http://mmae.iit.edu/~mpeet`