A Sum-of-Squares Approach to the Analysis of Zeno Stability in Polynomial Hybrid Systems

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August 8, 2013

Introduction

- Hybrid Systems can exhibit unique behaviors such as **Zeno behavior**.
  - Zeno behavior occurs when infinitely many discrete transitions occur in finite time.
  - Such behavior can cause simulations to fail, and may indicate modeling errors.
- Two main techniques for checking for Zeno behavior:
  - Checking simulation for increasingly fast switching times,
  - Deductively proving infinite transitions occur in finite time.
- **Objective**: To determine a means to use computational tools to test whether a hybrid system exhibits Zeno behavior.

Hybrid Automata

In this work, we model hybrid systems using **hybrid automata**:

Hybrid System

We say a hybrid system H is given by the tuple (Q, E, D, F, G, R) where

- Q is a finite collection of **discrete states** or indices.
- $E \subset Q \times Q$ is a collection of **edges** - provides all transitions that occur between discrete states.
- $D = \{D_q\}_{q \in Q}$ is a collection of **domains**.
- $F = \{f_q\}_{q \in Q}$ is a collection of **vector fields**.
- $G = \{G_e\}_{e \in E}$ is a collection of **guard sets**.
- $R = \{\phi_e\}_{e \in E}$ is a collection of **Reset Maps**.

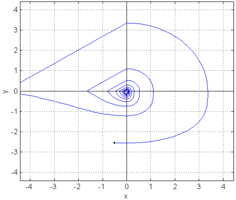

Figure : A Hybrid System Phase Portrait

Preliminaries: Cyclic Hybrid Systems

Cyclic Hybrid System

- A **cyclic hybrid system** is a hybrid system where each discrete state is the source of only one edge, and the target of one edge.

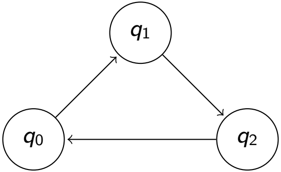

Figure : A Cyclic Hybrid Automaton

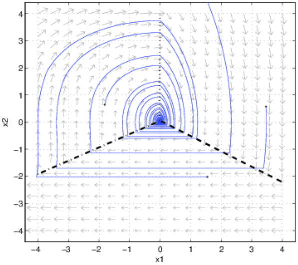

Figure : A Cyclic Hybrid System Phase Portrait

Executions, Zeno Executions, and Zeno Stability

Hybrid System Execution

- Executions: generalizations of solutions to non-smooth dynamical systems
- Describes both continuous flows and discrete transitions.

Zeno Execution

An execution of a hybrid system $H = (Q, E, D, F, G, R)$ is Zeno if infinitely many discrete transitions occur in a finite period of time.

Zeno Stability

Let $H = (Q, E, D, F, G, R)$ be a hybrid system. The set $z = \{z_q\}_{q \in Q}$ is **Zeno stable** if, for each $q \in Q$, there exist neighborhoods W_q, where $z_q \in W_q$, such that for any initial condition $x_0 \in \bigcup_{q \in Q} W_q$, the execution starting at that initial condition is Zeno, and converges to z.

Lyapunov Theorems for Zeno Stability

- We present a Lyapunov theorem below:

Theorem 1 (Based on results in [Lamperski & Ames CDC 2007],[Teel & Goebel CDC 2008])

Let $H = (Q, E, D, F, G, R)$ be a cyclic hybrid system, and let $z = \{z_q\}_{q \in Q}$. Let $\{W_q \subset D_q\}_{q \in Q}$, be a collection of neighborhoods of the $\{z_q\}_{q \in Q}$. Suppose that there exist continuously differentiable functions $V_q : W_q \to \mathbb{R}$, and positive constants $\{r_q\}_{q \in Q}$ and γ, where $r_q \in (0, 1]$, and $r_q < 1$ for some q and such that

$$V_q(x) - \alpha x^T x \geq 0 \quad \text{for all } x \in W_q \backslash z_q, q \in Q \tag{1}$$

$$V_q(z_q) = 0, \quad \text{for all } q \in Q \tag{2}$$

$$\nabla V_q^T(x) f_q(x) \leq -\gamma \quad \text{for all } x \in W_q,\ q \in Q \tag{3}$$

$$r_q V_q(x) \geq V_{q'}(\phi_e(x)) \quad \text{for all } e = (q, q') \in E \text{ and } x \in G_e \cap W_q. \tag{4}$$

then z is Zeno stable.

Sum-of-Squares Polynomials

Sum-of-Squares Polynomials

- Checking if a polynomial is positive is NP-hard - how do we work around that?
- We can check whether a polynomial is Sum-of-Squares (SOS) using semidefinite programming - this can be solved in polynomial time!
- A polynomial $p(x)$ is SOS if $p(x) = \sum_i f_i(x)^2$ where f_i are polynomials.
- Some Notation:
  - If $p(x)$ is a polynomial, we say $p \in \mathbf{R}[x]$.
  - If $p(x)$ is SOS, we say $p \in \Sigma_x$.
- If $p \in \Sigma_x$ and of degree $2d$, there exists a PSD matrix Q and a vector $Z(x)$ of monomials upto degree d such that $p(x) = Z(x)^T QZ(x)$.

Semialgebraic Sets and the Positivstellensatz

- Use of Positivstellensatz: allows us to convert local positivity conditions into global set inclusions.
- We use Sum-of-Squares programming to enforce set inclusions.

Semialgebraic Set

A Semialgebraic set S is a set defined by polynomial inequalities and equalities:

$$S := \{x : g_k(x) \geq 0, h_l(x) = 0, k = 1, ..., K, l = 1, ..., L\}$$

.

Positivstellensatz

Positivstellensatz is used as a generalization of the S-Procedure:
If

$$f - \sum_k s_k g_k - \sum_l t_l h_l \in \Sigma_x$$

then $f \in \mathbf{R}[x] \geq 0$ when $x \in X$ (as defined above), and where $s_k \in \Sigma_x$ and $t_l \in \mathbf{R}[x]$.

Semialgebraic Sets and the Positivstellensatz (continued)

Remark To use the above results, we assume that all vector fields and reset maps are polynomial, and that domains and guard sets are semialgebraic sets.

- We also assume that each neighborhood within which we analyze Zeno stability is a semialgebraic set.
  - Each D_q is defined as $D_q := \{x \in \mathbb{R}^n : g_{qk}(x) \geq 0,\ k = 1, \ldots\}$.
  - Each G_e is defined as $G_e := \{x \in \mathbb{R}^n : h_{e,0}(x) = 0, h_{e,k}(x) \geq 0,\ k = 1, \ldots\}$.
  - Each W_q is defined as $W_q := \{x \in \mathbb{R}^n : w_{qk} \geq 0,\ k = 1, \ldots\}$.

A Sum of Squares Result for Robust Zeno Stability

- Applying the Positivstellensatz to Equations **??**-**??**, we construct Feasibility Problem 1:

Feasibility Problem 1

For hybrid system $H = (Q, E, D, F, G, R)$, find

- $a_{qk},\ b_{qk},\ c_{qk},\ d_{qk},\ i_{qk},\ m_{e,k},\ j_{qk} \in \Sigma_x$, for $k = 1, 2, ...,\ q \in Q$;
- $V_q,\ m_{e,0} \in \mathbf{R}[x]$ for $e \in E$ and $q \in Q$.
- Constants $\alpha, \gamma > 0$, $\{r_q\}_{q\in Q} \in (0, 1]$ such that $r_q < 1$ for some $q \in Q$.

such that

$$V_q - \alpha x^T x - \sum_{k=1}^{K_{qw}} a_{qk} w_{qk}^{\nearrow \text{neighborhood}} - \sum_{k=1}^{K_q} b_{qk} g_{qk}^{\nearrow \text{domain}} \in \Sigma_x \quad \text{for all } q \in Q \quad (5)$$

A Sum of Squares Result for Zeno Stability

Feasibility Problem 1 (continued)

$$V_q(z_q) = 0 \quad \text{for all } q \in Q \tag{6}$$

$$-\nabla V_q^T f_q - \gamma - \sum_{k=1}^{K_{qw}} c_{qk} w_{qk}^{\nearrow \text{neighborhood}} - \sum_{k=1}^{K_q} d_{qk} g_{qk}^{\nearrow \text{domain}} \in \Sigma_x \quad \text{for all } q \in Q \tag{7}$$

$$r_q V_q - V_{q'}(\phi_e) - m_{e,0} h_{e,0}^{\nearrow \text{guard}} - \sum_{k=1}^{N_q} m_{e,k} h_{e,k}^{\nearrow \text{guard}} - \sum_{k=1}^{K_{qw}} i_{qk} w_{qk}^{\nearrow \text{neighborhood}}$$
$$- \sum_{k=1}^{K_q} j_{qk} g_{qk}^{\nearrow \text{domain}} \in \Sigma_x \quad \text{for all } e = (q, q') \in E. \tag{8}$$

A Sum of Squares Result for Zeno Stability

- Based on Feasibility Problem 1, we then obtain the following theorem:

Theorem 2

Consider a cyclic hybrid system $H = (Q, E, D, F, G, R)$, and let $z = \{z_q\}_{q \in Q}$. If Feasibility Problem 1 has a solution, then z is Zeno stable.

Remark: Theorem 2 can easily be applied to systems with parametric uncertainty:

- Let $\Delta := \{p \in \mathbb{R} : \rho_k(p) \geq 0\}$ be the set of uncertain parameters.
- Apply the Positivstellensatz to Theorem 1 with the uncertain set as well.
- All elements of Feasibility Problem 1 have dependency on both x and p.

Simulation Example: Gain Scheduling

We tested our results on the closed loop system of a plant controlled with gain scheduling:

Closed-Loop Plant with Gain Scheduling

We model the hybrid system with the tuple $H = (Q, E, D, F, G, R)$ where

- $Q = \{1, 2, 3\},\ E = \{(12), (23), (31)\}$
- $D_1 = \{x \in \mathbb{R}^2 : x_1 > 0, x_2 + \frac{1}{2}x_1 \geq 0\}$
  $D_2 = \{x \in \mathbb{R}^2 : x_2 - \frac{1}{2}x_1 \geq 0, x_2 + \frac{1}{2}x_1 < 0\}$
  $D_3 := \{x \in \mathbb{R}^2 : x_1 < 0, x_2 + \frac{1}{2}x_1 \geq 0\}$
- $f_1 = (x_2,\ -5x_1 - 1)^T;\ f_2 = \left(-x_1^2 - 3,\ 2x_2^2 - \frac{1}{2}x_1^2\right);\ f_3 = \left(x_2^2 + x_1,\ -3x_1\right)$
- $G_{12} := \left\{x \in \mathbb{R}^2 : x_2 \leq 0, \frac{1}{2}x_1 + x_2 = 0\right\}$
  $G_{23} := \left\{x \in \mathbb{R}^2 : x_2 \leq 0, \frac{1}{2}x_1 - x_2 = 0\right\}$
  $G_{31} := \left\{x \in \mathbb{R}^2 : x_2 > 0, x_1 = 0\right\}$
- Each reset map is given by $\phi_e(x) := x$
- **Goal**: To test whether the origin is Zeno stable.

Below, we provide a phase portrait of the system:

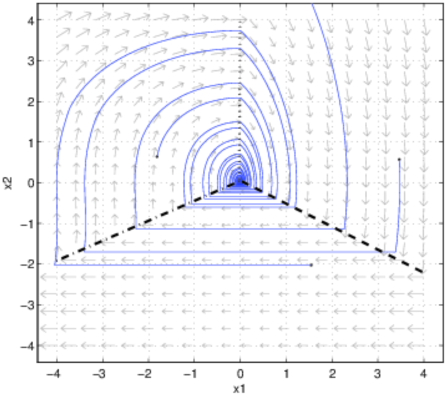

Figure : Hybrid System in Example 1. Dashed line indicates G_{12}, dash-dotted line indicates G_{23} and dotted line indicates G_{31}

Simulation Example: Gain Scheduling

System with Gain Scheduling (Results)

- We chose $W_q := D_q \cap \{x \in \mathbb{R}^2; |x| \leq 5\}$.
- We were able to prove Zeno stability of the origin for this hybrid system using 3 degree 8 polynomial Lyapunov functions to solve Feasibility problem 1.
- The degrees of the multiplier polynomials were fixed at 4.

Simulation Example: A Hybrid System with Parametric Uncertainty

We now consider a hybrid system with parametric uncertainty:

Hybrid System with Parametric Uncertainty

We consider a hybrid system with an uncertain parameter in the guard set.

- The uncertain parameter lies in $\Delta := \{p \in \mathbb{R} : p - C > 0\}$
- $Q = \{1, 2\},\ E = \{(12), (21)\}$
- $D_1 = \{x \in \mathbb{R}^2 : x + y \geq 0, x - py \geq 0\}$

  $D_2 := \mathbb{R}^2 \backslash D_1$
- $f_1 = (-0.1, 2)^T;\ f_2 = (-y - x^2, x)$
- $G_{12} := \left\{x \in \mathbb{R}^2 : x \geq 0, py - x = 0\right\}$

  $G_{21} := \left\{x \in \mathbb{R}^2 : x \geq 0, x + y = 0\right\}$
- Each reset map is given by $\phi_e(x) := x$
- **Goal**: To find a lower bound on C such that the origin is Zeno stable.

Simulation Example: A Hybrid System with Parametric Uncertainty

Below are two phase portraits of the hybrid system above with 2 different values of the uncertain parameter:

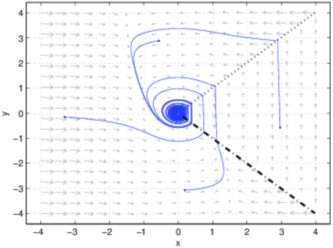

Figure : Trajectories of Hybrid System in Example 2 with p=1

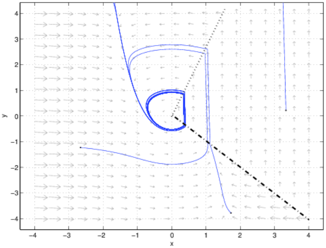

Figure : Trajectories of Hybrid System in Example 2 with p=.25.

Simulation Example: A Hybrid System with Parametric Uncertainty

Hybrid System with Parametric Uncertainty: Results

- The set of uncertain parameters is given by $\Delta := \{p \in \mathbb{R} : \rho(p) := p - C > 0\}$
- We search for Lyapunov functions of different degrees to find the bound:

Degree of V_1, V_2	**Lower bound on C**
8	2.11
10	1.87
12	1.73

Table : Lower bound on C for which z is Zeno stable obtained for different degrees of V_1 and V_2

Conclusions

- We developed a method to test whether a system exhibits Zeno behavior.
  - We framed Zeno behavior as a form of Stability
  - Used Sum-of-Squares programming to construct Lyapunov-like conditions to test Zeno stability of polynomial hybrid systems with uncertainties.
- We tested our technique on two hybrid systems
  - We were able to show Zeno stability for a hybrid system without parametric uncertainty.
  - We were able to obtain a lower bound on the range of uncertain parameters for which a hybrid system with parametric uncertainty was Zeno stable.

Ongoing Work

- Improving the computational cost of the technique
- Controller synthesis for hybrid systems such that the closed loop system does not exhibit Zeno behavior.

Thank you for your time!