Optimal Control Strategies for Systems with Input Delay using the PIE Framework

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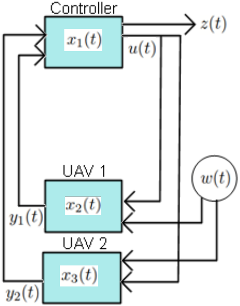
Networks: Delays in communication channels



Types of Delay: State delay (τ), input delay (h_i).

$$\begin{aligned} \dot{x}_i(t) &= a_i x_i(t) + b_{1i} w(t) + b_{2i} u(t - h_i) \\ z(t) &= C_1 x(t) + D_{12} u(t) \\ y_i(t) &= c_{2i} x_i(t - \tau_i) + d_{21i} w(t - \tau_i). \end{aligned}$$

Question: How to perform optimal control with communication delay?

INPUT DELAY: PROBLEM FORMULATION

Optimal Static State Feedback Problem with Input Delay

$$\begin{bmatrix}\dot{x}(t)\\ z(t)\end{bmatrix} = \begin{bmatrix}A_0 & B_1 & B_2\\ C_{10} & D_{11} & D_{12}\end{bmatrix}\begin{bmatrix}x(t)\\ w(t)\\ u(t)\end{bmatrix} + \begin{bmatrix}B_{2d}\\ D_{12d}\end{bmatrix} u(t-\tau)$$

Signals:

- The present state $x(t) \in \mathbb{R}^n$
- The disturbance or exogenous input, $w(t) \in \mathbb{R}^m$
- The controlled input, $u(t) \in \mathbb{R}^p$
- The regulated output, $z(t) \in \mathbb{R}^q$

Sources of Delay:

- Input Delay: $u(t-\tau)$

Not included:

- State delay: $x(t-\tau)$
- Disturbance Delay: $w(t-\tau)$

Static state-feedback problem: $\min\limits_{K_1,K_2} \sup\limits_{w \in L_2} \frac{\|z\|_{L_2}}{\|w\|_{L_2}}$ where

$$u(t) = K_1 x(t) + \int_{-\tau}^{0} K_2(s) \underbrace{\partial_s x(t+s)}_{\dot{x}(t+s)} ds$$

This includes controllers of the form

$$u(t) = K_1 x(t) + K_2 x(t-\tau) + \int_{-\tau}^{0} K_3(s) x(t+s) ds$$

Since $x(t+s) = x(t) - \int_s^0 \partial_s x(t+\theta) d\theta$ and hence $x(t-\tau) = x(t) - \int_{-\tau}^0 \partial_s x(t+s) ds$.

PARTIAL INTEGRAL EQUATIONs (PIEs)

ODE-PDE Representation of the Model

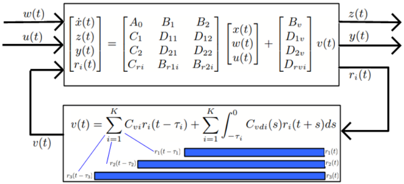


Figure: DDF to be converted

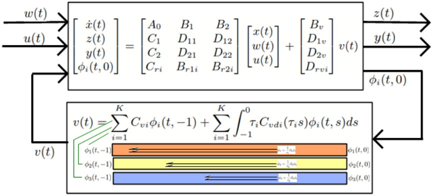


Figure: Equivalent ODE-PDE format

The ODE-PDE Representation:

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A_0 & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} v(t) \qquad \phi_i(t,0) = u(t)$$

$$\dot{\phi}(t,s) = \frac{1}{\tau}\phi_s(t,s), \qquad v(t) = \phi(t,-1)$$

- ϕ represents a pipe of length 1 with flow rate $\frac{1}{\tau}$, so $\phi(t,-1) = u(t-\tau)$.
- The conversion to ODE-PDE is otherwise trivial.

ODE-PDE System to PIE System

Ignoring Disturbances and Outputs for now

ODE Subsystem:

$$\dot{x}(t) = \begin{bmatrix} A_0 & B_2 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + B_{2d} v(t)$$

PDE Subsystem:

$$\phi(t,0) = u(t) \qquad \dot{\phi}(t,s) = \frac{1}{\tau}\phi_s(t,s) \qquad v(t) = \phi(t,-1)$$

Variable Substitution: $\phi \leftrightarrow \phi_s$

$$\phi(t,s) = \phi(t,0) - \int_s^0 \phi_s(t,\theta)d\theta = u(t) - \int_s^0 \phi_s(t,\theta)d\theta$$

Equivalent Partial Integral Subsystem: No boundary condition needed

$$\dot{u}(t) - \int_s^0 \dot{\phi}_s(t,\theta)d\theta = \frac{1}{\tau}\phi_s(t,s) \qquad v(t) = u(t) - \int_{-\tau}^0 \phi_s(t,\theta)d\theta$$

Equivalent Partial Integral System:

$$\underbrace{\begin{bmatrix} I & 0 \\ 0 & -\int_s^0 \end{bmatrix}}_{\mathcal{T}} \underbrace{\begin{bmatrix} \dot{x}(t) \\ \dot{\phi}_s(t,\cdot) \end{bmatrix}}_{\dot{\mathbf{x}}(t)} + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{\mathcal{T}_u} \dot{u}(t) = \underbrace{\begin{bmatrix} A_0 & -B_{2d}\int_{-1}^0 \\ 0 & I_\tau \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{bmatrix} x(t) \\ \phi_s(t,\cdot) \end{bmatrix}}_{\mathbf{x}(t)} + \underbrace{\begin{bmatrix} B_2 + B_{2d} \\ 0 \end{bmatrix}}_{\mathcal{B}_2} u(t)$$

The *-Algebra of Partial Integral (PI) Operators

$$\underbrace{\begin{bmatrix} I & 0 \\ 0 & -\int_s^0 \end{bmatrix}}_{\mathcal{T}\in\Pi_4} \underbrace{\begin{bmatrix} \dot{x}(t) \\ \dot{\phi}_s(t,\cdot) \end{bmatrix}}_{\dot{\mathbf{x}}(t)} + \underbrace{\begin{bmatrix} 0 & \emptyset \\ I & \emptyset \end{bmatrix}}_{\mathcal{T}_u\in\Pi_4} \dot{u}(t) = \underbrace{\begin{bmatrix} A_0 & -B_{2d}\int_{-1}^0 \\ 0 & I_\tau \end{bmatrix}}_{\mathcal{A}\in\Pi_4} \underbrace{\begin{bmatrix} x(t) \\ \phi_s(t,\cdot) \end{bmatrix}}_{\mathbf{x}(t)} + \underbrace{\begin{bmatrix} B_2 + B_{2d} & \emptyset \\ 0 & \emptyset \end{bmatrix}}_{\mathcal{B}_2\in\Pi_4} u(t)$$

Definition of a 4-PI Operator (Π_4) $\left(\mathcal{P}\left[\begin{smallmatrix} P, & Q_1 \\ Q_2, & \{R_i\} \end{smallmatrix}\right]\right)$**:** $\mathbb{R}\times L_2 \to \mathbb{R}\times L_2$

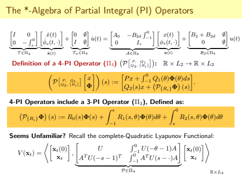
$$\left(\mathcal{P}\left[\begin{smallmatrix} P, & Q_1 \\ Q_2, & \{R_i\} \end{smallmatrix}\right]\begin{bmatrix} x \\ \mathbf{\Phi} \end{bmatrix}\right)(s) := \begin{bmatrix} Px + \int_{-1}^0 Q_1(\theta)\mathbf{\Phi}(\theta)ds \\ Q_2(s)x + \left(\mathcal{P}_{\{R_i\}}\mathbf{\Phi}\right)(s) \end{bmatrix}.$$

4-PI Operators include a 3-PI Operator (Π_3), Defined as:

$$\left(\mathcal{P}_{\{R_i\}}\mathbf{\Phi}\right)(s) := R_0(s)\mathbf{\Phi}(s) + \int_{-1}^s R_1(s,\theta)\mathbf{\Phi}(\theta)d\theta + \int_s^0 R_2(s,\theta)\mathbf{\Phi}(\theta)d\theta$$

Seems Unfamiliar? Recall the complete-Quadratic Lyapunov Functional:

$$V(\mathbf{x}_t) = \left\langle \begin{bmatrix} \mathbf{x}_t(0) \\ \mathbf{x}_t \end{bmatrix}, \underbrace{\begin{bmatrix} U & \int_{-1}^0 U(-\theta-1)A \\ A^T U(-s-1)^T & \int_{-1}^0 A^T U(s-\cdot)A \end{bmatrix}}_{\mathcal{P}\in\Pi_4} \begin{bmatrix} \mathbf{x}_t(0) \\ \mathbf{x}_t \end{bmatrix} \right\rangle_{\mathbb{R}\times L_2}$$



You may also recall the derivative also has this form. If

$$V(\mathbf{x}_t) = \left\langle \begin{bmatrix} \mathbf{x}_t(0) \\ \mathbf{x}_t \end{bmatrix}, \underbrace{\begin{bmatrix} M_{11} & \int_{-1}^0 M_{12}(\cdot) \\ M_{12}^T(s) & M_{22}(s) + \int_{-1}^0 N(s,\cdot) \end{bmatrix}}_{\mathcal{P}\in\Pi_4} \begin{bmatrix} \mathbf{x}_t(0) \\ \mathbf{x}_t \end{bmatrix} \right\rangle_{\mathbb{R}\times L_2}$$

then

$$\dot V(\mathbf{x}_t) = \left\langle \begin{bmatrix} \begin{bmatrix} \mathbf{x}_t(0) \\ \mathbf{x}_t(-1) \end{bmatrix} \\ \mathbf{x}_t \end{bmatrix}, \underbrace{\begin{bmatrix} D_{11} & \int_{-1}^0 D_{12}(s) \\ D_{12}(s)^T & -\dot M_{22}(s) - \int_{-1}^0 (\partial_s + \partial_\theta) N(s,\cdot) \end{bmatrix}}_{\mathcal{D}\in\Pi_4} \begin{bmatrix} \begin{bmatrix} \mathbf{x}_t(0) \\ \mathbf{x}_t(-1) \end{bmatrix} \\ \mathbf{x}_t \end{bmatrix} \right\rangle$$

where

$$D_{11} := \begin{bmatrix} G_{11} + G_{11}^T & G_{12} \\ G_{12}^T & -M_{22}(-1) \end{bmatrix}, \quad D_{12}(s) := \begin{bmatrix} A_0^T M_{12}(s) - \dot M_{12}(s) + N(0,s) \\ A_{1d}^T M_{12}(s) - N(-1,s) \end{bmatrix},$$

$$G_{11} = M_{11}A_0 + M_{12}(0) + \frac{1}{2}M_{22}(0) \qquad G_{12} = M_{11}A_{d1} - M_{12}(-1)$$

PIE Representation with Input Delay

Original System:

$$\begin{bmatrix}\dot{x}(t)\\ z(t)\end{bmatrix} = \begin{bmatrix}A_0 & B_1 & B_2\\ C_{10} & D_{11} & D_{12}\end{bmatrix}\begin{bmatrix}x(t)\\ w(t)\\ u(t)\end{bmatrix} + \begin{bmatrix}B_{2d}\\ D_{12d}\end{bmatrix} u(t-\tau)$$

The PIE version of the DDF system model (w/ input delay) is:

$$\begin{aligned}\mathcal{T}\dot{\mathbf{x}}(t) + \mathcal{T}_u\dot{u}(t) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t)\\ z(t) &= \mathcal{C}_1\mathbf{x}(t) + \mathcal{D}_{11}w(t) + \mathcal{D}_{12}u(t),\\ \mathbf{x}(t) &= \begin{bmatrix}x(t)\\ \partial_s\phi(t,s)\end{bmatrix}\end{aligned}$$

Partial Integral (PI) Parameters:

$$\mathcal{A} = \mathcal{P}\begin{bmatrix}A_0, & -B_{2d}\\ 0, & \left\{\frac{1}{\tau},0,0\right\}\end{bmatrix}, \qquad \mathcal{T} = \mathcal{P}\begin{bmatrix}I, & 0\\ 0, & \{0,0,-I\}\end{bmatrix}, \qquad \mathcal{T}_u = \mathcal{P}\begin{bmatrix}0, & \emptyset\\ I, & \{\emptyset\}\end{bmatrix}$$

$$\mathcal{B}_1 = \mathcal{P}\begin{bmatrix}B_1, & \emptyset\\ 0, & \{\emptyset\}\end{bmatrix}, \qquad \mathcal{B}_2 = \mathcal{P}\begin{bmatrix}B_2+B_{2d}, & \emptyset\\ 0, & \{\emptyset\}\end{bmatrix}, \qquad \mathcal{C}_1 = \mathcal{P}\begin{bmatrix}C_{10}, & -D_{12d}\\ \emptyset, & \{\emptyset\}\end{bmatrix},$$

$$\mathcal{D}_{11} = \mathcal{P}\begin{bmatrix}D_{11}, & \emptyset\\ \emptyset, & \{\emptyset\}\end{bmatrix}, \qquad \mathcal{D}_{12} = \mathcal{P}\begin{bmatrix}D_{12}+D_{12d}, & \emptyset\\ \emptyset, & \{\emptyset\}\end{bmatrix}, \qquad .$$

Duality and Optimal Control of PIEs

A Strong Duality Theorem for PIEs ($\mathcal{T}_u = 0$)

(A) Primal PIE:

$$\begin{aligned}\mathcal{T}\dot{\mathbf{x}}(t) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}w(t)\\ z(t) &= \mathcal{C}\mathbf{x}(t) + \mathcal{D}w(t)\end{aligned}$$

(B) Dual PIE/ Adjoint PIE:

$$\begin{aligned}\mathcal{T}^*\dot{\bar{\mathbf{x}}}(t) &= \mathcal{A}^*\bar{\mathbf{x}}(t) + \mathcal{C}^*\bar{w}(t)\\ \bar{z}(t) &= \mathcal{B}^*\bar{\mathbf{x}}(t) + \mathcal{D}^*\bar{w}(t)\end{aligned}$$

For a PIE and its Dual:

1. stability is equivalent; (A) is stable iff (B) is stable
2. L_2-gain is equivalent; $\gamma = \sup_{w \neq 0} \frac{\|z\|}{\|w\|} = \sup_{\bar{w} \neq 0} \frac{\|\bar{z}\|}{\|\bar{w}\|}$

H_∞-optimal static state-feedback control (no input delay*)

Partial Integral Equation Representation:

$$\mathcal{T}\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t), \qquad \mathbf{v}(0) = 0$$
$$z(t) = \mathcal{C}_1\mathbf{x}(t) + \mathcal{D}_{11} w(t) + \mathcal{D}_{12} u(t), \qquad u(t) = \mathcal{Z}\mathcal{P}^{-1}\mathbf{x}(t)$$

Theorem 1 (H_∞ optimal control, no input delay ($\mathcal{T}_u = 0$)).

Decision Variables: $\gamma, \mathcal{P}, \mathcal{Z}$

Optimization problem: $\min_{\gamma,\mathcal{Z},\mathcal{P}\succ 0} \gamma$

$$\begin{bmatrix} -\gamma I & \mathcal{D}_{11} & (\mathcal{C}_1\mathcal{P} + \mathcal{Z}\mathcal{D}_{12}\mathcal{Z})\mathcal{T}^* \\ \mathcal{D}_{11}^* & -\gamma I & \mathcal{B}_1^* \\ \mathcal{T}(\mathcal{C}_1\mathcal{P} + \mathcal{Z}\mathcal{D}_{12}\mathcal{Z})^* & \mathcal{B}_1 & \mathcal{T}\left(\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{Z}\right)^* + \left(\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{Z}\right)\mathcal{T}^* \end{bmatrix} \preccurlyeq 0.$$

Then $\|y\|_{L_2} \le \gamma\|\omega\|_{L_2}$.

- We have cast the optimal control problem as a Linear Operator Inequality.
- No source of conservatism

Problem: We have input delay! $\Rightarrow \mathcal{T}_u \neq 0$

Optimal Control with Input Delay: Young's Inequality

$$\mathcal{T}\dot{\mathbf{x}}(t) + \mathcal{T}_u\dot{u}(t) = \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t), \qquad \mathbf{v}(0) = 0$$
$$z(t) = \mathcal{C}_1\mathbf{x}(t) + \mathcal{D}_{11} w(t) + \mathcal{D}_{12} u(t), \qquad u(t) = \mathcal{Z}\mathcal{P}^{-1}\mathbf{x}(t) \quad (1)$$

Theorem 2 (H_∞ optimal control ($\mathcal{T}_u \neq 0$)).

Decision Variables: $\gamma, \mathcal{P}, \mathcal{Z}$

Optimization problem: $\min_{\gamma,\mathcal{Z},\mathcal{P}\succ 0} \gamma$

$$\begin{bmatrix} -\gamma & 0 & 0 & 0 & \mathcal{D}_1^* & \mathcal{B}_1^* \\ 0 & -\mathcal{P} & 0 & 0 & (\mathcal{D}_{12}\mathcal{Z})^* & 0 \\ 0 & 0 & -\mathcal{P} & 0 & 0 & \sqrt{2}(\mathcal{T}_u\mathcal{Z})^* \\ 0 & 0 & 0 & -\mathcal{P} & 0 & (\mathcal{B}_2\mathcal{Z})^* \\ \mathcal{D}_1 & \mathcal{D}_{12}\mathcal{Z} & 0 & 0 & -\gamma & \mathcal{H}_{12} \\ \mathcal{B}_1 & 0 & \sqrt{2}\mathcal{T}_u\mathcal{Z} & \mathcal{B}_2\mathcal{Z} & \mathcal{H}_{12}^* & \mathcal{H}_{22} \end{bmatrix} \preceq 0$$

$$\mathcal{H}_{12} = \mathcal{C}_1\mathcal{P}\mathcal{T}^* + \mathcal{C}_1\mathcal{Z}^*\mathcal{T}_u^* + \mathcal{D}_{12}\mathcal{Z}\mathcal{T}^*$$
$$\mathcal{H}_{22} = (\mathcal{T}\mathcal{P}\mathcal{A}^* + +\mathcal{T}_u\mathcal{Z}\mathcal{A}^* + \mathcal{T}\mathcal{Z}^*\mathcal{B}_2^*) + (\mathcal{T}\mathcal{P}\mathcal{A}^* + +\mathcal{T}_u\mathcal{Z}\mathcal{A}^* + \mathcal{T}\mathcal{Z}^*\mathcal{B}_2^*)^*$$

Then $\|y\|_{L_2} \leq \gamma\|\omega\|_{L_2}$.

Don't like Conservatism?
FILTER THE INPUT

Alternative Approach: Filter the Input

Filter Dynamics:

$$\boxed{\dot{x}_c(t) = -Rx_c(t) + Lu(t)}$$

Filter Output becomes the Input:

$$\boxed{\begin{bmatrix} \dot{x}_c(t) \\ \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -R & 0 & 0 & L \\ B_2 & A_0 & B_1 & 0 \\ D_{12} & C_{10} & D_{11} & 0 \end{bmatrix} \begin{bmatrix} x_c(t) \\ x(t) \\ w(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_{2d} \\ D_{12d} \end{bmatrix} x_c(t-\tau)}$$

Question: What is the effect of filtering the input?

$$\hat{G}(s) := \frac{\hat{x}_c(s)}{\hat{u}(s)} = \frac{L}{s+R}$$

- If $L = R$, then $\|\hat{G}\|_{H_\infty} = \sup_u \frac{\|x_c\|_{L_2}}{\|u\|_{L_2}} = 1$.
- If the closed-loop is stable and $\lim_{t\to\infty} u(t) = \lim_{t\to\infty} x_c(t)$.

Filtered ODE-PDE System to PIE System

Ignoring Disturbances and Outputs for now

ODE Subsystem:

$$\begin{bmatrix}\dot{x}_c(t)\\ \dot{x}(t)\end{bmatrix} = \begin{bmatrix}-R & 0 & L\\ B_2 & A_0 & 0\end{bmatrix}\begin{bmatrix}x_c(t)\\ x(t)\\ u(t)\end{bmatrix} + \begin{bmatrix}0\\ B_{2d}\end{bmatrix} v(t)$$

PDE Subsystem:

$$\phi(t,0) = x_c(t) \qquad \dot{\phi}(t,s) = \frac{1}{\tau}\phi_s(t,s) \qquad v(t) = \phi(t,-1)$$

Variable Substitution: $\phi \leftrightarrow \phi_s$

$$\phi(t,s) = \phi(t,0) - \int_s^0 \phi_s(t,\theta)d\theta = x_c(t) - \int_s^0 \phi_s(t,\theta)d\theta$$

Equivalent Partial Integral Subsystem: No boundary condition needed

$$\dot{x}_c(t) - \int_s^0 \dot{\phi}_s(t,\theta)d\theta = \frac{1}{\tau}\phi_s(t,s) \qquad v(t) = x_c(t) - \int_{-\tau}^0 \phi_s(t,\theta)d\theta$$

Equivalent Partial Integral Equation System:

$$\underbrace{\begin{bmatrix}\begin{bmatrix}I & 0\\ 0 & I\end{bmatrix} & \begin{bmatrix}0\\ 0\end{bmatrix}\\ \begin{bmatrix}I & 0\end{bmatrix} & -\int_s^0\end{bmatrix}}_{\mathcal{T}}\underbrace{\begin{bmatrix}\dot{x}_c(t)\\ \dot{x}(t)\\ \dot{\phi}_s(t,\cdot)\end{bmatrix}}_{\dot{\mathbf{x}}(t)} = \underbrace{\begin{bmatrix}\begin{bmatrix}-R & 0\\ B_2 + B_{2d} & A_0\end{bmatrix} & -\begin{bmatrix}0\\ B_{2d}\end{bmatrix}\int_{-1}^0\\ \begin{bmatrix}0 & 0\end{bmatrix} & 1/\tau\end{bmatrix}}_{\mathcal{A}}\underbrace{\begin{bmatrix}x_c(t)\\ x(t)\\ \phi_s(t,\cdot)\end{bmatrix}}_{\mathbf{x}(t)} + \underbrace{\begin{bmatrix}\begin{bmatrix}L\\ 0\end{bmatrix}\\ 0\end{bmatrix}}_{\mathcal{B}_2} u(t)$$

PIE Representation with Filtered Input Delay

$$\begin{bmatrix} \dot{x}_c(t) \\ \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -R & 0 & 0 & L \\ B_2 & A_0 & B_1 & 0 \\ D_{12} & C_{10} & D_{11} & 0 \end{bmatrix} \begin{bmatrix} x_c(t) \\ x(t) \\ w(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_{2d} \\ D_{12d} \end{bmatrix} x_c(t-\tau)$$

The PIE version of the DDF system model (w/ filtering) is:

$$\begin{aligned} \mathcal{T}\dot{\mathbf{x}}(t) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t) \\ z(t) &= \mathcal{C}_1 \mathbf{x}(t) + \mathcal{D}_{11} w(t) + \mathcal{D}_{12} u(t), \quad \mathbf{x}(t) = \begin{bmatrix} x_c(t) \\ x(t) \\ \partial_s \phi(t,s) \end{bmatrix} \end{aligned}$$

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where $\{\mathcal{T}, \mathcal{A}, \cdots, \mathcal{D}_{22}\} \subset \Pi_4$ are given by:

$$\mathcal{A} = \mathcal{P}\left[\begin{smallmatrix} \begin{bmatrix} -R & 0 \\ B_2 + B_{2d} & A_0 \end{bmatrix}, & -\begin{bmatrix} 0 \\ B_{2d} \end{bmatrix} \\ 0, & \{1/\tau, 0, 0\} \end{smallmatrix}\right], \qquad \mathcal{T} = \mathcal{P}\left[\begin{smallmatrix} I, & 0 \\ [I \;\; 0], & \{0, 0, -I\} \end{smallmatrix}\right],$$

$$\mathcal{B}_1 = \mathcal{P}\left[\begin{smallmatrix} \begin{bmatrix} 0 \\ B_1 \end{bmatrix}, & \emptyset \\ 0, & \{\emptyset\} \end{smallmatrix}\right], \qquad \mathcal{B}_2 = \mathcal{P}\left[\begin{smallmatrix} \begin{bmatrix} L \\ 0 \end{bmatrix}, & \emptyset \\ 0, & \{\emptyset\} \end{smallmatrix}\right],$$

$$\mathcal{C}_1 = \mathcal{P}\left[\begin{smallmatrix} [D_{12d} \;\; C_{10}], & -D_{12d} \\ \emptyset, & \{\emptyset\} \end{smallmatrix}\right] \qquad \mathcal{D}_{11} = \mathcal{P}\left[\begin{smallmatrix} D_{11}, & \emptyset \\ \emptyset, & \{\emptyset\} \end{smallmatrix}\right], \qquad \mathcal{D}_{12} = \mathcal{P}\left[\begin{smallmatrix} D_{12}, & \emptyset \\ \emptyset, & \{\emptyset\} \end{smallmatrix}\right],$$

Chili Cook-Off: Young's Inequality or Filtering?

Numerical Examples 1 and 2

Numerical Example 1: This system is open-loop stable.

$$\dot{x}(t)=\begin{bmatrix}0 & 1\\ -1.25 & -3\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}w(t)+\begin{bmatrix}0\\1\end{bmatrix}u(t-\tau) \qquad z(t)=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}x(t)+\begin{bmatrix}0\\.1\end{bmatrix}u(t)$$

Table: Closed Loop H_∞ gain of Example 1.

$\tau \rightarrow$	1	2	3
$\gamma_{\min}$ w/o filter	.3286	.3333	.3333
$\gamma_{\min}$ w filter	.2718	.3103	.3270

Note: The only case where filtering always beat sub-optimal controllers.

Numerical Example 2: This system is open-loop neutrally stable.

$$\dot{x}(t)=\begin{bmatrix}0 & 0\\ 1 & -5\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}w(t)+\begin{bmatrix}1\\0\end{bmatrix}u(t-\tau) \qquad z(t)=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}x(t)+\begin{bmatrix}0\\.1\end{bmatrix}u(t)$$

Table: Closed Loop H_∞ gain of Example 2.

$\tau \rightarrow$	1	2	3
$\gamma_{\min}$ w/o filter	1.0797	.4933	.4736
$\gamma_{\min}$ w filter	.2361	.4544	.6481

This example is interesting in that the suboptimal controller performs better at higher delay and for $\tau = 3$ outperforms the filtered controller.

Numerical Examples 1 and 2

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$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 1 & -5 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t-\tau) \qquad z(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ .1 \end{bmatrix} u(t)$$

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This example is interesting in that the suboptimal controller performs better at higher delay and for $\tau = 3$ outperforms the filtered controller.

For Example 2, in [Li, 1999] a robust controller was found with $\gamma = 1.56$ for $\tau = .24$ (a result with no weighting on the control effort). The corresponding gain with input-delayed controller was $\gamma = .0891$ and with filter was $\gamma = .0357$.

Numerical Examples 3 and 4

Numerical Example 3 This system is open-loop unstable.

$$\dot{x}(t) = \begin{bmatrix} -0.8 & -0.01 \\ 1 & 0.1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} 0.4 \\ 0.1 \end{bmatrix} u(t-\tau) \qquad z(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ .1 \end{bmatrix} u(t)$$

Table: Closed Loop H_∞ gain of Example 3.

$\tau \rightarrow$	1	2	3
$\gamma_{\min}$ w/o filter	.7372	1.683	2.6044
$\gamma_{\min}$ w filter	.7924	2.1811	3.9598

Numerical Example 4 This sytem has 6 states and is open-loop stable.

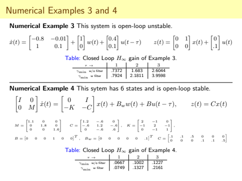
$$\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & I \\ -K & -C \end{bmatrix} x(t) + B_w w(t) + Bu(t-\tau), \qquad z(t) = Cx(t)$$

$$M = \begin{bmatrix} 1.1 & 0 & 0 \\ 0 & 1.8 & 0 \\ 0 & 0 & 1.6 \end{bmatrix} \quad C = \begin{bmatrix} 1.2 & -.6 & 0 \\ -.6 & 1.2 & -.6 \\ 0 & -.6 & .6 \end{bmatrix}, \quad K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T, \quad B_w = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & .1 \end{bmatrix}^T \quad C = \begin{bmatrix} .1 & .1 & .5 & 0 & 0 & 0 \\ 0 & 0 & 0 & .1 & .1 & .5 \end{bmatrix}$$

Table: Closed Loop H_∞ gain of Example 4.

$\tau \rightarrow$	1	2	3
$\gamma_{\min}$ w/o filter	.0667	.1002	.1227
$\gamma_{\min}$ w filter	.0749	.1327	.2161



Numerical Examples 3 and 4

Numerical Example 3 This system is open-loop unstable.

$$\dot{x}(t) = \begin{bmatrix} -0.8 & -0.01 \\ 1 & 0.1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} 0.4 \\ 0.1 \end{bmatrix} u(t-\tau) \qquad z(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ .1 \end{bmatrix} u(t)$$

Table: Closed Loop H_∞ gain of Example 3.

$\tau \rightarrow$	1	2	3
$\gamma_{\min}$ w/o filter	.7372	1.683	2.6044
$\gamma_{\min}$ w filter	.7924	2.1811	3.9598

Numerical Example 4 This sytem has 6 states and is open-loop stable.

$$\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & I \\ -K & -C \end{bmatrix} x(t) + B_w w(t) + Bu(t-\tau), \qquad z(t) = Cx(t)$$

$$M = \begin{bmatrix} 1.1 & 0 & 0 \\ 0 & 1.8 & 0 \\ 0 & 0 & 1.6 \end{bmatrix} \quad C = \begin{bmatrix} 1.2 & -.6 & 0 \\ -.6 & 1.2 & -.6 \\ 0 & -.6 & .6 \end{bmatrix}, \quad K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T, \quad B_w = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & .1 \end{bmatrix}^T \quad C = \begin{bmatrix} .1 & .1 & .5 & 0 & 0 & 0 \\ 0 & 0 & 0 & .1 & .1 & .5 \end{bmatrix}$$

Table: Closed Loop H_∞ gain of Example 4.

$\tau \rightarrow$	1	2	3
$\gamma_{\min}$ w/o filter	.0667	.1002	.1227
$\gamma_{\min}$ w filter	.0749	.1327	.2161

For Example 4, the minimum achievable closed-loop L_2-gain for 3 values of delay are listed in Table 4. For comparison, at $\tau = .15$, [Du, 2005] obtained an L_2-gain of .624 and for $\gamma = 1$ the maximum allowable delay was $.164$. Note that the very small closed-loop gains are partially a result of the failure to weight the control effort in the optimal control formulation.

Input Delays Make Optimal Control via PIEs Conservative

State delays are easier to handle

PIETOOLS 2022a Implementation:

- Combine Input-delays, state delays, process delays, PDEs
- See `http://control.asu.edu/pietools`
  - User manual, documentation, etc.
- Provides a standardized representation of DDEs/DDFs/PIEs
- A standardized input format
- More than 50 examples in the libraries

Advantages of Filtering:

- Reduces Noise
- No Conservatism

Advantages of Input Delay:

- Faster Responses
- Often better performance
- Not that conservative

Thank you for your attention

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