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Optimal Set Containment Using Sublevel Set Volume Minimization

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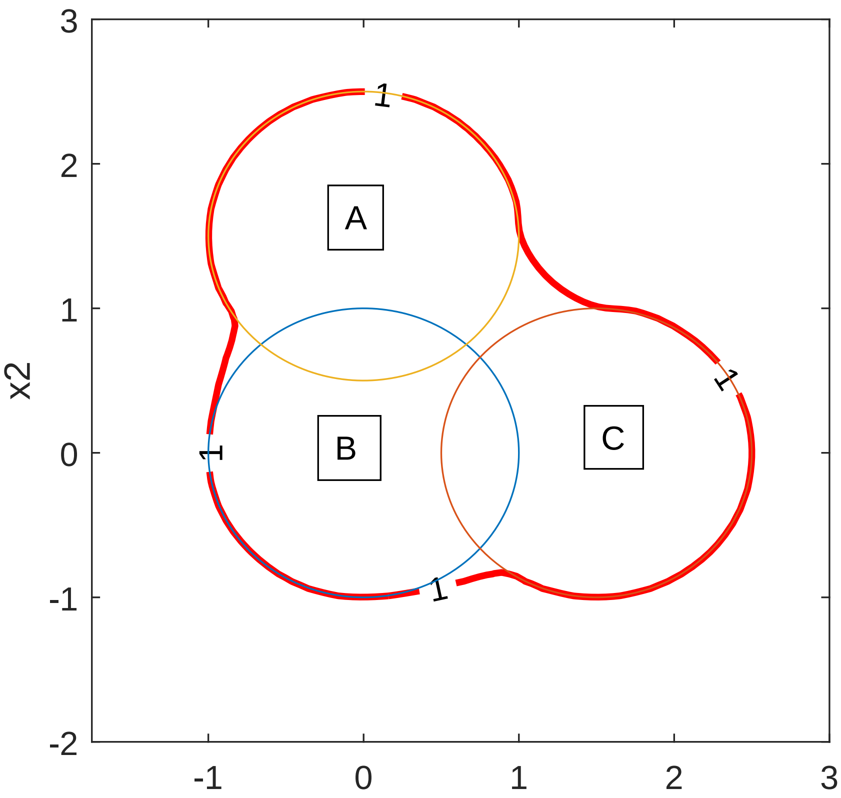
Optimal Set Containment

Goal

For $S \subset \mathbb{R}^n$ find $V : \mathbb{R}^n \to \mathbb{R}$ such that $S \subseteq \{x \in \mathbb{R}^n : V(x) < 1\}$.

Example:

$$V(x_1,x_2) = 0.15361x_1^6 + 0.2062x_1^5x_2 + 0.57102x_1^4x_2^2 + 0.49633x_1^3x_2^3 + 0.57102x_1^2x_2^4 + 0.2062x_1x_2^5 + 0.15361x_2^6 - 0.644x_1^5 - 1.011x_1^4x_2 - 1.6852x_1^3x_2^2 - 1.6852x_1^2x_2^3 - 1.011x_1x_2^4 - 0.644x_2^5 + 0.47172x_1^4 + 1.043x_1^3x_2 + 0.67683x_1^2x_2^2 + 1.043x_1x_2^3 + 0.47172x_2^4 + 0.73992x_1^3 + 1.0435x_1^2x_2 + 1.0435x_1x_2^2 + 0.73992x_2^3 - 0.67654x_1^2 - 1.347x_1x_2 - 0.67654x_2^2 - 0.15349x_1 - 0.15349x_2 + 0.99181.$$



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1. Applications:
    - Region of Attraction (ROA) estimation.
    - Relaxing Optimization Problems.
    - Finding domains where determinism fails.
2. General problem formulation and sublevel set volume minimization.
3. Formulation of SOS program for approximation of:
    - Unions of semialgebriac sets.
    - Attractor sets.
    - Reachable sets.

Regions Of Attraction

Consider the ODE,

$$\boxed{\begin{aligned}&\dot{x}(t) = f(x(t))\\ &\text{Given} \quad x(0) = x_0.\end{aligned}} \qquad (1)$$

Where $f : \mathbb{R}^n \to \mathbb{R}$ and $x_0 \in \mathbb{R}^n$.

The Solution Map

We say $\phi_f(x_0, t)$ is the solution map of the ODE (1) if,

$$\frac{\delta}{\delta t}\phi_f(x,t) = f(\phi_f(x,t))$$
$$\phi_f(x,0) = x.$$

The Region of Attraction

The Region of Attraction (ROA) for the ODE (1) is defined by,

$$ROA_f := \{x \in \mathbb{R}^n : \lim_{t \to \infty} \phi_f(x,t) = 0\}. \qquad (2)$$

Estimation of ROA's Using Trajectory Data

Given data, $D = \{x_1, ..., x_N\} \subset ROA_f$, we can find $V : \mathbb{R}^n \to \mathbb{R}$ such that $D \subseteq \{x \in \mathbb{R}^n : V(x) < 1\}$.

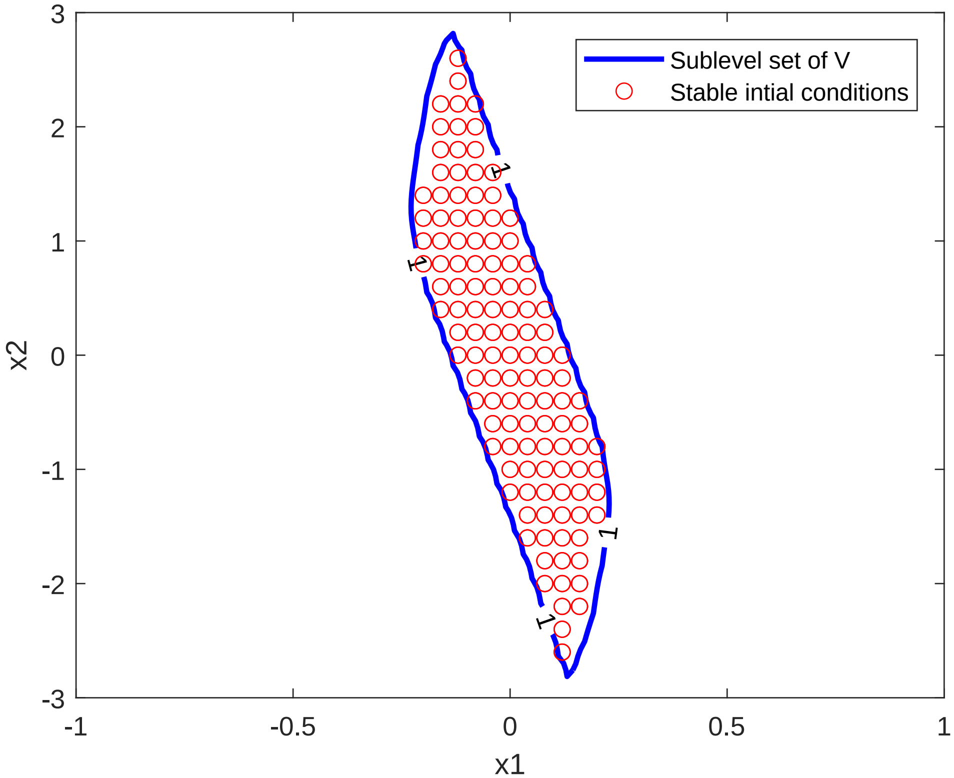
Example: Single Machine Infinite Bus System

Consider the nonlinear, non-polynomial, switching system:

$$\begin{bmatrix} \dot{\theta}(t) \\ \dot{\omega}(t) \end{bmatrix} = \begin{cases} \begin{bmatrix} \omega(t) \\ \frac{1}{M}\left(P_m - \frac{\nu E_s}{X_t}\sin(\theta(t)) - D\omega(t)\right) \end{bmatrix} & \forall \omega \in [-3,3], \delta \in [-\pi, \pi] \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{otherwise} \end{cases}$$

where $M = 0.0212$, $X_t = 0.28$, $P_m = 1$, $\nu = 1$, $E_s = 1.21$, $D = 0.02$.

Estimation of ROA of Single Machine Infinite Bus



Stable initial conditions were generated using a numerical ODE solver or from the real time output of the machine.

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Relaxing Optimization Problems

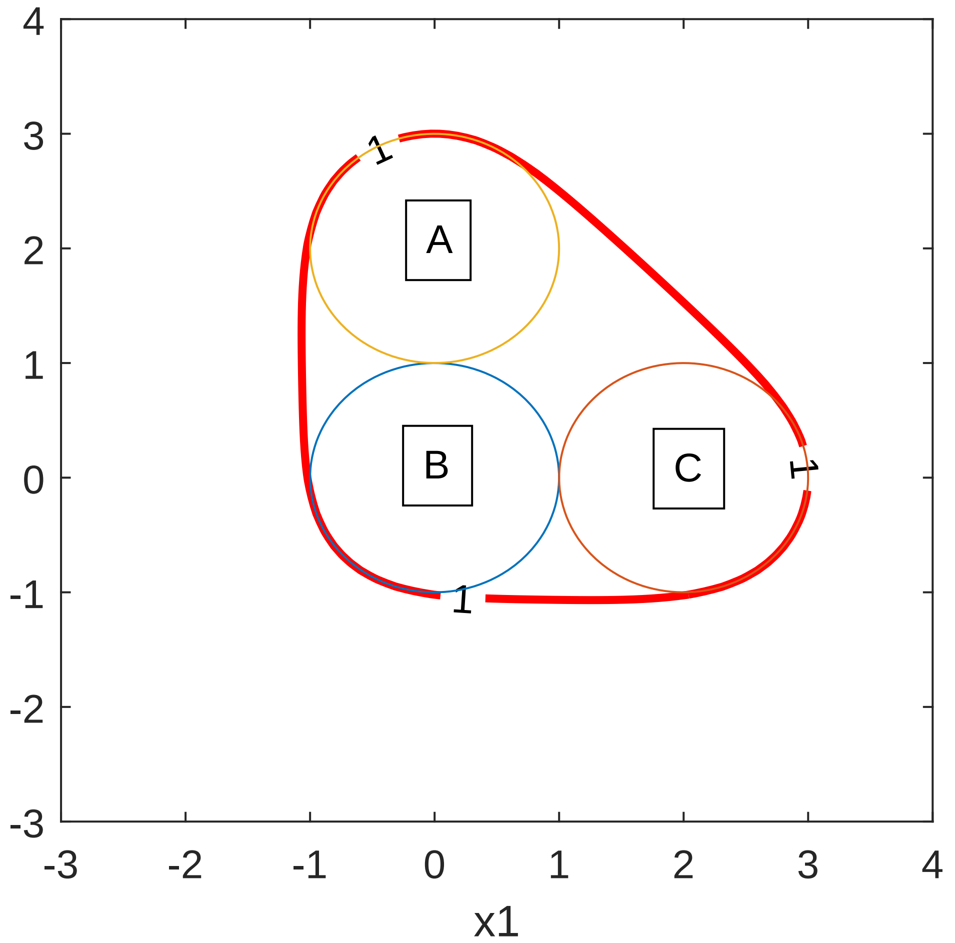
$$\boxed{\min_{x \in X} f(x)} \qquad (3)$$

$$\boxed{\min_{x \in C} f(x)} \qquad (4)$$

Fact

Suppose $X \subseteq C$ then $\min_{x\in C} f(x) \le \min_{x\in X} f(x)$.

- Replacing X with a set, C, such that $X \subset C$ allows us to provide a bound for $\min_{x\in X} f(x)$.
- Finding a set C that is convex can make the bound of $\min_{x\in X} f(x)$ tractable to compute.


4
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A
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1
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1

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The Butterfly Effect

Chaos: The Butterfly Effect

When a small change in the initial conditions for a system results in a large change in a later stage.

Classical Example

A tornado can be influenced by small perturbations such as a butterfly flapping its wings several weeks earlier.

How Chaos Undermines Modeling Physical Systems Mathematically

Experimental error

You can never measure phenomena in nature without error.

Definition: Chaotic Region

A compact set $A \subset \mathbb{R}^n$ is a chaotic region of $\dot{x}(t) = f(x(t)), x(0) = x_0$ if $\forall x_0 \in A$ the limit $\lim_{t\to\infty} \phi(x_0, t)$ is neither convergent, divergent or recurrent.

ODE's loose the ability to predict future natural phenomena inside chaotic regions!

Our Goal is to estimate chaotic regions of ODE's

The Lorenz System

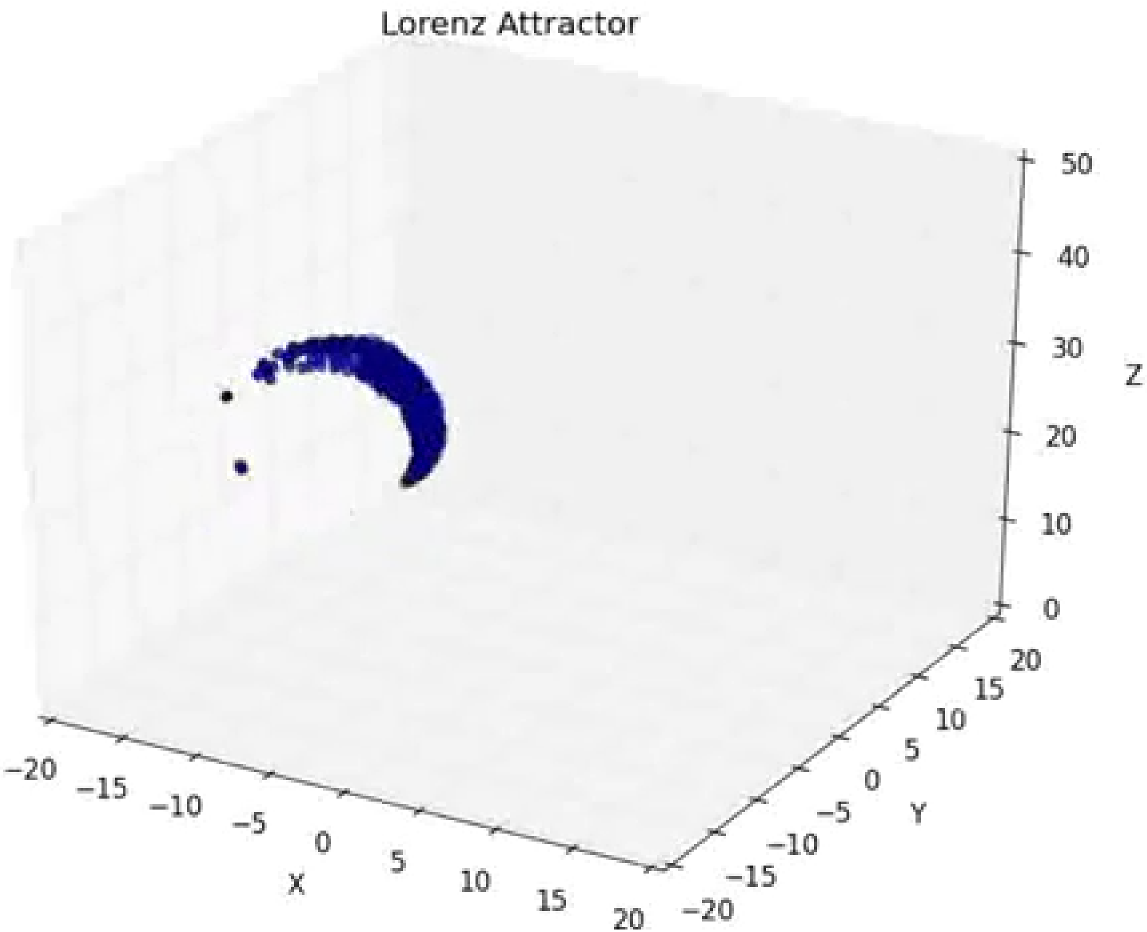
In 1963 E.N. Lorenz proposed a 3D nonlinear model convection rolls in the atmosphere.

$$\dot{x}_1(t) = \sigma(x_2(t) - x_1(t))$$
$$\dot{x}_2(t) = \rho x_1(t) - x_2(t) - x_1(t)x_3(t)$$
$$\dot{x}_3(t) = x_1(t)x_2(t) - \beta x_3(t)$$

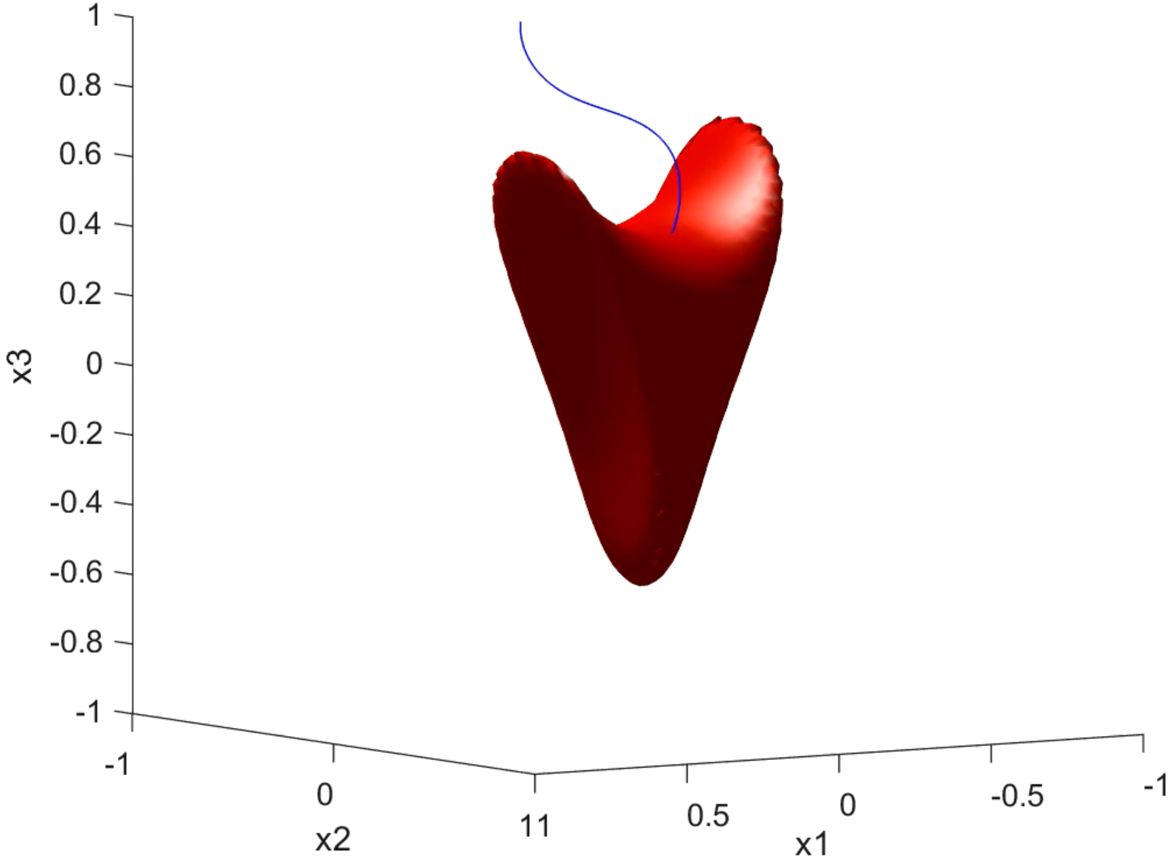
- $x_1(t) =$ rate of convection
- $x_2(t) =$ horizontal temperature variation
- $x_3(t) =$ vertical temperature variation
- $\sigma =$ Prandtl number
- $\rho =$ Rayleigh number
- $\beta =$ aspect ratio

For the classical parameter choice $\sigma = 10$, $\rho = 28$ and $\beta = \frac{8}{3}$ the Lorenz system exhibits chaotic properties.

Chaos Simulation of Lorenz System



We Are Able to Generate an Approximation of the Lorenz Attractor



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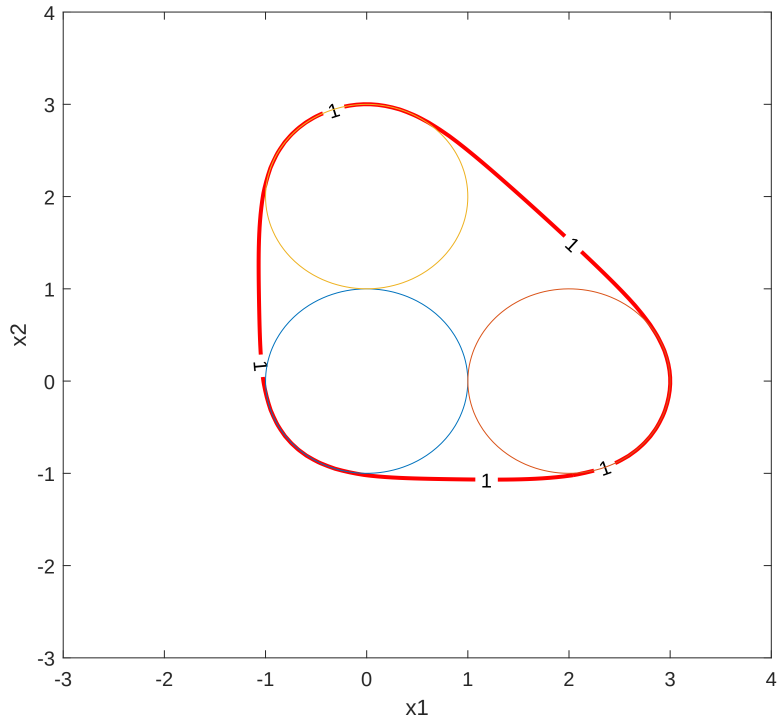
An Optimization Problem for Outer Set Approximation

$$\min_{X \in C} \{D(X,Y)\}$$
$$\text{subject to: } Y \subseteq X$$

where,

- $D : P(\mathbb{R}^n) \times P(\mathbb{R}^n) \to \mathbb{R}$ is some metric that measures the distance between two subsets of $\mathbb{R}^n$.
- $Y \subset \mathbb{R}^n$.
- $C \subset P(\mathbb{R}^n) := \{X : X \subset \mathbb{R}^n\}$.



Example

- $Y = B((0,0),1) \cup B((0,2),1) \cup B((2,0),1)$.
- $C = \{X \subset \mathbb{R}^n : X = \{x \in \mathbb{R}^n : g(x) \leq 1\}, g \in \sum_{sos}, X \text{ is convex}\}$.

Volume is a Metric for Set Approximation

Definition: Volume

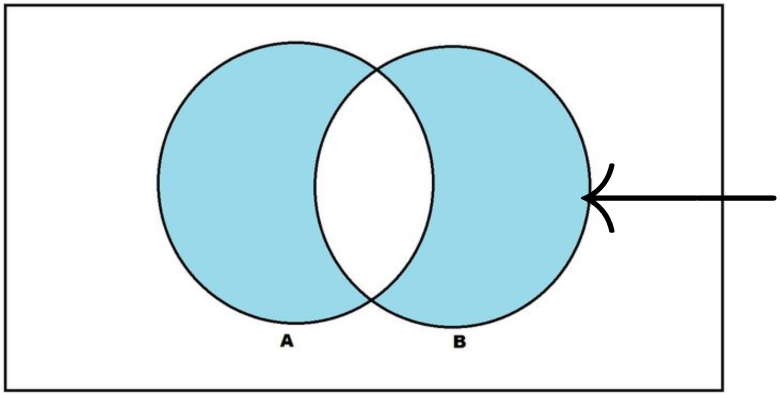
For $A \subset \mathbb{R}^n$ we define

$$vol\{A\} = \int_{\mathbb{R}^n} \mathbf{1}_A(x)dx$$

Definition: Volume Metric

For $A, B \subset \mathbb{R}^n$ we define,

$$D_V(A,B) = vol\{(A/B) \cup (B/A)\}$$



Blue Area=$D_V(A,B)$

Outer Set Approximation Using the Volume Metric

$$\min_{X \in C} \{D_V(X,Y)\} \quad \text{subject to: } Y \subseteq X \qquad (5)$$

$$\min_{X \in C} \{vol\{X\}\} \quad \text{subject to: } Y \subseteq X \qquad (6)$$

Lemma: The above optimization problems are equivalent

If X_1 solves (5) and X_2 solves (6) then $X_1 = X_2$.

To make the problem tractable we consider Sum-of-Square (SOS) polynomial sublevel set outer approximations parameterized by a positive matrix $A \in S^n_{++}$,

$$\{x \in \mathbb{R}^n : z_d(x)^T A z_d(x) < 1\}$$

where $z_d : \mathbb{R}^n \to \mathbb{R}^N$ is a monomial vector of degree d.

Sublevel set volume minimization is hard!

Determinants are Related to Volumes of Ellipses

For an invertible square matrix $A \in GL(n,\mathbb{R})$ and a fixed function $g : \mathbb{R}^n \to \mathbb{R}$,

$$\boxed{\det(A) \propto vol\{x \in \mathbb{R}^n : g(A^{-1}x) < 1\}.}$$

$$\Longrightarrow$$

$$\boxed{\det(A) \propto vol\{x \in \mathbb{R}^n : x^T A x < 1\}.}$$

We can formulate a convex objective function that is minimized by the matrix with minimum feasible determinant.

Lemma: logdet is convex

$f : S^n_{++} \to \mathbb{R}$ given by $f(X) = -\log\det(X)$ is convex.

$-\log\det$ objective functions increase eigenvalues

Fact

Minimizing

$$vol\{x \in \mathbb{R}^n : x^T A x < 1\}$$

is equivalent to minimizing $-\log\det A$.

Question

Can we use $-\log\det A$ to minimize

$$vol\{x \in \mathbb{R}^n : z_d(x)^T A z_d(x) < 1\}?$$

For $A \in S^n_{++}$ there exists unitary $T \in \mathbb{R}^{n\times n}$ and a diagonal matrix $\Lambda \in \mathbb{R}^{n\times n}$ such that $A = T^T \Lambda T$.

Therefore,

$$-\log\det A = -\sum_{i=1}^n \log(\Lambda_{i,i}).$$

$\implies$ Increasing the eigenvalues of A minimizes $-\log\det A$.

Increasing Eigenvalues Decreases Volume

Consider the following set

$$\{x \in \mathbb{R}^n : V(x) < 1\},$$

where $V(x) = z_d(x)^T A z_d(x)$.

As $A \in S^n_{++}$ it follows

$$V(x) = (T z_d(x))^T \Lambda T z_d(x) = \sum_{i=1}^{n} \Lambda_{i,i} (p_i(x))^2,$$

for some $p_i : \mathbb{R}^n \to \mathbb{R}$.

Minimizing $-\log \det A$ results in:
$\implies$ Larger eigenvalues of A
$\implies$ Larger value of $V(x) = z_d(x)^T A z_d(x)$ for all $x \in \mathbb{R}^n$
$\implies$ Less elements in $\{x \in \mathbb{R}^n : z_d(x)^T A z_d(x) < 1\}$
$\implies$ Smaller $vol\{x \in \mathbb{R}^n : z_d(x)^T A z_d(x) < 1\}$.

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SOS Approximation of Unions of Semialgebriac Sets

Optimization for approximation of semialgebriac sets

$$\min_{A \in S^n_{++}} \{-\log \det A\}$$

subject to: $Y \subseteq \{x \in \mathbb{R}^n : z_d(x)^T A z_d(x) < 1\}$

where $Y = \cup_{i=1}^m S_i$ and $S_i = \{x \in \mathbb{R}^n : g_{i,1}(x) \le 0, \ldots, g_{i,m}(x) \le 0\}$.

Tightened SOS optimization problem

$$\min_{A \in S^N_{++}} \{-\log \det A\} \quad \text{subject to,}$$

$$(1 - z_d^T A z_d) + \sum_{j=1}^{k} s_{i,j} g_{i,j} \in \sum_{sos} \quad \forall i \in \{1, \ldots, m\}$$

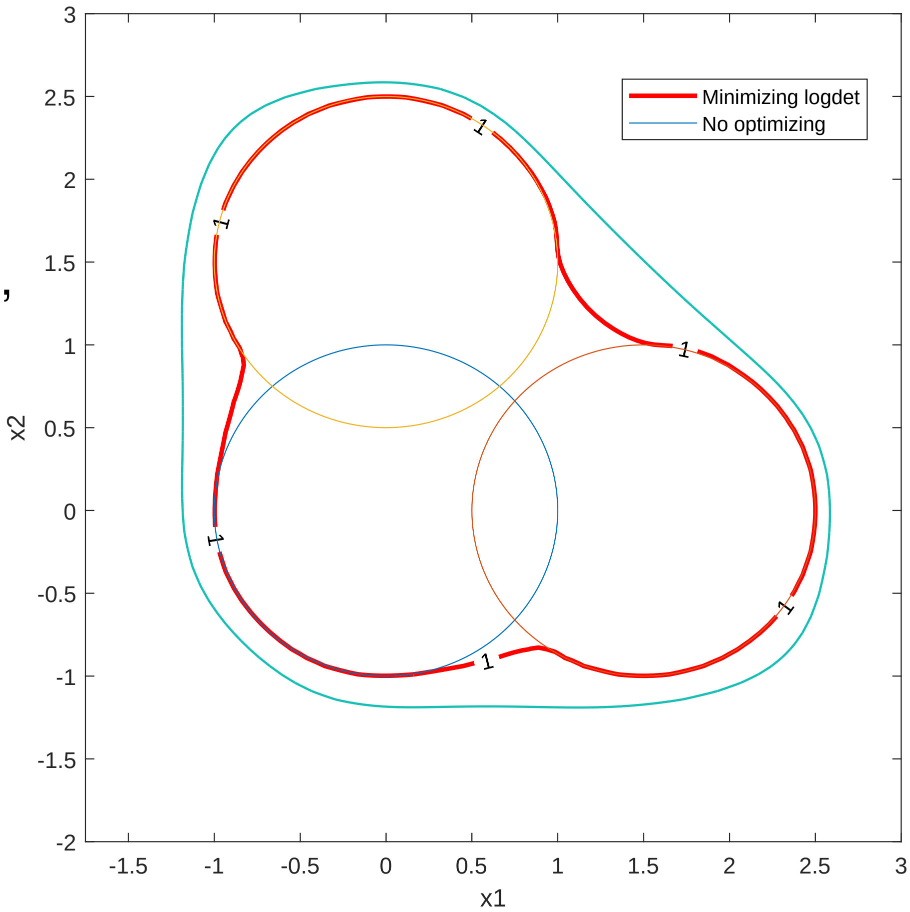
$$s_{i,j} \in \sum_{sos} \quad \forall i, j$$

Numerics: Sublevel Set Approximation of balls

- Outer approx for $S = \cup_{i=1}^{3} S_i$ where,

$$S_1 = \{(x_1,x_2) : x_1^2 + x_2^2 \leq 1\}$$
$$S_2 = \{(x_1,x_2) : (x_1 - 1.5)^2 + (x_2)^2 \leq 1\}$$
$$S_3 = \{(x_1,x_2) : (x_1)^2 + (x_2 - 1.5)^2 \leq 1\}.$$

- degree=8.

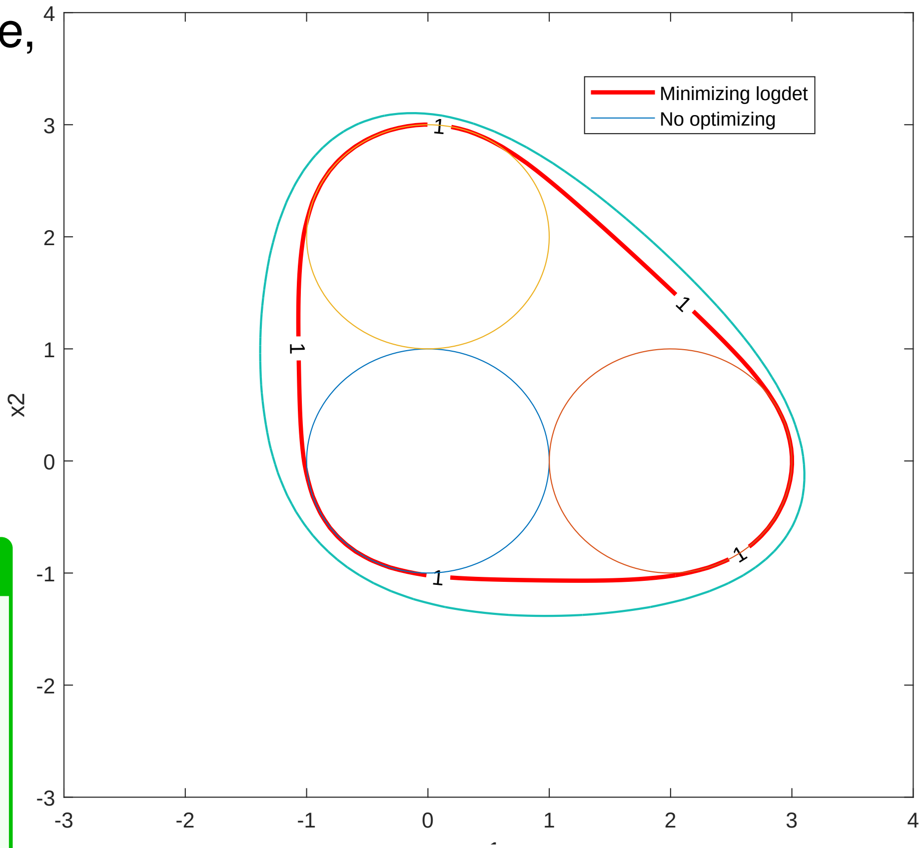


Numerical Examples

- Outer approx for $S = \cup_{i=1}^{3} S_i$ where,

$$S_1 = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$$
$$S_2 = \{(x_1, x_2) : (x_1 - 2)^2 + (x_2)^2 \leq 1\}$$
$$S_3 = \{(x_1, x_2) : (x_1)^2 + (x_2 - 2)^2 \leq 1\}.$$

- degree=6.



Convex Constraint

Adding the constraint $\nabla^2(z_d^T A z_d) \in \Sigma_{SOS}$ to the optimization problem ensures the function $V(x) = z_d(x)^T A z_d(x)$ is convex and thus its 1-sublevel set is also convex.

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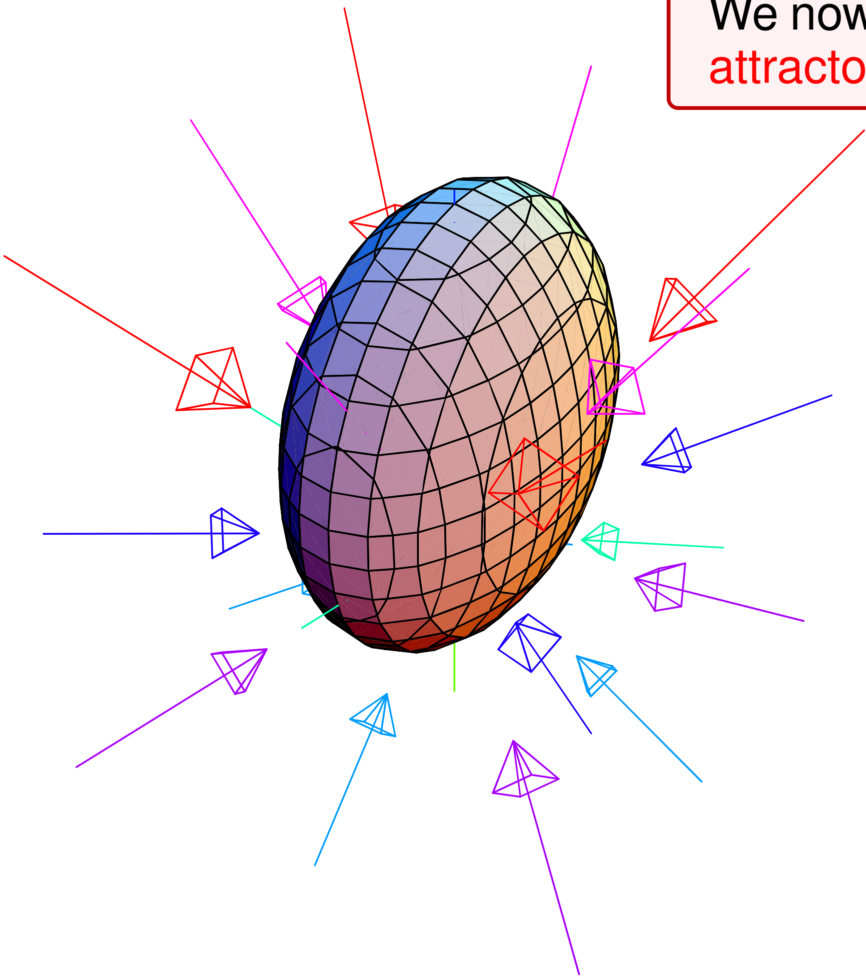
Attractor Sets

$$\min_{X \in S^n_{++}} \{-\log \det X\}$$

subject to: $Y \subseteq \{x \in \mathbb{R}^n : z_d(x)^T X z_d(x) < 1\}$

We now consider the problem when Y is an attractor set.



Definition: Attractor set

For a dynamical system $\dot{x}(t) = f(x(t))$ we say a set $A \subset \mathbb{R}^n$ is an attractor set if $\forall x_0 \in \mathbb{R}^n$ there exists $T > 0$ such that $x(t) \in A$ for all $t > T$.

Definition: Minimal Attractor

A is the minimal attractor of a dynamical system $\dot{x}(t) = f(x(t))$ if it is an attractor set and has no proper subset that is also an attractor set.

Lyapunov Theory to Enforce Attractor Set Containment Constraints

$$\min_{X \in S^n_{++}} \{-\log \det X\}$$

subject to: $Y \subseteq \{x \in \mathbb{R}^n : z_d(x)^T X z_d(x) < 1\}$

Lyapunov theory

Consider some ODE of the $\dot{x}(t) = f(x(t))$. Suppose there exists $V : \mathbb{R}^n \to \mathbb{R}$ such that,

$V(x) > 0$ for all $x \notin D$

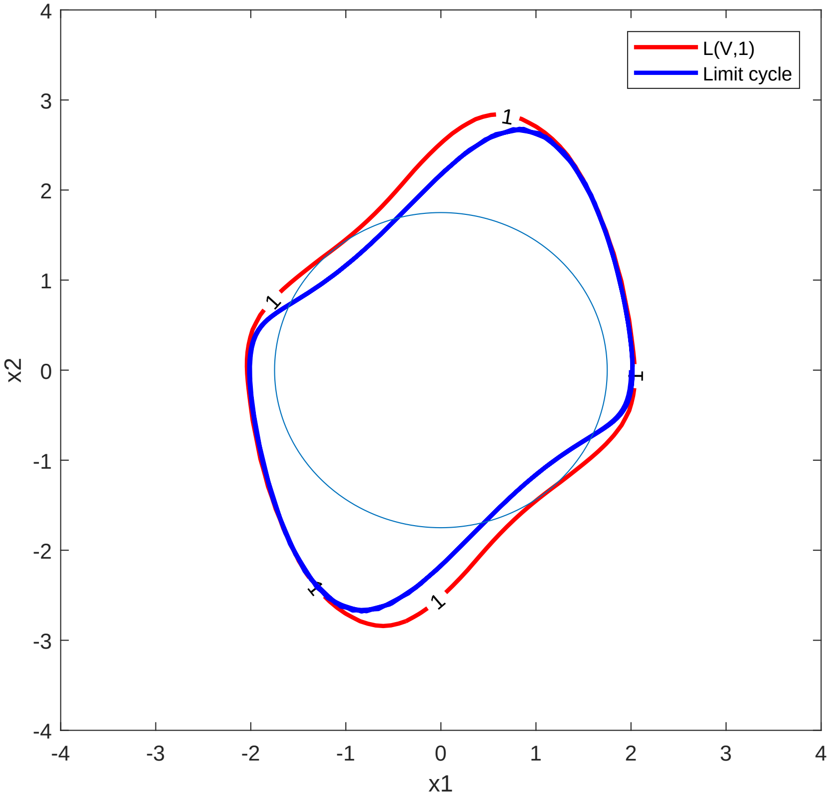
$\nabla V(x)^T f(x) < 0$ for all $x \notin D$.

Then if $\gamma > 0$ is such that $D \subset \{x \in \mathbb{R}^n : V(x) < \gamma\}$ we have that

$$A \subset \{x \in \mathbb{R}^n : V(x) < \gamma\}$$

where A is the minimal attractor set.

Numerical Example: Van der Pol Oscillator



Consider the Van-der-Pol system:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -x_1(t) + x_2(t)(1 - x_1^2(t)) \end{bmatrix}.$$

This system has a 2D attractor which is a limit cycle shown in blue.

Unlike the Lorenz attractor this attractor is non-chaotic as the solution map is recurrent.

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Reachable Sets

Consider the ODE,

$$\begin{array}{|l|}\hline \dot{x}(t) = f(x(t), \mathbf{u}(t)) \\ \text{Given} \quad x(0) = x_0. \\ \hline \end{array} \qquad (7)$$

Where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $\mathbf{u}(t) : \mathbb{R} \to \mathbb{R}^m$, and $x_0 \in \mathbb{R}^n$.

The Solution Map

We say $\phi_f(x_0, t, \mathbf{u})$ is the solution map of the ODE (7) if,

$$\frac{\delta}{\delta t}\phi_f(x, t, \mathbf{u}) = f(\phi_f(x, t, \mathbf{u}), , \mathbf{u}(t))$$
$$\phi_f(x, 0, \mathbf{u}) = x.$$

The Forward Reachable Set

The forward reachable set for the ODE (7) is defined by,

$$FR_f(X_0, Y, S) := \{y \in \mathbb{R}^n \; : \; \exists x \in X_0, \mathbf{u} : \mathbb{R} \to \mathbb{R}^m, \text{and } t \in S$$
$$\text{such that } \phi_f(x, t, \mathbf{u}) = y \text{ and } \mathbf{u}(s) \in Y \quad \forall s \in [0, t]\}.$$

$$\min_{X \in S^n_{++}} \{-\log \det X\}$$

subject to: $Y \subseteq \{x \in \mathbb{R}^n : z_d(x)^T X z_d(x) < 1\}$

Lyapunov like theorem

Consider some ODE of the $\dot{x}(t) = f(x(t), \mathbf{u}(t))$.
For some $X_0 \subset \mathbb{R}^n$, $T > 0$, $Y \subset \mathbb{R}^{m_u}$, and $\gamma \geq 0$, suppose there exists a function $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that

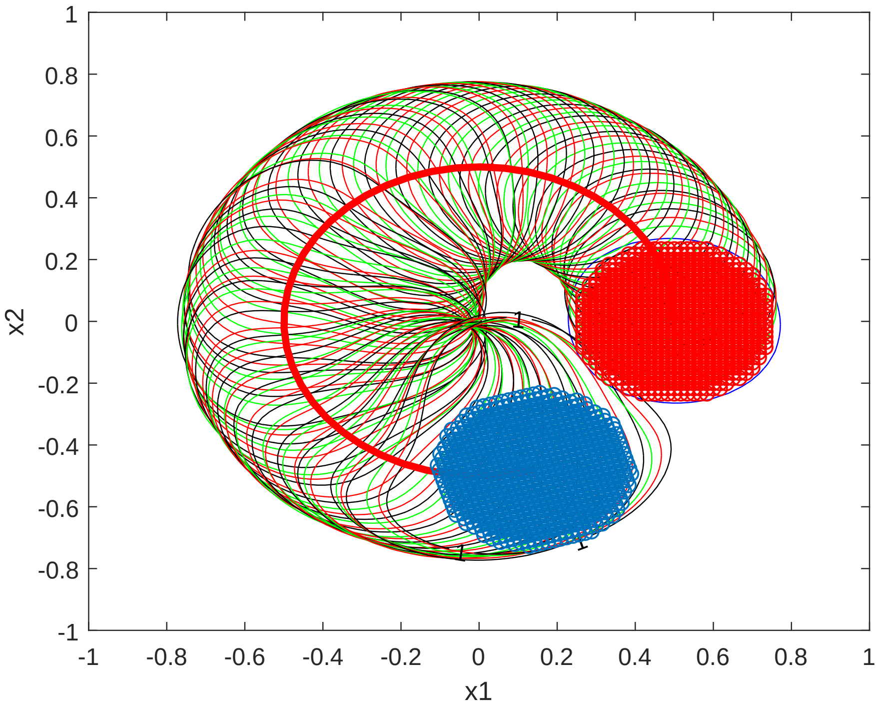
$$V(x,0) \leq 1 \text{ for all } x \in X_0. \quad (8)$$

$$\frac{\partial V}{\partial t}(x,t) + \nabla_x V(x,t)^T f(x,u) \leq 0, \quad (9)$$

$$\text{for all } x \in X_c, t \in [0,T], u \in Y.$$

Then $FR_f(X_0, Y, S) \subseteq \{x \in \mathbb{R}^n : V(x,T) \leq 1 + \gamma\}$.

Sublevel Set Approximation of Reachable Sets



Consider the linear system:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \qquad (10)$$

System has eigenvalues $\pm i$ so produces circular trajectories.

We can also find reachable sets of nonlinear systems in higher dimensions!

Thanks for your time!