# LMI parametrization of Lyapunov Functions for Infinite-Dimensional Systems: A Framework

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# Linear Ordinary Differential Equations

**Consider:** A System of Linear Ordinary Differential Equations

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad\qquad x \in \mathbb{R}^n$$
$$y(t) = Cx(t) + Du(t)$$

**Questions: Stability and Control**

1. Stability: If $u = 0$, do all solutions satisfy $\lim_{t\to\infty} x(t) = 0$
2. Control: Find $K$ so if $u(t) = Kx(t)$, all solutions satisfy $\lim_{t\to\infty} x(t) = 0$
3. Observation: Find map $x, y, u \to \hat{x}$ and $K$ so if $u(t) = K\hat{x}(t)$, all solutions satisfy $\lim_{t\to\infty} x(t) = 0$

# Linear Matrix Inequalities

**Key:** System Performance is captured by quadratic Lyapunov Functions.

The key is that **ANY** quadratic Lyapunov Function can be represented as

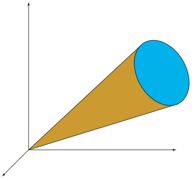
$$V(x) = x^T P x$$

**The SDP Formulation:**

$$\max_{x \in \mathbb{R}^{n \times n}} \ \text{trace}(JX)$$

$$\text{subject to} \quad H_i X G_i + G_i^T X H_i^T \succeq 0$$

where $X \succeq 0$ means $X$ is a positive semidefinite matrix.



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- Stability Use $P \succ 0$, $A^T P + PA \preceq 0$ to show $V(x) = x^T P x$ is a LF.
- $H_\infty$ state feedback Use $Y \succ 0$ and

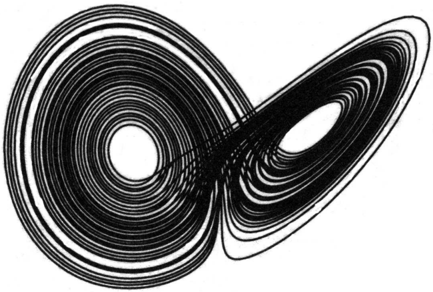
$$\begin{bmatrix} YA^T + AY + Z^T B_2^T + B_2 Z & B_1 & YC_1^T + Z^T D_{12}^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} \preceq 0$$

to show $V(x) = x^T Y x$ is a LF which proves $K = ZY^{-1}$ implies $\sup_w \frac{\|y\|}{\|w\|} \leq \gamma$.

# Nonlinear Ordinary Differential Equations

**Consider:** A System of Nonlinear Ordinary Differential Equations

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = g(x(t))$$



**Again with the Questions:** **Things we can do with LFs**

1. Stability: If $u = 0$, do all solutions satisfy $\lim_{t\to\infty} x(t) = 0$ (or something...)
2. Gain: Find $\gamma$ so $\frac{\|y\|_{L_2}}{\|u\|_{L_2}} \leq \gamma$.

The key is that **ANY** Lyapunov Function can be represented as

$$V(x) = Z(x)^T P Z(x)$$

where $P > 0$ and $Z : \mathbb{R}^m \to \mathbb{R}^n$ is a nonlinear map. ($Z$ can just be a vector of monomials)

**Q:** Really?

- Yes [Peet, 2008][Peet and Papachristodoulou, 2012](For exp. stability of $\dot{x} = f(x)$ on X where f is Lip.)

**Q:** Why?

- If $P > 0$, then $P = P^{\frac{1}{2}} P^{\frac{1}{2}}$, so
  $V(x) = Z(x)^T P Z(x) = (Z(x) P^{\frac{1}{2}})^T (P^{\frac{1}{2}} Z(x)) = q(x)^T q(x) \geq 0$.

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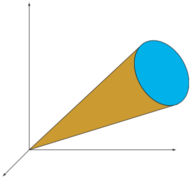
**The SOS Formulation:**

$$\max_{c \in \mathbb{R}^n} \; b^T c$$

subject to $\quad h(x) := c^T g(x) + d(x) \geq 0 \qquad$ for all $x \in \mathbb{R}^n$

where "$\geq 0$ for all $x \in \mathbb{R}^n$"
really means "$= Z(x)^T P Z(x)$ with $P \succeq 0$".



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# Lyapunov Functions and Positive Matrices

**Positive Matrices parameterize LFs with a square root:**

Linear ODE's: Quadratic LFs are N+S.

$$V(x) = \langle x, Px\rangle_{\mathbb{R}^n} = \langle P^{\frac{1}{2}}x, P^{\frac{1}{2}}x\rangle_{\mathbb{R}^n} = \langle q_l(x), q_l(x)\rangle_{\mathbb{R}^n}$$

Nonlinear ODE's: LFs are N+S.

$$V(x) = \langle Z(x), PZ(x)\rangle_{\mathbb{R}^{\text{big}}} = \langle P^{\frac{1}{2}}Z(x), P^{\frac{1}{2}}Z(x)\rangle_{\mathbb{R}^{\text{big}}} = \langle q_{nl}(x), q_{nl}(x)\rangle_{\mathbb{R}^{\text{big}}}$$

both LFs just map to Euclidean norms. The **Difference** is

- in $q_l(x) = P^{\frac{1}{2}}x$ the map is linear
- $q_{nl}(x) = P^{\frac{1}{2}}Z(x)$ is nonlinear

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**Q:** What about L-K Functions?

$$V(x) = \langle x, \mathcal{P}x\rangle_{L_2} = \int_{-\tau}^{0} x(s)^T M(s)x(s)ds + \int_{-\tau}^{0}\int_{-\tau}^{0} x(s)^T N(s,\theta)x(\theta)\, ds\, d\theta$$

Choose the right basis $Z : X \to L_2$ and $P \geq 0$ will define the map...

**SemiGroup Concept:** Let $\mathcal{A}$: $D(\mathcal{A}) \to X$ be an **OPERATOR** and

$$\dot{x}(t) = \mathcal{A}x(t).$$

The system is a *strongly continuous semigroup* (SCS) on $X$ with domain $D(\mathcal{A})$ if

- There is a *Solution Map*: $\mathcal{T}(t) : X \to X$ such that $\mathcal{T}(0)x = x$ and

$$\mathcal{T}(0)x = x \qquad \text{and} \qquad \frac{\partial}{\partial t}\mathcal{T}(t)x = \mathcal{A}\mathcal{T}(t)x \qquad \text{for any } x \in D(\mathcal{A}).$$

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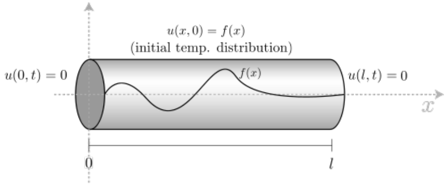
**Example:** The heat equation $u_t = u_{xx}$

- $u(0,t) = 0$ and $u(1,t) = 0$

yields

$$(\mathcal{A}u)(s) = \frac{\partial^2}{\partial x^2}u(s)$$

with $X = L_2$ and
$D(\mathcal{A}) = \{u \in W^2 \,:\, u(0) = u(1) = 0\}$



- Quadratic Lyapunov functions are Necessary and Sufficient for Stability
- The key is that **ANY** quadratic Lyapunov function can be represented as

$$V(x) = \langle x, \mathcal{P}x \rangle_X$$

for some positive operator $\mathcal{P} \succ 0$.

**Theorem 1.**

*If $\mathcal{A}$ generates a SCS on $X$ with domain $D_A$ then*

$$\dot{x}(t) = \mathcal{A}x(t)$$

*is exponentially stable IFF there exists a positive $P \in \mathcal{L}(X \to X)$ such that*

$$\langle x, (\mathcal{A}^*\mathcal{P} + \mathcal{P}\mathcal{A})x \rangle_X < \|x\|_X^2$$

*for all $x \in D_A$.*

We say $\mathcal{P} \succ 0$ if $\mathcal{P}$ is a positive on its domain (an operator inequality).

# Positive Matrices Parameterize Positive Operators

ANY positive operator has a square root:

$$V(x) = \langle x, \mathcal{P}x \rangle_{L_2} = \langle \mathcal{P}^{\frac{1}{2}} x, \mathcal{P}^{\frac{1}{2}} x \rangle_{L_2}$$

Let $\mathcal{Z} : X \to L_2$ be any operator(or vector of operators), then if

$$(\mathcal{P}^{\frac{1}{2}} x)(s) = Q^{\frac{1}{2}} (\mathcal{Z}x)(s)$$

for $Q = Q^{\frac{1}{2}} Q^{\frac{1}{2}} \in \mathbb{R}^{n \times n}$, we have

$$\begin{aligned}
\langle x, \mathcal{P}x \rangle_{L_2} &= \langle \mathcal{P}^{\frac{1}{2}} x, \mathcal{P}^{\frac{1}{2}} x \rangle_{L_2} = \int_\Gamma \langle (\mathcal{P}^{\frac{1}{2}} x)(s), (\mathcal{P}^{\frac{1}{2}} x)(s) \rangle_{\mathbb{R}^n} ds \\
&= \int_\Gamma \langle Q^{\frac{1}{2}} (\mathcal{Z}x)(s), Q^{\frac{1}{2}} (\mathcal{Z}x)(s) \rangle_{\mathbb{R}^n} ds \\
&= \int_\Gamma \langle (\mathcal{Z}x)(s), Q(\mathcal{Z}x)(s) \rangle_{\mathbb{R}^n} ds
\end{aligned}$$

SOoooo...... positive matrices can ALSO parameterize positive operators on $L_2$

- $L_2$ is a bit special (Sobolev variants OK too).

# Linear Systems with Delay

## Consider:

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{K} A_i x(t - \tau_i)$$

$D(A) = \{x \in \mathbb{R}^n \times W^2,\ x_1 = x_2(0)\}$

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**[Datko, 1970]:** For delay systems, **ANY** LF can be represented as

$$V(x) = \langle x, \mathcal{P}x \rangle_{L_2} = \int_{-\tau_K}^{0} x(s)^T M(s) x(s) + \int_{-\tau_K}^{0} \int_{-\tau_K}^{0} x(s)^T N(s,\theta) x(\theta) ds d\theta.$$

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**A Class of Operators**:

$$(\mathcal{P}x)(s) = M(s)x(s) + \int_{-\tau_K}^{0} N(s,t) x(t) dt$$

- $M(s)$ is the multiplier of a **Multiplier Operator**.
- $N(s,t)$ is the kernel of an **Integral Operator**.

# Quadratic L-K Functions

The Datko result implies that ANY LKF is of the form

$$V(x) = \langle x, \mathcal{P}x \rangle_{L_2} = \int_{-\tau}^{0} \langle (\mathcal{Z}x)(s), Q(\mathcal{Z}x)(s) \rangle_{\mathbb{R}^m} ds$$

where

$$(\mathcal{Z}x)(s) = \begin{bmatrix} Z_1(s)x(s) \\ \int_{-\tau}^{0} Z_2(s,\theta)x(\theta)d\theta \end{bmatrix}$$

**Theorem 2.**

*For any functions $Z_1$ and $Z_2$, suppose that*

$$\begin{aligned} M(s) &= Z_1(s)^T Q_{11} Z_1(s) \\ N(s,\theta) &= Z_1(s) Q_{12} Z_2(s,\theta) + Z_2(\theta,s)^T Q_{21} Z_1(\theta) \\ &\qquad + \int_{-\tau}^{0} Z_2(\omega,s)^T Q_{22} Z_2(\omega,\theta)\, d\omega \end{aligned}$$

*where*

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \geq 0.$$

*Then $\langle x, \mathcal{P}x \rangle_{L_2} \geq 0$ for all $x \in L_2[-\tau, 0]$.*

# Quadratic L-K Functions

Lets run through that. Let

$$V(x) = \int_{-\tau}^{0} x(s)^T M(s) x(s) + \int_{-\tau}^{0}\int_{-\tau}^{0} x(s)^T N(s,\theta) x(\theta) ds d\theta.$$

where $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} = \begin{bmatrix} D & H \end{bmatrix}^T \begin{bmatrix} D & H \end{bmatrix} \geq 0$ and

$$M(s) = Z_1(s)^T Q_{11} Z_1(s)$$

$$N(s,\theta) = Z_1(s) Q_{12} Z_2(s,\theta) + Z_2(\theta,s)^T Q_{21} Z_1(\theta) + \int_{-\tau} Z_2(\omega,s)^T Q_{22} Z_2(\omega,\theta)\, d\omega$$

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Then $V(x) = \langle \mathcal{P}^{\frac{1}{2}} x, \mathcal{P}^{\frac{1}{2}} x \rangle \geq 0$ where

$$(\mathcal{P}^{\frac{1}{2}} x)(s) = D Z_1(s) x(s) + \int_{-\tau} H Z_2(s,\theta) x(\theta) d\theta$$

**Note:** The choice of $Z_1$ and $Z_2$ makes a big difference!

# Special Case 1: Piecewise-Continuous Functions

Consider a **Multi-Delay** system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau_1) + A_2 x(t - \tau_2)$$

and the L-K function

$$V(x) = \int_{-\tau_2}^{0} x(s)^T M(s) x(s) + \int_{-\tau_2}^{0} \int_{-\tau_2}^{0} x(s)^T N(s,\theta) x(\theta) ds d\theta.$$

The most OBVIOUS choice is to make $Z_1(s)$ and $Z_2(s,\theta)$ vectors of monomials.

- Then $M$ and $N$ are polynomials.

Problem:

- Converse theory says $M(s)$ and $N(s,\theta)$ may be *discontinuous* at $s, \theta = -\tau_1$.
  - Not polynomial.

# Special Case 1: Piecewise-Continuous Functions

To make $M$ and $N$ discontinuous, define the vector of indicator functions $J = \begin{bmatrix} I_1 & \cdots & I_K \end{bmatrix}^T$ where

$$I_i(t) = \begin{cases} 1 & t \in [-\tau_i, -\tau_{i-1}] \\ 0 & \text{otherwise,} \end{cases} \qquad i = 1, \cdots, K$$

Now if $Z_{1p}(s)$ and $Z_{2p}(s,\theta)$ are vectors of monomials we set

$$Z_1(s) = Z_{1p}(s) \otimes J(s), \qquad Z_2(s,\theta) = Z_{2p}(s,\theta) \otimes J(s) \otimes J(\theta).$$

Now $V(x) \geq 0$ if, for $Q \geq 0$, we have

$$\begin{aligned}
M(s) &= \begin{cases} M_i(s) & s \in [-\tau_i, -\tau_{i-1}] \end{cases} \\
N(s,\theta) &= \begin{cases} N_{ij}(s,\theta) & s \in [-\tau_i, -\tau_{i-1}] \text{ and } \theta \in [-\tau_j, -\tau_{j-1}] \end{cases} \\
M_i(s) &= Z_d(s)^T Q_{11,ii} Z_d(s) \\
N_{ij}(s,\theta) &= Z_{1p}(s) Q_{12,i,(i-1)K+j} Z_{1p}(s,\theta) + Z_{2p}(\theta,s)^T Q_{21,(j-1)K+i,j} Z_{1p}(\theta) \\
&\quad + \sum_{k=1}^{K} \int_{-\tau_k}^{-\tau_{k-1}} Z_{2p}(\omega_k,s)^T Q_{22,i+(k-1)K,j+(k-1)K} Z_{2p}(\omega_k,\theta)\, d\omega_k
\end{aligned}$$

# Special Case 2: Semi-Separable Functions

What about L-K Functions of the Form

$$V(x) = \int_{-\tau}^{0} x(s)^T M(s) x(s) + \int_{-\tau}^{0}\int_{s}^{0} x(s)^T N(s,\theta) x(\theta) d\theta\, ds \quad ?$$

This is actually a special case of functions

$$N(s,\theta) = \begin{cases} N_1(s,\theta) & s \geq \theta \\ N_2(s,\theta) & s < \theta \end{cases}$$

**Q:** How do we parameterize these L-K Functions?
**A:** Use the indicators

$$I_s(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

And use the functions

$$Z_1(s) = Z_{1p}(s),\ Z_2(s,\theta) = \begin{bmatrix} Z_{2p}(s,\theta) I_s(s-\theta) \\ Z_{2p}(s,\theta) I_s(\theta - s) \end{bmatrix}$$

**For Multiple Spatial Domains:** if $t$ is multidimensional (e.g. $t \in \mathbb{R}^n$), then the inequality ($t \geq 0$) is understood to represent a complete ordering on $\Gamma$ (e.g. $t \geq 0$ if $c^T t \geq 0$ for arbitrary vector $c$).

# Special Case 2: Semi-Separable Functions

Now we have

$$M(s) = Z_{1p}(s)^T Q_{11} Z_{1p}(s), \quad N(s,t) = \begin{cases} N_1(s,t) & s \geq t \\ N_2(s,t) & s < t, \end{cases}$$

where

$$N_1(s,t) = Z_{1p}(s)^T Q_{12} Z_{2p}(s,t) + Z_{2p}(t,s) Q_{31} Z_{1p}(t) + \int_s^0 Z_{2p}(\theta,s)^T Q_{22} Z_{2p}(\theta,t) d\theta$$
$$+ \int_t^s Z_{2p}(\theta,s)^T Q_{32} Z_{2p}(\theta,t) d\theta + \int_0^t Z_{2p}(\theta,s)^T Q_{33} Z_{2p}(\theta,t) d\theta.$$

and

$$N_2(s,t) = Z_{1p}(s)^T Q_{13} Z_{2p}(s,t) + Z_{2p}(t,s) Q_{21} Z_{1p}(t) + \int_t^0 Z_{2p}(\theta,s)^T Q_{22} Z_{2p}(\theta,t) d\theta$$
$$+ \int_s^t Z_{2p}(\theta,s)^T Q_{23} Z_{2p}(\theta,t) d\theta + \int_0^s Z_{2p}(\theta,s)^T Q_{33} Z_{2p}(\theta,t) d\theta.$$

The multi-dimensional version is more complicated.

# Non-Quadratic L-K Functions

Lets do nonlinear time-delay systems

$$\dot{x}(t) = f(x(t), x(t-\tau_1))$$

For any $Z_1 : [-\tau, 0] \times \mathbb{R}^n \to \mathbb{R}^{m_1}$ and $Z_2 : [-\tau, 0] \times [-\tau, 0] \times \mathbb{R}^n \to \mathbb{R}^{m_2}$, let

$$\begin{aligned} V(x) = &\int_{-\tau}^{0} Z_1(s, x(s))^T Q_{11} Z_1(s, x(s)) ds \\ &+ \int_{-\tau}^{0}\int_{-\tau}^{0} Z_1(s, x(s)) Q_{12} Z_2(s, \theta, x(\theta)) ds d\theta \\ &+ \int_{-\tau}^{0}\int_{-\tau}^{0} Z_2(\theta, s, x(s))^T Q_{21} Z_1(\theta, x(\theta)) ds d\theta \\ &+ \int_{-\tau}^{0}\int_{-\tau}^{0}\int_{-\tau}^{0} Z_2(\omega, s, x(s))^T Q_{22} Z_2(\omega, \theta, x(\theta))\, d\omega ds d\theta, \end{aligned}$$

where

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \geq 0.$$

Then $V(x) \geq 0$ for all $x \in L_2[-\tau, 0]$.

# Spacing Functions for Time-Delay Systems

$$V(x)=\int_{-\tau}^{0}\begin{bmatrix}x(0)\\x(s)\end{bmatrix}^T M(s)\begin{bmatrix}x(0)\\x(s)\end{bmatrix}+\int_{-\tau}^{0}\int_{-\tau}^{0}\begin{bmatrix}x(0)\\x(s)\end{bmatrix}^T N(s,\theta)\begin{bmatrix}x(0)\\x(\theta)\end{bmatrix}d\theta\, ds$$

**Theorem 3.**

*For any $Z_1$, $Z_2$ and ANY $R_{11}$, $R_{12}$, $R_{21}$, let*

$$M(s)=Z_1(s)^T Q_{11} Z_1(s)+\begin{bmatrix}T(s)+\frac{1}{\tau}\int_{-\tau}^{0}\int_{-\tau}^{0}R_{11}(\omega,t)d\omega dt & \int_{-\tau}^{0}R_{12}(\omega,s)d\omega\\ \int_{-\tau}^{0}R_{21}(s,\omega)d\omega & 0\end{bmatrix}$$

$$N(s,\theta)=Z_1(s)Q_{12}Z_2(s,\theta)+Z_2(\theta,s)^T Q_{21}Z_1(\theta)$$
$$+\int_{-\tau}^{0}Z_2(\omega,s)^T Q_{22}Z_2(\omega,\theta)\,d\omega-\begin{bmatrix}R_{11}(s,\theta) & R_{12}(s,\theta)\\ R_{21}(s,\theta) & 0\end{bmatrix}d\omega$$

*where $\int_{-\tau}^{0}T(s)=0$ and $Q=\begin{bmatrix}Q_{11} & Q_{12}\\ Q_{21} & Q_{22}\end{bmatrix}\geq 0$.*

*Then $V(x)\geq 0$ for all $x\in\mathcal{C}[-\tau,0]$.*

# Derivatives for Time-Delay Systems

$$\dot{V}(x) = \int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(-\tau) \\ x(s) \end{bmatrix}^T D(s) \begin{bmatrix} x(0) \\ x(-\tau) \\ x(s) \end{bmatrix} + \int_{-\tau}^{0}\int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(-\tau) \\ x(s) \end{bmatrix}^T N(s,\theta) \begin{bmatrix} x(0) \\ x(-\tau) \\ x(s) \end{bmatrix} d\theta\, ds$$

where $\{D, E\} = \mathcal{L}(\{M, N\})$.

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`DelayTOOLS`: A mod pack for SOSTOOLS

- download from `http://control.asu.edu/software`.

**PseudoCode:**

1. `[M,N]=sosjointpos_mat_ker_ndelay`
2. `[D,E]=L(M, N)`
3. `[Q,R]=sosjointpos_mat_ker_ndelay`
4. `sosmateq((D,E) + (Q,R))`

NOTE: you have to define $L$ yourself...

# Numerical Tests

Consider the Standard Problem

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & .1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x\left(t - \frac{\tau}{2}\right) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t - \tau)$$

Compare the new and old Stability Tests for TDS

| | SOS | | SOS-joint | |
|---|---|---|---|---|
| $d$ | $\tau_{\min}$ | $\tau_{\max}$ | $\tau_{\min}$ | $\tau_{\max}$ |
| 1 | .20247 | 1.354 | .20247 | 1.3711 |
| 2 | .20247 | 1.3722 | .20247 | 1.3722 |

More accurate, but not that exciting........

- Also less Numerical Error (Stable)

# Numerical Tests - Duality

The Real Improvement is in the Dual Stability Test

- Developed for Controller Synthesis
- Imposes structure on $M$ and $N$

$$(\mathcal{P}x)(s) := \begin{bmatrix} (\tau Q_2(0,0) + Q_1(0))x_1 + \int_{-\tau}^{0} Q_2(0,s)x_2(s)ds \\ \tau Q_2(s,0)x_2(0) + Q_1(s)x_2(s) + \int_{-\tau}^{0} Q_2(s,\theta)x_2(\theta)d\theta \end{bmatrix}$$

The advantage is that now $\mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}^* \preceq 0$ implies stability.

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$$\dot{x}(t) = -x(t-\tau) \qquad \text{which is stable for } \tau \in \left[0, \frac{\pi}{2}\right].$$

- The DS condition w/o joint positivity proves stability on $\tau \in [0, .7] \subset [0, 1.57]$.
  - Required polynomials of degree $8$
- The DS condition with joint positivity yields $\tau = \frac{\pi}{2}$ to 6 decimal places.

# Conclusions:

**Delayed and PDE systems:**

- Creates an optimization framework for studying infinite-dimensional problems.
  - Made Possible by support from **NSF CAREER CMMI-1151018** and the **Chateaubriand Program** (Also **NSF CAREER CMMI-1100376**).

**Unanswered Questions**

- The next big thing in computation?
- A uniform representation for PDEs.

**Papers, Algorithms, Lecture Notes, etc. are available for download at:**

`http://control.asu.edu`