Analysis of Systems with State-Dependent Delay

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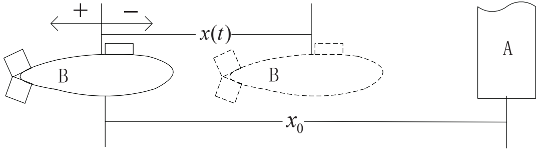
Systems with State-Dependent Delay

State-Dependent Delay arises which communication distance changes with time.

- Any Moving System.
  - Sonar (Speed of Sound)
  - EM (Speed of Light)

Example: Position Measurement using Sonar



- Propagation Distance: $2\left(\hat{x}(t) = x_0 + x(t)\right)$
- Delay is $\tau(t) = 2\frac{x_0 + x(t)}{c}$.

Feedback Dynamics: neglecting inertia...

$$\dot{x}(t) = ax\left(t - \frac{2x_0}{c} - \frac{2}{c}x(t)\right)$$

Stability of Systems with State-Dependent Delay

Consider the general class of systems with state-dependent delay:

$$\dot{x}(t) = f\Big(x(t), x\Big(t - g(x(t))\Big)\Big)$$

Note: State $x(t)$ enters into the *independent argument*, t.

- Contrast to the fixed-delay case: $\dot{x}(t) = f\Big(x\big(t - \tau\big)\Big)$

Linear/Affine Form:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau_0 - b^T x(t))$$

Question: Given $A \in \mathbb{R}^n$, $\tau_0 > 0$ and $-\tau \in \mathbb{R}^n$,

- Determine whether the system is stable.
- Estimate the rate of decay.

Why State-Dependent Delay?

The most common source of delay is **Propagation Time**.

- A time-delay system is the interconnection of an ODE with a transport PDE in the feedback channel.

$$\dot{x}(t) = A_0 x(t) + \sum A_i x(t - \tau_i)$$

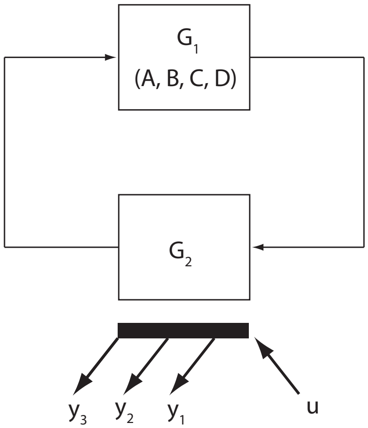
ODE: The system G_1

$$\dot{x}_1(t) = Ax_1(t) + Bu_1(t)$$
$$y_1(t) = Cx_1(t) + Du_1(t)$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} A_0 & \begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix} \\ \hline I & 0 \end{array}\right]$$

PDE: The system G_2

$$\frac{\partial}{\partial t} x_2(t,s) = \frac{\partial}{\partial s} x_2(t,s) \quad x_2(t,0) = u_2(t),$$
$$y_2(t) = \begin{bmatrix} x_2(-\tau_1) & \cdots & x_2(-\tau_K) \end{bmatrix}^T$$



Of course, the solution is just $x_2(t,s) = u_2(t-s)$.

Systems with State-Dependent Delay

Restrictions on the Model:

- For physical systems:
  - Delay is always positive
  - Delay is usually affine in the state
  - Lower and upper bounds for the delay
- Other types of systems are often ill-posed or otherwise pathological [Verriest, 2013]
  - Global Stability is not a well-defined problem for this model.

Assumptions: In this talk, we use scalar systems of the form

$$\dot{x}(t) = ax(t - b - cx(t))$$

where $a, c \in \mathbb{R}$, $b > 0$.

- To ensure $b + cx(t) > 0$, we bound the state as $\|x(t)\| \leq d$.
  - Also leads to the bound $b + cx(t) \leq \tau_m$.

Note: The extension to $\mathbb{R}^n$ is not hard.

Lyapunov-Krasovskii Functionals

For linear fixed delay systems:

$$\dot{x}(t) = Ax(t) + Bx(t-\tau)$$

Theorem 1.

The discrete-delay system is stable if and only if there exist continuous functions M and N such that $V(\phi) \geq \alpha\|\phi\|$ and $\dot{V}(\phi) \leq 0$ where

$$V(\phi) = \int_{-\tau}^{0} \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix}^T M(s) \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix} ds + \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s)^T N(s,\theta)\phi(\theta) ds d\theta$$

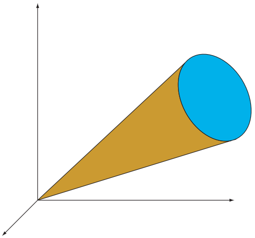
Note: The functional is parameterized by unknown functions M and M.
Problem: How to numerically compute M and N such that

$$V(\phi) > \alpha\|\phi\|$$
$$\dot{V}(\phi) < 0$$

Answer: Convex Optimization

Problem:

$$\max \ bx$$
$$\text{subject to} \quad Ax \in C$$



The problem is *convex optimization* if

- C is a convex cone.
- b and A are affine.

Computational Tractability: Convex Optimization over C is, in general, tractable if

- The set membership test for $y \in C$ is in P.
- x is finite dimensional.

Parametrization of Lyapunov-Krasovskii Functionals

Problem 1: Is the set of decision variables finite-dimensional?

- Decision variables are the functions M and N

$$V(\phi) = \int_{-\tau\left(\phi(t)\right)}^{0} Z\left(\begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix}\right)^T M(s) Z\left(\begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix}\right) ds + \int_{-\tau\left(\phi(t)\right)}^{0}\int_{-\tau\left(\phi(t)\right)}^{0} Z\Big(\phi(s)\Big)^T N(s,\theta) Z\Big(\phi(\theta)\Big) ds d\theta$$

Solution: Suppose M and N are polynomials of bounded degree.

$$M(s) = {c_1}^T Z(s), \qquad N(s,\theta) = {c_2}^T Z(s,\theta)$$

Problem 2: How to enforce positivity of the L-K functional?

- Need constraints on c_1 and c_2.

Positivity Constraints for Polynomials

Sum-of-Squares: $p(x) \geq 0$ if

$$p(x) = \sum_i g_i(x)^2, \qquad \text{denoted } p \in \Sigma_s$$

Lemma 2 (Parametrization of Sums-of-Squares).

Given multivariate polynomial p of degree $2d$, $p \in \Sigma_s$ ($p(x) \geq 0$ for all $x \in \mathbb{R}^n$) if and only if there exists a positive matrix $M \in \mathbb{S}^q$ such that

$$p(x) = Z_d(x)^T M Z_d(x)$$

where z_d is the vector of monomials of degree d or less.

Lemma 3 (Polynomial Positivity on a Subset of $\mathbb{R}^n$).

Given polynomial p, $p(x) \geq 0$ for all $x \in \{x : g_i(x) \geq 0\}$ if there exist $s_i \in \Sigma_s$ such that

$$p(s) = s_0(x) + \sum_i g_i(x) s_i(x)$$

Positivity Constraints for Lyapunov Functionals

Lemma 4.

Suppose there exist $S \in \Sigma_s$ *and polynomial* T *such that*

$$V(\phi(0), \phi(s), s) - \phi(0)^2 = S(\phi(0), \phi(s), s) + T(\phi(0), s)$$

with $\int_{-\tau(x(t))}^{0} T(\phi(0), s) ds = 0$*. Then*

$$\int_{-\tau\left(\phi(t)\right)}^{0} V(\phi(0), \phi(s), s) ds \geq \alpha \phi(0)^2$$

for any $\phi \in \mathcal{C}[-\tau_m, 0]$

We can tighten this a bit

- Restrict $s \in [-\tau(\phi(s)), 0]$

$$\{s, \phi \,:\, g_1(s, \phi(s)) = -s(s + \tau(\phi(s))) \geq 0\}$$

- Restrict $\|\phi(s)\| \leq d$

$$\{s, \phi \,:\, g_2(s, \phi(s)) = d - \phi(s)^2 \geq 0\}$$

Positivity Constraints for Lyapunov Functionals

Lemma 5.

Suppose there exist $S, S_1, S_2, S_3 \in \Sigma_s$ *and polynomial* T *such that*

$$\begin{aligned} &V(\phi(0), \phi(s), s) - \phi(0)^2 \\ &= S + g_1(s, \phi(s)) S_1 \\ &\qquad + g_2(s, \phi(s)) S_2 + g_2(s, \phi(0)) S_3 + T(\phi(0), s) \end{aligned}$$

with $\int_{-\tau(x(t))}^{0} T(\phi(0), s) ds = 0$*. Then*

$$\int_{-\tau\left(\phi(t)\right)}^{0} V(\phi(0), \phi(s), s) ds \geq \alpha \phi(0)^2$$

for any $\|\phi\|_\infty \leq d$*.*

Where Recall

- S, S_1, S_2, S_3, T and V are the decision variables
  - Represented using positive matrices.
- Constraints are equalities and matrix positivity (SDP).

Positivity Constraints for Lyapunov Functionals

Lemma 6.

Suppose there exists a positive matrix $Q \geq 0$ *such that*

$$V_2(\phi(s), \phi(\theta), s, \theta) = Z(s, \phi(s))^T Q Z(\theta, \phi(\theta))$$

Then

$$\int_{-\tau\left(\phi(t)\right)}^{0} \int_{-\tau\left(\phi(t)\right)}^{0} V_2(\phi(s), \phi(\theta), s, \theta)\, ds\, d\theta \geq 0$$

for any $\phi \in \mathcal{C}[-\tau_m, 0]$.

Positivity Constraints

- Constraints are equalities and matrix positivity (SDP).

Convex Optimization

- Positive Lyapunov Functionals are represented using vectors and matrix positivity constraints.

The Derivative

Now Recall the dynamics:

$$\dot{x}(t) = ax(t - \tau_0 + bx(t)) \qquad\qquad \tau(x(t)) = \tau_0 + bx(t)$$

With Lyapunov Functional

$$V(t) = \int_{-\tau\left(x(t)\right)}^{0} V_1\Big(x(0), x(s), s\Big)ds + \int_{-\tau\left(x(t)\right)}^{0} \int_{-\tau\left(x(t)\right)}^{0} V_2\Big(x(s), s, x(\theta), \theta\Big)\, ds\, d\theta$$

The derivative is

$$\dot{V}(t) = \int_{-\tau\left(x(t)\right)}^{0} V_3\Big(x(t), x(t - \tau(x(t))), x(t+s), s\Big)ds + \int_{-\tau\left(x(t)\right)}^{0} \int_{-\tau\left(x(t)\right)}^{0} V_4\Big(x(t+s), x(t+\theta), s, \theta\Big)\, ds\, d\theta$$

where

$$V_4(x_\theta, x_\xi, \theta, \xi) = \frac{\partial}{\partial\theta} V_2(x_\theta, x_\xi, \theta, \xi) + \frac{\partial}{\partial\xi} V_2(x_\theta, x_\xi, \theta, \xi). \qquad (1)$$

The Derivative

$$\dot{x}(t) = ax(t - \tau_0 + bx(t)) \qquad\qquad \tau(x(t)) = \tau_0 + bx(t)$$

$$\begin{aligned}\dot{V}(t) = &\int_{-\tau\left(x(t)\right)}^{0} V_3\Big(x(t), x(t-\tau(x(t))), x(t+s), s\Big) ds \\ &+ \int_{-\tau\left(x(t)\right)}^{0}\int_{-\tau\left(x(t)\right)}^{0} V_4\Big(x(t+s), x(t+\theta), s, \theta\Big)\, ds\, d\theta\end{aligned}$$

$$\begin{aligned}V_3(x_t, x_\tau, x_s, s) =& \frac{1}{\tau(x_t)}(abx_\tau - 1)V_1(x_t, x_\tau, -\tau(x_\tau)) + \frac{1}{\tau(x_t)}V_1(x_t, x_t, 0) \\ &+ ax_\tau \frac{\partial}{\partial x}V_1(x_t, x_s, s) - \frac{\partial}{\partial \theta}V_1(x_t, x_s, \theta) \\ &+ (2abx_\tau - 2)V_2(x_\tau, x_s, -\tau(x_t), \theta) + 2V_2(x_t, x_s, 0, \theta)\end{aligned}$$

Stability Test

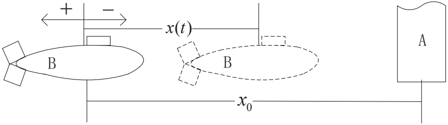
Suppose there exist $S_i \in \Sigma_s$ for $i = 1, 2, 3$, $L_i \in \Sigma_s$ for $i = 1, 2, 3, 4$, some $\epsilon_j > 0$ for $j = 1, 2$, polynomial $R_1(x_0, \theta)$, $R_2(x_0, x_1, \theta)$ and matrices $M, N > 0$, such that

1) $V_1(x_0, x_2, \theta) + R_1(x_0, \theta) - \epsilon_1 x_0^2 - \sum_{i=1}^{3} S_i(x_0, x_2, \theta) g_i(x_0, x_2, \theta) \in \Sigma_s$,
2) $-V_3(x_0, x_1, x_2, \theta) + R_2(x_0, x_1, \theta) - \epsilon_2 x_0^2 - \sum_{i=1}^{4} L_i(x_0, x_1, x_2, \theta) g_i(x_0, x_1, x_2, \theta) \in \Sigma_s$,
3) $V_2(x_2, x_3, \theta, \xi) = Z_d^T(x_2, \theta) M Z_d(x_3, \xi)$,
4) $V_4(x_2, x_3, \theta, \xi) = Z_d^T(x_2, \theta) N Z_d(x_3, \xi)$,
5) $\int_{-\tau(x_0)}^{0} R_1(x_0, \theta) d\theta = 0$,
6) $\int_{-\tau(x_0)}^{0} R_2(x_0, x_1, \theta) d\theta = 0$,

Then the system is asymptotically stable for all $x_t \in \Omega$, where Ω is defined as

$$\Omega := \{x_t \in \mathbb{C} : \|x_t\| \leq \tau_0/(2b)\}.$$

Numerical Validation

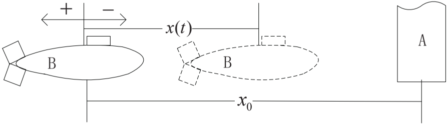


$$\dot{x}(t) = ax(t - \tau_0 + bx(t))$$

	a=−0.1	a=−0.5	a=−1
τ_0=0.1	$b \in [4e\text{-}4,\ 1]$	$b \in [1e\text{-}4,\ 1]$	$b \in [2e\text{-}4,\ 1]$
τ_0=0.5	$b \in [6e\text{-}4,\ 2]$	$b \in [3e\text{-}4,\ 2]$	$b \in [3e\text{-}3,\ 0.02]$
τ_0=1	$b \in [7e\text{-}4,\ 3]$	$b \in [8e\text{-}4,\ 2]$	$b \in [3e\text{-}3,\ 0.02]$

Table: The minimum and maximum stable values of b for a fixed a and τ_0.

Numerical Validation



$$\dot{x}(t) = ax(t - \tau_0 + bx(t))$$

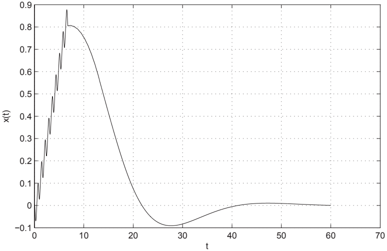


Figure: Simulation Results using $a = -0.1, b = 1, \tau_0 = 6$, and initial condition $\phi(\theta) = 0.5 \sin(\theta)$

Conclusions:

A Difficult Problem:

- Lyapunov Stability Test
  - Convexifies the problem
  - Relies on SDP.
  - Complexity depends on Accuracy.
- Practical Implications
  - The effect is small, but finite.

Numerical Code Produced:

- Not currently posted
- Must generalize to multidimensional systems
- Future Work
  - Joint Positivity
  - Controller Synthesis

Will be Available for download at
`http://control.asu.edu`