# Full-State Feedback of Delayed Systems using SOS: A New Theory of Duality

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# Control of Linear Systems with Delays

Consider an autonomous linear Discrete-Time system.

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{K} A_i x(t - \tau_i) + Bu(t) \qquad \text{for all } \; t \geq 0$$

Stability Analysis of linear discrete-delay systems is a CLOSED PROBLEM.

- Lets move on to optimal control.
- Analysis of PDEs and other DPS is still open.

We would like to use LMI and SOS methods to design controllers for this system.

- LMI methods optimize positive matrices
- SOS methods optimize positive polynomials

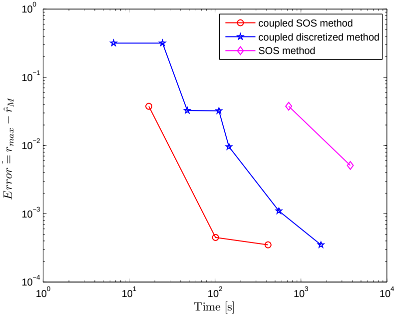



Figure: Comparison of asymptotic algorithms for maximum stable delay

# Differential Form of Delay System

A linear time-delay system can be represented without delay as the interconnection of an ODE and a simple transport PDE with point actuation and point observation.

**ODE:** The system $G_1$

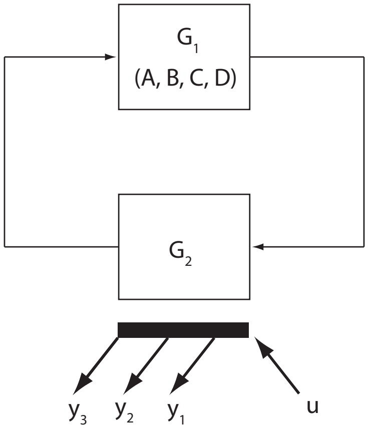
$$\begin{aligned}\dot{x}_1(t) &= Ax_1(t) + Bu_1(t)\\ y_1(t) &= Cx_1(t) + Du_1(t)\end{aligned}$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} A_0 & \begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix} \\ \hline I & 0 \end{array}\right]$$

**PDE:** The system $G_2$

$$\frac{\partial}{\partial t} x_2(t,s) = \frac{\partial}{\partial s} x_2(t,s) \quad x_2(t,0) = u_2(t),$$

$$y_2(t) = \begin{bmatrix} x_2(-\tau_1) \\ \vdots \\ x_2(-\tau_K) \end{bmatrix}$$




Of course, the solution is just $x_2(t, s) = u_2(t - s)$.

# Operator Representation

Solving for $u$ and $y$, we get the differential operator

$$\dot{x}(t) = \mathcal{A}x(t)$$

where

$$(\mathcal{A}x)(s) = \begin{bmatrix} A_0 x_1 + \sum_{i=1}^K A_i x_2(-\tau_i) \\ \frac{d}{ds} x_2(s) \end{bmatrix}.$$

and where the combined state is $x \in X := \mathbb{R}^n \times L_2$ with inner product $\langle u, v \rangle = \langle u_1, y_1 \rangle_{\mathbb{R}^n} + \langle u_2, y_2 \rangle_{L_2}$.

Let $\mathcal{A}$ be the infinitesimal generator of a $\mathcal{C}_0$ semigroup $T(t) : L_2 \to L_2$ on Hilbert space $X$ with domain $\mathcal{D}(A) := \{x \in \mathbb{R}^n \times W^2\,,\, x_1 = x_2(0)\}$.

- $\mathcal{D}(A)$ defines properties of the solution.

**Theorem 1 (e.g. Curtain and Zwart).**

*$T(t) : X \to X$ is exponentially stable if and only if there exists a positive linear operator, $P$, such that*

$$\langle Az, Pz \rangle_X + \langle Pz, Az \rangle_X = -\langle z, z \rangle_X \qquad \text{for all } z \in \mathcal{D}(A)$$

# Controller Synthesis

Now suppose we add an input to the time-delay system

$$\dot{x}_1(t) = A_0 x_1(t) + \sum_{i=1}^{K} A_i x_2(t, -\tau_i) + Bu(t)$$

In differential form, this is

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t)$$

where $\mathcal{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$ and $u(t) \in \mathbb{R}^m$.

**Static State-Feedback** $u(t) = Kx(t)$.

- $K$ can be any operator $K : \mathbb{R}^n \times L_2 \to \mathbb{R}^m$.
- Here recall the state of a TDS is $X = \mathbb{R}^n \times L_2$.

This approach is in contrast to output feedback of the form

$$u(t) = Kx(t) \qquad \text{or} \qquad u(t) = Kx(t - \tau).$$

We will return to this subject later.

# An Intractable Controller Synthesis Condition

**Lemma 2.**

*Suppose that $\dot{x} = \mathcal{A}x$ generates a strongly continuous semigroup on $L_2$ with domain $\mathcal{D}(A)$ and $B : U \to \mathcal{D}(A)$. Further suppose there exists a positive operator $P : L_2 \to L_2$ which is self-adjoint with respect to the $L_2$ inner product and an operator $K : \mathcal{D}(A) \to U$ such that*

$$\langle (P\mathcal{A} + PBK)\, x, x\rangle + \langle x, (P\mathcal{A} + PBK)\, x\rangle \leq -\langle x, x\rangle$$

*for all $x \in X$. Then $\dot{x}(t) = (\mathcal{A} + BK)\, x$ generates an exponentially stable semigroup.*

The theorem requires the existence of two variables

- The Lyapunov operator, $P$
- The Controller, $K$

The constraints have a bilinear term $PBK$, making the conditions difficult to verify using current algorithms.

We need the dual formulation:

$$AP + PA^* + BKP + PK^*B^* < 0$$

# Dual Stability Condition

**Theorem 3.**

*Suppose that $\mathcal{A}$ generates a strongly continuous semigroup on $L_2$ with domain $\mathcal{D}(A)$. Further suppose there exists a positive operator $P : \mathcal{D}(A) \to \mathcal{D}(A)$ which is self-adjoint with respect to $L_2$ and*

$$\langle \mathcal{A}Px, x\rangle + \langle x, \mathcal{A}Px\rangle \le -\langle x, x\rangle$$

*for all $x \in \mathcal{D}(A)$. Then the dynamical system $\dot{x}(t) = \mathcal{A}x$ generates an exponentially stable semigroup.*

The key constraints on $P$ are

- **Self-Adjoint** - $\langle Px, y\rangle = \langle x, Py\rangle$ for any $x, y \in L_2$.
- **Positive** - $\langle x, Px\rangle > 0$ for any $x \neq 0$.
- **Invertible** - We need to be able to *find* the inverse.
- *****Preserves the Space***** - must map $\mathcal{D}(A) \to \mathcal{D}(A)$.
  - This is harder than the Curtain+Zwart primal condition.

# Proof Outline

Consider an operator $P > 0$, with

$$\langle APz, z\rangle + \langle z, PAz\rangle = -\langle z, z\rangle \qquad \text{for all } z \in \mathcal{D}(A)$$

Since $P : \mathcal{D}(A) \to \mathcal{D}(A)$ is a positive operator, it has a positive inverse $P^{-1} : \mathcal{D}(A) \to \mathcal{D}(A)$.

Thus for any $y \in \mathcal{D}(A)$, let $x = P^{-1}y \in \mathcal{D}(A)$. Then $y = Px$ and

$$\begin{aligned}
&\langle \mathcal{A}y, P^{-1}y\rangle + \langle P^{-1}y, \mathcal{A}y\rangle \\
&= \langle \mathcal{A}Px, x\rangle + \langle x, \mathcal{A}Px\rangle \\
&\le -\langle x, x\rangle = -\langle P^{-1}y, P^{-1}y\rangle \\
&\le -\alpha\langle y, y\rangle
\end{aligned}$$

# A Parametrization of Operators

**Problem:** What is the structure of the operator $P$?
**PDE and Delay Systems:** Many transport and diffusion systems are stable iff there exists some $P > 0$ with $\mathcal{A}^* P + P\mathcal{A} < 0$ where

$$(Px)(s) = M(s)x(s) + \int_{-\tau_K}^{0} N(s,\theta)x(\theta)d\theta.$$

This $P$ defines the "complete-quadratic functional"

$$V(x) = \langle Px, x \rangle = \int_{-\tau_K}^{0} x(s)^T M(s)x(s) + \int_{-\tau_K}^{0}\int_{-\tau_K}^{0} x(s)^T N(s,\theta)x(\theta)dsd\theta$$

Unfortunately,

- This operator does not map $\mathcal{D}(A) \to \mathcal{D}(A)$.

# A Class of Structure-Preserving Operators

In order to ensure that $P : \mathcal{D}(A) \to \mathcal{D}(A)$, and $P = P^*$, we suppose that $P$ has the form

$$(Px)(s) := \begin{bmatrix} (\bar{P}x)(0) \\ (\bar{P}x)(s) \end{bmatrix} = \begin{bmatrix} (\tau Q_2(0,0) + Q_1(0))x_1 + \int_{-\tau}^0 Q_2(0,s)x_2(s)ds \\ \tau Q_2(s,0)x_2(0) + Q_1(s)x_2(s) + \int_{-\tau}^0 Q_2(s,\theta)x_2(\theta)d\theta \end{bmatrix}$$

Where

$$(\bar{P}x)(x) = \tau Q_2(s,0)x_2(0) + Q_1(s)x_2(s) + \int_{-\tau}^0 Q_2(s,\theta)x_2(\theta)d\theta$$

for some continuous functions $Q_1$ and $Q_2$.

### Lemma 4.

*Suppose that $Q_2(s,\theta) = Q_2(\theta,s)^T$ and $Q_1(s) \in \mathbb{S}^n$. Then $P$ maps $\mathcal{D}(A) \to \mathcal{D}(A)$ and is self-adjoint with respect to $L_2$.*

**Note:** $(P^{-1}x)(s) = \begin{bmatrix} (\bar{P}^{-1}x)(0) \\ (\bar{P}^{-1}x)(s) \end{bmatrix}$.

# Previous Work

## SDP and sum-of-Squares Conditions

In previous work we gave conditions on $M$ and $N$.

### Theorem 5.

*Let $M$ be piecewise-continuous, then following are equivalent*

1. $\int_{-h}^{0} \begin{bmatrix} x(0) \\ x(s) \end{bmatrix}^T M(s) \begin{bmatrix} x(0) \\ x(s) \end{bmatrix} ds \ge \epsilon \|x\|^2$
2. $\int_{-h}^{0} T(s) ds = 0$ *and* $M(s) + \begin{bmatrix} T(s) & 0 \\ 0 & 0 \end{bmatrix} \succeq \epsilon' I$ *for some* $T$

### Theorem 6 (Denoted $N \in \Sigma_k$).

*Suppose $N(s,t)$ is a polynomial of degree $2d$ and $Z_d$ is a polynomial basis of degree $d$. The following are equivalent:*

- $\int_{-h}^{0} \int_{-h}^{0} x(s)^T N(s,t) x(t) ds dt \ge 0$ *for all* $x \in \mathcal{C}$
- *There exists a* $Q \ge 0$ *such that* $N(s,t) + N(t,s)^T = Z_d(s)^T Q Z_d(t)$

# A Dual LMI for stability via SOS

**Theorem 7.**

*Suppose there exist polynomials $Q_1, Q_2, T$ such that the following hold*

$$\begin{bmatrix} \tau Q_2(0,0) + Q_1(0) & \tau Q_2(0,s) \\ \tau Q_2(s,0) & Q_1(s) \end{bmatrix} + \begin{bmatrix} T(s) & 0 \\ 0 & 0 \end{bmatrix} - \epsilon I \in \Sigma_s,$$

$$- \begin{bmatrix} S_{11} + S_{11}^T & *^T & *^T \\ S_{12}^T & S_{22} & *^T \\ S_{13}(s)^T & 0 & \dot{Q}_1(s) \end{bmatrix} + \begin{bmatrix} U_{11}(s) & U_{21}(s)^T & 0^T \\ U_{21}(s) & U_{22}(s) & 0^T \\ 0 & 0 & 0 \end{bmatrix} - \epsilon I \in \Sigma_s,$$

$$S_{11} = A_0(\tau Q_2(0,0) + Q_1(0)) + \tau A_1 Q_2(-\tau, 0) + \frac{1}{2\tau} Q_1(0), \quad S_{12} = A_1 Q_1(-\tau),$$

$$S_{22} = -\frac{1}{\tau} Q_1(-\tau), \quad S_{13}(s) = \tau A_0 Q_2(0,s) + \tau A_1 Q_2(-\tau, s) + \tau \dot{Q}_2(s,0)^T,$$

$$\int_{-\tau}^{0} \begin{bmatrix} U_{11}(s) & *^T \\ U_{21}(s) & U_{22}(s) \end{bmatrix} ds = 0, \qquad \int_{-\tau}^{0} T(s) ds = 0,$$

$$\frac{d}{ds} Q_2(s,\theta) + \frac{d}{d\theta} Q_2(s,\theta) \in \Sigma_k, \qquad Q_2(s,\theta) \in \Sigma_k.$$

*Then the system defined by $\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau)$ is exponentially stable.*

# Accuracy of Dual Stability Condition

Consider the simple delayed system

$$\dot{x}(t) = -x(t-\tau)$$

which is known to be stable for $\tau \in [0, \frac{\pi}{2}]$.

- The dual stability condition is only able to prove stability for $\tau \in [0, .7]$.
  - Required polynomials of degree $8$
- Primal condition using SOS yields $\tau = \frac{\pi}{2}$ to 6 decimal places.
- Other system work better, but a gap remains.
- A result of structure imposed on the operator.

However, controllers based on this operator are not so conservative.

# Controllers

Again, we would like to add a controller

$$\dot{x}_1(t) = A_0 x_1(t) + \sum_{i=1}^{K} A_i x_2(t, -\tau_i) + Bu(t)$$

**Static State-Feedback** $u(t) = Kx$.

- Recall the state of a TDS is $X = \mathbb{R}^n \times L_2$.
- Hence we can expect $K : \mathbb{R}^n \times L_2 \to \mathbb{R}^m$.

# An LOI for Full-State Feedback Synthesis

**Corollary 8 (Full-State Feedback).**

*Suppose that $\dot{x} = \mathcal{A}x$ generates a strongly continuous semigroup on $L_2$ with domain $\mathcal{D}(A)$ and $B : U \to \mathcal{D}(A)$. Further suppose there exists a positive operator $P : \mathcal{D}(A) \to \mathcal{D}(A)$ which is self-adjoint with respect to the $L_2$ inner product and an operator $Z : \mathcal{D}(A) \to U$ such that*

$$\langle (\mathcal{A}P + BZ)\, x, x\rangle + \langle x, (\mathcal{A}P + BZ)\, x\rangle \leq -\langle x, x\rangle$$

*for all $x \in X$. Let $K = ZP^{-1}$. Then the dynamical system $\dot{x}(t) = (\mathcal{A} + BK)\, x$ generates an exponentially stable semigroup.*

# Full-State Feedback Controllers

Recall the question of controller

$$\dot{x}_1(t) = A_0 x_1(t) + \sum_{i=1}^{K} A_i x_2(t, -\tau_i) + Bu(t)$$

**Static State-Feedback:** $u(t) = Kx$.

- $K$ is recovered as $K = ZP^{-1}$.
- Hence structure of $K : \mathbb{R}^n \times L_2 \to$ is inherited from $P^{-1}$ and $Z$.
- Let

$$(Zx)(s) = Z_0 x_1 + Z_1 x_2(-\tau) + \int_{-\tau}^{0} Z_2(s) x_2(s) ds$$

But what is the structure of $P^{-1}$???

**Previously (TDS 2009):** An explicit inverse of the positive operator $P > 0$.

## Theorem 9.

*Consider the linear operator $P$ defined by*

$$Px(s) = M(s)x(s) + \int_I N(s,\theta)x(\theta)d\theta,$$

*where $M(s) > 0$ for all $s \in I$ and $N$ has a representation $N(s,\theta) = Z(s)^T R Z(\theta)$ where $R > 0$. Define the linear operator $\hat{P}$ by*

$$\hat{P}x(s) = M(s)^{-1}x(s) + \int_I \hat{N}(s,\theta)x(\theta)d\theta$$

*Where*

$$\hat{N}(s,\theta) = M(s)^{-1}Z(s)^T Q Z(\theta) M(\theta)^{-1} \qquad Q = -R(S^{-1} + R)^{-1}S^{-1}$$

$$S = \int_I Z(s)M(s)^{-1}Z(s)^T ds.$$

*Then $\hat{P}Px = P\hat{P}x = x$ for any integrable function $x$.*

However, in this paper, our operator has an extra term:

$$(Px)(s) = \tau Q_2(s,0)x(0) + Q_1(s)x(s) + \int_{-\tau}^0 Q_2(s,\theta)x(\theta)d\theta$$

## Theorem 10 (Expanded Operator Inversion Formula).

*Consider*

$$(Px)(s) := L(s)x(0) + M(s)x(s) + \int_I N(s,\theta)x(\theta)\,d\theta$$

*where* $M(s) > 0$ *for all* $s \in I$ *and* $N$ *has a representation* $N(s,\theta) = Z(s)^T T\, Z(\theta)$ *where* $T > 0$. *Then*

$$(P^{-1}x)(s) := Y_0(s)x(0) + Y_1(s)x(s) + \int_I Y_2(s,\theta)x(\theta)\,d\theta.$$

*where*

$$Y_0(s) = -H(s)(I+J)^{-1}M^{-1}(0), \quad Y_2(s,\theta) = R(s,\theta) - H(s)(I+J)^{-1}R(0,\theta),$$

$$Y_1(s) = M^{-1}(s), \qquad H(s) = M^{-1}(s)L(s) + \int_I R(s,\theta)L(\theta)\,d\theta,$$

$$R(s,\theta) = M(s)^{-1}Z(s)^T Q Z(\theta)M(\theta)^{-1}, \qquad Q = -T(S^{-1}+T)^{-1}S^{-1},$$

$$S = \int_I Z(s)M(s)^{-1}Z(s)^T ds, \qquad J := Q(0)K(0) + \int_I R(0,s)K(s)\,ds.$$

- *This formula requires* $\rho(J) < 1$.
- *This formula can be implemented in Matlab/Maple/***Mathematica**.

# A Full-State Feedback Controller

Now we know what the controller $K = ZP^{-1}$ will look like!

**Lyapunov Operator:**

$$(\bar{P}^{-1}x)(s) = Y_0(s)x_1 + Y_1(s)x_2(s) + \int_{-\tau}^{0} Y_2(s,\theta)x(\theta)d\theta$$

**Pseudo-Variable,** $Z$**:**

$$(Zx)(s) = Z_0 x_1 + Z_1 x_2(-\tau) + \int_{-\tau}^{0} Z_2(s)x_2(s)ds$$

Then the controller $u(t)$ has the form:

$$u(t) = Kx = K_0 x_1(t) + K_1 x_2(t-\tau) + \int_{-\tau}^{0} K_2(s)x_2(t+s)ds$$

where

$$K_0 = Z_0 Y_0(0) + Z_1 Y_0(-\tau) + \int_{-\tau}^{0} Z_2(s)Y_0(s)ds + Z_0 Y_1(0)$$

$$K_1 = Z_1 Y_1(-\tau)$$

$$K_2(s) = Z_0 Y_2(0,s) + Z_1 Y_2(-\tau,s) + Z_2(s)Y_1(s) + \int_{-\tau}^{0} Z_2(\theta)Y_2(\theta,s)d\theta.$$

Contrast with output feedback forms $u(t) = Kx(t)$ or $u(t) = Kx(t-\tau)$.

## Theorem 11.

*Suppose there exist matrices $Z_0$, $Z_1$ and polynomials $Q_1, Q_2, Z_2, U, T$ such that the following hold*

$$\begin{bmatrix}\tau Q_2(0,0)+Q_1(0)+T(s) & \tau Q_2(0,s)\\ \tau Q_2(s,0) & Q_1(s)\end{bmatrix}-\epsilon I\in\Sigma_s,$$

$$-\begin{bmatrix}S_{11}+S_{11}^T+L_{11}+L_{11}^T & *^T & *^T\\ S_{21}+L_{12}^T & S_{22} & *^T\\ S_{31}(s)+L_{13}(s)^T & 0 & \dot{Q}_1(s)\end{bmatrix}+\begin{bmatrix}U_{11}(s) & U_{21}(s)^T & 0\\ U_{21}(s) & U_{22}(s) & 0\\ 0 & 0 & 0\end{bmatrix}-\epsilon I\in\Sigma_s,$$

$$S_{11}=A_0(\tau Q_2(0,0)+Q_1(0))+\tau A_1 Q_2(-\tau,0)+\frac{1}{2\tau}Q_1(0),\qquad S_{21}=Q_1(-\tau)^T A_1^T,$$

$$S_{22}=-\frac{1}{\tau}Q_1(-\tau),\qquad S_{31}(s)=\tau Q_2(0,s)^T A_0^T+\tau Q_2(-\tau,s)^T A_1^T+\tau\dot{Q}_2(s,0),$$

$$L_{11}=B_0Z_0,\qquad L_{12}=B_0Z_1,\qquad L_{13}=\tau B_0 Z_2(s)$$

$$\int_{-\tau}^0\begin{bmatrix}U_{11}(s) & *^T\\ U_{21}(s) & U_{22}(s)\end{bmatrix}ds=0,\qquad \int_{-\tau}^0 T(s)ds=0,$$

$$\frac{d}{ds}Q_2(s,\theta)+\frac{d}{d\theta}Q_2(s,\theta)\in\Sigma_k,\qquad Q_2(s,\theta)\in\Sigma_k.$$

*Then $\dot{x}(t)=A_0x(t)+A_1x(t-\tau)+B_0u(t)$ is full-state feedback stabilizable.*

# Full-state Feedback Controller: Numerical Example

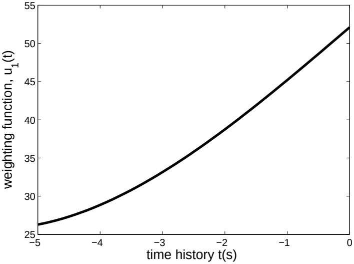
Consider a numerical example.

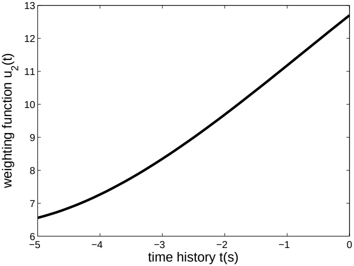
$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -2 & -.5 \\ 0 & -1 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Using a value of $\tau = 5s$, we compute the following controller:

$$u(t) = \begin{bmatrix} -3601 \\ -944 \end{bmatrix}^T x(t) + \begin{bmatrix} -.00891 \\ .872 \end{bmatrix}^T x(t-\tau)$$

$$+ \int_{-5}^{0} \begin{bmatrix} 52.1 + 6.98s + .00839s^2 - .0710s^3 \\ 12.7 + 1.50s - .0407s^2 - .0190s^3 \end{bmatrix}^T x(t+s)ds$$



# Numerical Example

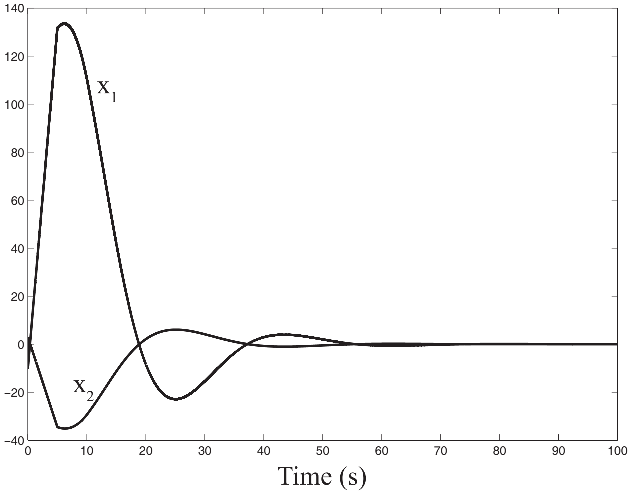



Figure: Trajectory of a delayed system ($\tau = 5s$) with full-state feedback

# Conclusions:

- A Dual approach to controller synthesis
  - Convexifies the problem
  - Can be applied to any Lyapunov-Krasovskii-based approach.
  - NOT limited to SOS.
  - Biggest technical hurdle is operator inversion.
- Practical Implications
  - First numerical solution to **Full-State Feedback** of multi-state delayed systems.

# Numerical Code Produced:

- Operator Inversion Code
  - Code in Mathematica
  - Complicated multi-state systems require polynomial approximation before inversion of $M(s) > 0$ as per Theorem 9.

Available for download at
`http://control.asu.edu`

- Controller Synthesis Code
  - Uses DelayTools toolbox of functions
  - Coupled with Mathematica Code (Must be run separately)
- Simulation Code
  - Approximates distributed delay with 10 discrete delays
  - Very slow for large delays