LMI parametrization of Lyapunov Functions for Infinite-Dimensional Systems: A Framework

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Consider: A System of Linear Ordinary Differential Equations

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

Questions: Stability and Control

1. Stability: If \( u = 0 \), do all solutions satisfy \( \lim_{t \to \infty} x(t) = 0 \)
2. Control: Find \( K \) so if \( u(t) = Kx(t) \), all solutions satisfy \( \lim_{t \to \infty} x(t) = 0 \)
3. Observation: Find map \( x, y, u \to \hat{x} \) and \( K \) so if \( u(t) = K\hat{x}(t) \), all solutions satisfy \( \lim_{t \to \infty} x(t) = 0 \)
Key: **System Performance** is captured by quadratic Lyapunov Functions.

The key is that **ANY** quadratic Lyapunov Function can be represented as

\[ V(x) = x^T P x \]

**The SDP Formulation:**

\[
\max_{x \in \mathbb{R}^{n \times n}} \text{trace}(JX) \\
\text{subject to} \quad H_i X G_i + G_i^T X H_i^T \succeq 0
\]

where \( X \succeq 0 \) means \( X \) is a positive semidefinite matrix.

- **Stability** Use \( P > 0, A^T P + PA \preceq 0 \) to show \( V(x) = x^T P x \) is a LF.
- **\( H_\infty \) state feedback** Use \( Y > 0 \) and

\[
\begin{bmatrix}
YA^T + AY + Z^T B_2^T + B_2 Z \\
B_1^T \\
C_1 Y + D_{12} Z \\
B_1 \\
egamma I \\
D_{11} \\
egamma I
\end{bmatrix} \preceq 0
\]

\[
Y C_1^T + Z^T D_{12}^T
\]

\[
D_{11} \quad -\gamma I
\]

\[
-\gamma I
\]

\[
\frac{\|y\|}{\|w\|} \leq \gamma.
\]
Consider: A System of Nonlinear Ordinary Differential Equations

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)) \\
y(t) &= g(x(t))
\end{align*}
\]

Again with the Questions: Things we can do with LF

1. Stability: If \( u = 0 \), do all solutions satisfy \( \lim_{t \to \infty} x(t) = 0 \) (or something...)

2. Gain: Find \( \gamma \) so \( \frac{\|y\|_{L^2}}{\|u\|_{L^2}} \leq \gamma \).
The key is that ANY Lyapunov Function can be represented as

\[ V(x) = Z(x)^T P Z(x) \]

where \( P > 0 \) and \( Z : \mathbb{R}^m \to \mathbb{R}^n \) is a nonlinear map. (\( Z \) can just be a vector of monomials)

Q: Really?
- Yes [Peet, 2008][Peet and Papachristodoulou, 2012] (For exp. stability of \( \dot{x} = f(x) \) on \( X \) where \( f \) is Lip.)

Q: Why?
- If \( P > 0 \), then \( P = P^{\frac{1}{2}} P^{\frac{1}{2}} \), so
  \[ V(x) = Z(x)^T P Z(x) = (Z(x)P^{\frac{1}{2}})^T (P^{\frac{1}{2}} Z(x)) = q(x)^T q(x) \geq 0. \]

The SOS Formulation:

\[
\max_{c \in \mathbb{R}^n} b^T c
\]

subject to \( h(x) := c^T g(x) + d(x) \geq 0 \) for all \( x \in \mathbb{R}^n \)

where "\( \geq 0 \) for all \( x \in \mathbb{R}^n \)" really means "\( = Z(x)^T P Z(x) \) with \( P \succeq 0 \)."
Lyapunov Functions and Positive Matrices

Positive Matrices parameterize LFs with a square root:
Linear ODE's: Quadratic LFs are N+S.

\[ V(x) = \langle x, Px \rangle_{\mathbb{R}^n} = \langle P^{\frac{1}{2}} x, P^{\frac{1}{2}} x \rangle_{\mathbb{R}^n} = \langle q_l(x), q_l(x) \rangle_{\mathbb{R}^n} \]

Nonlinear ODE's: LFs are N+S.

\[ V(x) = \langle Z(x), PZ(x) \rangle_{\mathbb{R}^{\text{big}}} = \langle P^{\frac{1}{2}} Z(x), P^{\frac{1}{2}} Z(x) \rangle_{\mathbb{R}^{\text{big}}} = \langle q_{nl}(x), q_{nl}(x) \rangle_{\mathbb{R}^{\text{big}}} \]

both LFs just map to Euclidean norms. The **Difference** is

- in \( q_l(x) = P^{\frac{1}{2}} x \) the map is linear
- \( q_{nl}(x) = P^{\frac{1}{2}} Z(x) \) is nonlinear

---

Q: What about L-K Functions?

\[ V(x) = \langle x, P x \rangle_{L_2} = \int_{-\tau}^{0} x(s)^T M(s) x(s) ds + \int_{-\tau}^{0} \int_{-\tau}^{0} x(s)^T N(s, \theta) x(\theta) ds d\theta \]

Choose the right basis \( Z : X \to L_2 \) and \( P \geq 0 \) will define the map...
SemiGroup Concept: Let $A: \mathcal{D}(A) \to X$ be an OPERATOR and 
\[ \dot{x}(t) = Ax(t). \]

The system is a strongly continuous semigroup (SCS) on $X$ with domain $\mathcal{D}(A)$ if

- There is a Solution Map: $\mathcal{T}(t): X \to X$ such that $\mathcal{T}(0)x = x$ and
  \[ \mathcal{T}(0)x = x \quad \text{and} \quad \frac{\partial}{\partial t} \mathcal{T}(t)x = A\mathcal{T}(t)x \quad \text{for any } x \in \mathcal{D}(A). \]

Example: The heat equation $u_t = u_{xx}$

- $u(0, t) = 0$ and $u(1, t) = 0$

yields

\[ (Au)(s) = \frac{\partial^2}{\partial x^2} u(s) \]

with $X = L_2$ and 
$D(A) = \{ u \in W^2 : u(0) = u(1) = 0 \}$
A Convex Approach to Stability of PDEs and DDEs
A Converse Lyapunov Theorem

- Quadratic Lyapunov functions are Necessary and Sufficient for Stability
- The key is that **ANY** quadratic Lyapunov function can be represented as

\[ V(x) = \langle x, \mathcal{P}x \rangle_x \]

for some positive operator \( \mathcal{P} \succ 0 \).

**Theorem 1.**

If \( A \) generates a SCS on \( X \) with domain \( D_A \) then

\[ \dot{x}(t) = Ax(t) \]

is exponentially stable IFF there exists a positive \( P \in \mathcal{L}(X \to X) \) such that

\[ \langle x, (A^*\mathcal{P} + \mathcal{P}A)x \rangle_X < \|x\|_X^2 \]

for all \( x \in D_A \).

We say \( \mathcal{P} \succ 0 \) if \( \mathcal{P} \) is a positive on its domain (an operator inequality).
Positive Matrices Parameterize Positive Operators

ANY positive operator has a square root:

\[ V(x) = \langle x, \mathcal{P}x \rangle_{L^2} = \langle \mathcal{P}^{\frac{1}{2}}x, \mathcal{P}^{\frac{1}{2}}x \rangle_{L^2} \]

Let \( \mathcal{Z} : X \rightarrow L^2 \) be any operator (or vector of operators), then if

\[ (\mathcal{P}^{\frac{1}{2}}x)(s) = Q^{\frac{1}{2}}(\mathcal{Z}x)(s) \]

for \( Q = Q^{\frac{1}{2}}Q^{\frac{1}{2}} \in \mathbb{R}^{n \times n} \), we have

\[
\langle x, \mathcal{P}x \rangle_{L^2} = \langle \mathcal{P}^{\frac{1}{2}}x, \mathcal{P}^{\frac{1}{2}}x \rangle_{L^2} = \int_{\Gamma} \langle (\mathcal{P}^{\frac{1}{2}}x)(s), (\mathcal{P}^{\frac{1}{2}}x)(s) \rangle_{\mathbb{R}^n} ds \\
= \int_{\Gamma} \langle Q^{\frac{1}{2}}(\mathcal{Z}x)(s), Q^{\frac{1}{2}}(\mathcal{Z}x)(s) \rangle_{\mathbb{R}^n} ds \\
= \int_{\Gamma} \langle (\mathcal{Z}x)(s), Q(\mathcal{Z}x)(s) \rangle_{\mathbb{R}^n} ds 
\]

SOoooo...... positive matrices can ALSO parameterize positive operators on \( L^2 \)

- \( L^2 \) is a bit special (Sobolev variants OK too).
Consider:

\[ \dot{x}(t) = A_0 x(t) + \sum_{i=1}^{K} A_i x(t - \tau_i) \]

\[ D(A) = \{ x \in \mathbb{R}^n \times W^2, \ x_1 = x_2(0) \} \]

[Datko, 1970]: For delay systems, \textbf{ANY} LF can be represented as

\[ V(x) = \langle x, \mathcal{P} x \rangle_{L^2} = \int_{-\tau_K}^{0} x(s)^T M(s) x(s) + \int_{-\tau_K}^{0} \int_{-\tau_K}^{0} x(s)^T N(s, \theta) x(\theta) ds d\theta. \]

A Class of Operators:

\[ (\mathcal{P} x)(s) = M(s) x(s) + \int_{-\tau_K}^{0} N(s, t) x(t) dt \]

- \( M(s) \) is the multiplier of a \textbf{Multiplier Operator}.
- \( N(s, t) \) is the kernel of an \textbf{Integral Operator}.
Quadratic L-K Functions

The Datko result implies that ANY LKF is of the form

\[ V(x) = \langle x, Px \rangle_{L_2} = \int_{-\tau}^{0} \langle (Zx)(s), Q(Zx)(s) \rangle_{\mathbb{R}^m} ds \]

where

\[ (Zx)(s) = \begin{bmatrix} Z_1(s)x(s) \\ \int_{-\tau}^{s} Z_2(s, \theta)x(\theta)d\theta \end{bmatrix} \]

**Theorem 2.**

For any functions \( Z_1 \) and \( Z_2 \), suppose that

\[
M(s) = Z_1(s)^T Q_{11} Z_1(s) \\
N(s, \theta) = Z_1(s)Q_{12} Z_2(s, \theta) + Z_2(\theta, s)^T Q_{21} Z_1(\theta) \\
+ \int_{-\tau}^{0} Z_2(\omega, s)^T Q_{22} Z_2(\omega, \theta) d\omega
\]

where

\[ Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \geq 0. \]

Then \( \langle x, Px \rangle_{L_2} \geq 0 \) for all \( x \in L_2[-\tau, 0] \).
Quadratic L-K Functions

Let's run through that. Let

\[ V(x) = \int_{-\tau}^{0} x(s)^T M(s)x(s) + \int_{-\tau}^{0} \int_{-\tau}^{0} x(s)^T N(s, \theta)x(\theta) \, ds \, d\theta. \]

where \( Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} = [D \quad H]^T [D \quad H] \geq 0 \) and

\[ M(s) = Z_1(s)^T Q_{11} Z_1(s) \]
\[ N(s, \theta) = Z_1(s)Q_{12}Z_2(s, \theta) + Z_2(\theta, s)^T Q_{21} Z_1(\theta) + \int_{-\tau}^{0} Z_2(\omega, s)^T Q_{22} Z_2(\omega, \theta) \, d\omega \]

Then \( V(x) = \langle \mathcal{P}^{\frac{1}{2}} x, \mathcal{P}^{\frac{1}{2}} x \rangle \geq 0 \) where

\[ (\mathcal{P}^{\frac{1}{2}} x)(s) = DZ_1(s)x(s) + \int_{-\tau}^{0} HZ_2(s, \theta)x(\theta) \, d\theta \]

Note: The choice of \( Z_1 \) and \( Z_2 \) makes a big difference!
Special Case 1: Piecewise-Continuous Functions

Consider a **Multi-Delay** system

\[ \dot{x}(t) = A_0 x(t) + A_1 x(t - \tau_1) + A_2 x(t - \tau_2) \]

and the L-K function

\[
V(x) = \int_{-\tau_2}^{0} x(s)^T M(s) x(s) + \int_{-\tau_2}^{0} \int_{-\tau_2}^{0} x(s)^T N(s, \theta) x(\theta) ds d\theta.
\]

The most OBVIOUS choice is to make \( Z_1(s) \) and \( Z_2(s, \theta) \) vectors of monomials.

- Then \( M \) and \( N \) are polynomials.

**Problem:**

- Converse theory says \( M(s) \) and \( N(s, \theta) \) may be *discontinuous* at \( s, \theta = -\tau_1 \).
  - Not polynomial.
Special Case 1: Piecewise-Continuous Functions

To make $M$ and $N$ discontinuous, define the vector of indicator functions

$$J = [I_1 \cdots I_K]^T$$

where

$$I_i(t) = \begin{cases} 
1 & t \in [-\tau_i, -\tau_{i-1}] \\
0 & \text{otherwise},
\end{cases} \quad i = 1, \cdots, K$$

Now if $Z_{1p}(s)$ and $Z_{2p}(s, \theta)$ are vectors of monomials we set

$$Z_1(s) = Z_{1p}(s) \otimes J(s), \quad Z_2(s, \theta) = Z_{2p}(s, \theta) \otimes J(s) \otimes J(\theta).$$

Now $V(x) \geq 0$ if, for $Q \geq 0$, we have

$$M(s) = \begin{cases} 
M_i(s) & s \in [-\tau_i, -\tau_{i-1}]
\end{cases}$$

$$N(s, \theta) = \begin{cases} 
N_{ij}(s, \theta) & s \in [-\tau_i, -\tau_{i-1}] \text{ and } \theta \in [-\tau_j, -\tau_{j-1}]
\end{cases}$$

$$M_i(s) = Z_d(s)^T Q_{11,ii} Z_d(s)$$

$$N_{ij}(s, \theta) = Z_{1p}(s)Q_{12, i, (i-1)K+j} Z_{1p}(s, \theta) + Z_{2p}(\theta, s)^T Q_{21, (j-1)K+i, j} Z_{1p}(\theta)$$

$$+ \sum_{k=1}^{K} \int_{-\tau_k}^{-\tau_{k-1}} Z_{2p}(\omega_k, s)^T Q_{22, i+(k-1)K, j+(k-1)K} Z_{2p}(\omega_k, \theta) d\omega_k$$
Special Case 2: Semi-Separable Functions

What about L-K Functions of the Form

$$V(x) = \int_{-\tau}^{0} x(s)^T M(s)x(s) + \int_{-\tau}^{0} \int_{s}^{0} x(s)^T N(s, \theta)x(\theta)d\theta ds$$

This is actually a special case of functions

$$N(s, \theta) = \begin{cases} N_1(s, \theta) & s \geq \theta \\ N_2(s, \theta) & s < \theta \end{cases}$$

Q: How do we parameterize these L-K Functions?
A: Use the indicators

$$I_s(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

And use the functions

$$Z_1(s) = Z_1p(s), \quad Z_2(s, \theta) = \begin{bmatrix} Z_2p(s, \theta)I_s(s - \theta) \\ Z_2p(s, \theta)I_s(\theta - s) \end{bmatrix}$$

For Multiple Spatial Domains: if $t$ is multidimensional (e.g. $t \in \mathbb{R}^n$), then the inequality $(t \geq 0)$ is understood to represent a complete ordering on $\Gamma$ (e.g. $t \geq 0$ if $c^T t \geq 0$ for arbitrary vector $c$).
Special Case 2: Semi-Separable Functions

Now we have

\[ M(s) = Z_p(s)^T Q_{11} Z_p(s), \quad N(s, t) = \begin{cases} N_1(s, t) & s \geq t \\ N_2(s, t) & s < t \end{cases}, \]

where

\[
N_1(s, t) = Z_p(s)^T Q_{12} Z_p(s, t) + Z_p(t, s) Q_{31} Z_p(t) + \int_s^0 Z_p(\theta, s)^T Q_{22} Z_p(\theta, t) d\theta \\
+ \int_t^s Z_p(\theta, s)^T Q_{32} Z_p(\theta, t) d\theta + \int_0^t Z_p(\theta, s)^T Q_{33} Z_p(\theta, t) d\theta.
\]

and

\[
N_2(s, t) = Z_p(s)^T Q_{13} Z_p(s, t) + Z_p(t, s) Q_{21} Z_p(t) + \int_t^0 Z_p(\theta, s)^T Q_{22} Z_p(\theta, t) d\theta \\
+ \int_s^t Z_p(\theta, s)^T Q_{23} Z_p(\theta, t) d\theta + \int_0^s Z_p(\theta, s)^T Q_{33} Z_p(\theta, t) d\theta.
\]

The multi-dimensional version is more complicated.
Non-Quadratic L-K Functions

Lets do nonlinear time-delay systems

\[ \dot{x}(t) = f(x(t), x(t - \tau_1)) \]

For any \( Z_1 : [-\tau, 0] \times \mathbb{R}^n \rightarrow \mathbb{R}^{m_1} \) and \( Z_2 : [-\tau, 0] \times [-\tau, 0] \times \mathbb{R}^n \rightarrow \mathbb{R}^{m_2} \), let

\[
V(x) = \int_{-\tau}^{0} Z_1(s, x(s))^T Q_{11} Z_1(s, x(s)) ds \\
+ \int_{-\tau}^{0} \int_{-\tau}^{0} Z_1(s, x(s)) Q_{12} Z_2(s, \theta, x(\theta)) ds d\theta \\
+ \int_{-\tau}^{0} \int_{-\tau}^{0} Z_2(\theta, s, x(s))^T Q_{21} Z_1(\theta, x(\theta)) ds d\theta \\
+ \int_{-\tau}^{0} \int_{-\tau}^{0} \int_{-\tau}^{0} Z_2(\omega, s, x(s))^T Q_{22} Z_2(\omega, \theta, x(\theta)) d\omega ds d\theta,
\]

where

\[
Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \geq 0.
\]

Then \( V(x) \geq 0 \) for all \( x \in L_2[-\tau, 0] \).
Spacing Functions for Time-Delay Systems

\[ V(x) = \int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(s) \end{bmatrix}^T M(s) \begin{bmatrix} x(0) \\ x(s) \end{bmatrix} + \int_{-\tau}^{0} \int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(s) \end{bmatrix}^T N(s, \theta) \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} \, d\theta \, ds \]

**Theorem 3.**

For any \( Z_1, Z_2 \) and ANY \( R_{11}, R_{12}, R_{21} \), let

\[
M(s) = Z_1(s)^T Q_{11} Z_1(s) + \begin{bmatrix}
T(s) + \frac{1}{\tau} \int_{-\tau}^{0} \int_{-\tau}^{0} R_{11}(\omega, t) \, d\omega \, dt & \int_{-\tau}^{0} R_{12}(\omega, s) \, d\omega \\
\int_{-\tau}^{0} R_{21}(s, \omega) \, d\omega & 0
\end{bmatrix}
\]

\[
N(s, \theta) = Z_1(s)Q_{12} Z_2(s, \theta) + Z_2(\theta, s)^T Q_{21} Z_1(\theta)
\]

\[
+ \int_{-\tau}^{0} Z_2(\omega, s)^T Q_{22} Z_2(\omega, \theta) \, d\omega - \begin{bmatrix}
R_{11}(s, \theta) & R_{12}(s, \theta) \\
R_{21}(s, \theta) & 0
\end{bmatrix} \, d\omega
\]

where \( \int_{-\tau}^{0} T(s) = 0 \) and \( Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \geq 0 \).

Then \( V(x) \geq 0 \) for all \( x \in C[-\tau, 0] \).
Derivatives for Time-Delay Systems

\[
\dot{V}(x) = \int_{-\tau}^{0} \left[ \begin{array}{c} x(0) \\ x(-\tau) \\ x(s) \end{array} \right]^T D(s) \left[ \begin{array}{c} x(0) \\ x(-\tau) \\ x(s) \end{array} \right] + \int_{-\tau}^{0} \int_{-\tau}^{0} \left[ \begin{array}{c} x(0) \\ x(-\tau) \\ x(s) \end{array} \right]^T N(s, \theta) \left[ \begin{array}{c} x(0) \\ x(-\tau) \\ x(s) \end{array} \right] d\theta ds
\]

where \( \{D, E\} = \mathcal{L}(\{M, N\}) \).

DelayTOOLS: A mod pack for SOSTOOLS
- download from http://control.asu.edu/software

PseudoCode:

1. \([M, N]=\text{sosjointpos\_mat\_ker\_ndelay}\)
2. \([D, E]=\text{L}(M, N)\)
3. \([Q, R]=\text{sosjointpos\_mat\_ker\_ndelay}\)
4. \text{sosmateq}((D, E) + (Q, R))

NOTE: you have to define \( L \) yourself...
Consider the Standard Problem

\[
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x \left( t - \frac{\tau}{2} \right) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t - \tau)
\]

Compare the new and old Stability Tests for TDS

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<th>( \tau_{\text{max}} )</th>
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More accurate, but not that exciting........

- Also less **Numerical Error** (Stable)
Numerical Tests - Duality

The Real Improvement is in the **Dual Stability Test**

- Developed for Controller Synthesis
- Imposes structure on \( M \) and \( N \)

\[
(\mathcal{P}x)(s) := \begin{bmatrix}
(\tau Q_2(0, 0) + Q_1(0))x_1 + \int_{-\tau}^{0} Q_2(0, s)x_2(s)ds \\
\tau Q_2(s, 0)x_2(0) + Q_1(s)x_2(s) + \int_{-\tau}^{0} Q_2(s, \theta)x_2(\theta)d\theta
\end{bmatrix}
\]

The advantage is that now \( \mathcal{AP} + \mathcal{PA}^* \preceq 0 \) implies stability.

\[
\dot{x}(t) = -x(t - \tau)
\]

which is stable for \( \tau \in \left[0, \frac{\pi}{2} \right] \).

- The DS condition w/o joint positivity proves stability on \( \tau \in [0, .7] \subset [0, 1.57] \).
  - Required polynomials of degree 8
- The DS condition with joint positivity yields \( \tau = \frac{\pi}{2} \) to 6 decimal places.
Conclusions:

Delayed and PDE systems:
- Creates an optimization framework for studying infinite-dimensional problems.
  - Made Possible by support from NSF CAREER CMMI-1151018 and the Chateaubriand Program (Also NSF CAREER CMMI-1100376).

Unanswered Questions
- The next big thing in computation?
- A uniform representation for PDEs.

Papers, Algorithms, Lecture Notes, etc. are available for download at:

http://control.asu.edu