Constructing Lyapunov-Krasovskii Functionals For Linear Time Delay Systems

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***Abstract*— We present an algorithmic methodology for constructing Lyapunov-Krasovskii (L-K) functionals for linear time-delay systems, using the sum of squares decomposition of multivariate polynomials to solve the related infinite dimensional Linear Matrix Inequalities (LMIs). The resulting functionals retain the structure of the complete L-K functional and yield conditions that approach the true delay-dependent stability bounds. The method can also be used to construct parameter-dependent L-K functionals for certifying stability under parametric uncertainty.**

I. INTRODUCTION

Functional Differential Equations (FDEs) are an important modeling tool for systems involving transport and propagation of data or with aftereffect. Recently, research in the area [5], [8] has been intensified as the simplest adequate models for Internet congestion control schemes [9], [16] are in the form of nonlinear FDEs.

The presence of delays can have an effect on system stability and performance, so ignoring them may lead to design flaws and incorrect analysis conclusions. Stability is classified as delay-independent if it is retained irrespective of the size of the delays, and delay-dependent if it is lost at a certain delay value. In general the former condition is more conservative as in most cases bounds on the expected value of the delay exist.

The investigation of the stability properties of linear time-delay systems is usually performed using "frequency-domain" tests which are suitable for systems with a small number of heterogeneous delays. When there are many heterogeneous delays involved, an attractive alternative is the use of *time-domain* (Lyapunov-based) methodologies, which amount to constructing simple Lyapunov-Krasovskii (L-K) or Lyapunov-Razumikhin (L-R) certificates by solving an appropriate set of Linear Matrix Inequalities (LMIs) [6]. The stability conditions that can be obtained in this way are often conservative - even though the existence of complete quadratic L-K functionals necessary and sufficient for stability is well known, and so is their structure [4]. The reason is that use of the complete functional yields infinite dimensional LMI conditions that are difficult to verify algorithmically with current tools; researchers have concentrated on other structures that yield simple finite dimensional LMI conditions but which inevitably produce conservative conditions for stability [12]. A discretization scheme of the infinite dimensional LMIs, based on the complete Lyapunov functional was introduced by Gu [4], wherein the resulting sufficient conditions were written as a set of finite dimensional LMIs. This approach carries a high computational cost as delays closer to the stability boundary are tested. Moreover the method is quite complicated to set up and cannot be generalized to nonlinear time delay systems.

In this paper we investigate delay-dependent stability of linear time-delay systems with or without parametric uncertainty by solving the infinite dimensional LMIs that the complete L-K functional conditions produce algorithmically using the sum of squares (SOS) [13] methodology. The introduction of SOS techniques has paved the way for analysis of nonlinear systems with polynomial vector fields [11], and the construction of parameter-dependent Lyapunov functions for linear systems with parameters. For nonlinear time delay systems, this topic has been addressed in [10], [14]. The case of linear time-delay systems merits special consideration due to the existence of a known complete quadratic Lyapunov functional structure necessary and sufficient for stability. We show that the existence of a complete L-K functional is equivalent to feasibility of certain infinite dimensional LMIs which can be solved using the SOS approach with relatively little conservatism. By increasing the order of the polynomial variables, our estimate on the delay bound approaches the true value.

In section II, we will review the stability conditions for linear time-delay systems, and present the complete L-K functional. In section III we will give a brief introduction to the SOS decomposition and how it can be computed for multivariate polynomials. In section IV we will present the main result that can be used to test delay-dependent stability. In section V we will show how

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robustness analysis for linear time-delay systems with parametric uncertainty can be performed. We close the paper with some conclusions.

A. Notation

Notation is standard; $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. For $\eta \in [0, +\infty)$, $C = C([-\eta, 0], \mathbb{R}^n)$ denotes the Banach space of continuous functions defined on an interval $[-\eta, 0]$, taking values in $\mathbb{R}^n$ with the topology of uniform convergence, and with a norm $\|\phi\| = \max_{\theta\in[-\eta,0]} |\phi(\theta)|$, where $|\cdot|$ is an arbitrary norm in $\mathbb{R}^n$.

II. Problem statement and past results

Consider the following linear system with delayed states:

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t - \tau_i) \qquad (1)$$

with an initial condition $x(\theta) = \phi(\theta)$, $\theta \in [-\tau, 0]$, where $\tau = \max\{\tau_1, \ldots, \tau_k\}$, $\phi \in C$. The inhomogeneous delays τ_i are assumed to be constant, $x(t) \in \mathbb{R}^n$ and A_i, $i = 1, \ldots, k$ are known real constant matrices of appropriate dimensions. In a later section we will investigate the case in which the A_i are parameter-dependent. The following theorem (Lyapunov-Krasovskii) can be used to prove asymptotic stability of the above system.

Theorem 1: [4] The system described by Equation (1) is asymptotically stable if there exists a bounded quadratic Lyapunov functional $V(x_t)$ such that for some $\epsilon > 0$, it satisfies:

$$V(x_t) \geq \epsilon \|x_t(0)\|^2 \qquad (2)$$

and its derivative along the system trajectory satisfies

$$\dot{V}(x_t) \leq -\epsilon \|x_t(0)\|^2. \qquad (3)$$

As mentioned in the Introduction, we identify two types of stability: delay-independent and delay-dependent. In the first case the stability property is retained irrespective of the size of the delay, whereas in the second one the stability property is a function of the delay size, seen as a parameter.

This theorem is a natural extension of the Lyapunov theory for ordinary differential equations (ODEs) to systems described by FDEs. It respects the fact that the state is infinite dimensional and proposes a certificate for stability that is a functional rather than a function, which is the case for systems described by ODEs. It is well known that a Lyapunov function necessary and sufficient for stability of the generic linear ODE system of the form $\dot{x} = Ax$ is $V = x^T P x$ where P is a positive definite matrix satisfying $A^T P + PA < -Q$ for some positive definite matrix Q. Given a particular $Q > 0$, the P that satisfies the above conditions is unique, and can be found numerically by solving an LMI, also known as a feasibility semidefinite program [2]. In the same spirit, the search for structures that are 'complete', i.e produce necessary and sufficient conditions for stability in the case of time-delay systems has produced some important results in the past few years. For the case of what is called strong delay-independent stability (for definitions and details see [1]), the class of such Lyapunov functions has been completely characterized [1].

Example 2: For system (1) with $k = 1$, a Lyapunov-Krasovskii candidate that would yield a delay-independent condition is

$$V(x_t) = x_t(0)^T P x_t(0) + \int_{-\tau}^{0} x_t(\theta)^T S x_t(\theta) d\theta$$

Sufficient conditions on $V(x_t)$ to be positive definite are $P > 0$, $S \geq 0$. For $\dot{V}(x_t) < 0$ we require

$$\begin{bmatrix} A^T P + PA + S & PA_1 \\ A_1^T P & -S \end{bmatrix} < 0,$$

i.e. the conditions for stability (see Theorem 1) can be written as a Linear Matrix Inequality (LMI) with P and S as the unknowns. The above structure nonetheless may not be adequate to prove delay-independent stability of a particular system. The structure presented in [1] would be the next choice.

As far as delay-dependent stability is concerned, the structure of the Lyapunov-Krasovskii functionals necessary and sufficient for delay-dependent stability are known, but difficult to construct. For this reason, researchers have concentrated their attention on finding structures for which an algorithmic approach can be used [6]. Inevitably the maximum delays that could be tested in this way were conservative.

The complete Lyapunov functional, which is known to be necessary and sufficient for delay-dependent stability of the linear system is known [4] and has the following form:

$$\begin{aligned} V(x_t) &= x_t^T(0) P x_t(0) + x_t^T(0) \int_{-\tau}^{0} P_1(\theta) x_t(\theta) d\theta \\ &+ \int_{-\tau}^{0} x_t^T(\theta) P_1^T(\theta) d\theta x_t(0) \\ &+ \int_{-\tau}^{0} \int_{-\tau}^{0} x_t^T(\theta) P_2(\theta, \xi) x_t(\xi) d\xi d\theta \\ &+ \int_{-\tau}^{0} x_t^T(\theta) Q x_t(\theta) d\theta \qquad (4) \end{aligned}$$

with appropriate continuity conditions on P, P_1 and P_2. The conditions $V > 0$ and $\dot{V} < 0$ can be thought of as infinite-dimensional versions of the standard finite dimensional LMI problem where $x_t(\theta)$ is now the state at time t and summations have been replaced by integrations. An approach to solve the infinite dimensional LMIs was considered in [4] by discretising the kernels, yielding a set of finite-dimensional LMIs with size dependent on the discretisation level: finer discretisation can test delay approaching the true bounds at the expense of increasing computational effort. Setting up the discretisation scheme in this algorithm is quite involved; moreover, it returns a complicated discretised certificate, and there exists no obvious way to generalise the procedure to non-linear time-delay systems, issues that the approach we propose in this paper can address.

In this paper, we consider a structure which is a slight generalization of the complete quadratic L-K functional (4). We construct certificates in which the kernels (matrices P_1, P_2 etc) are polynomials in the variables θ, ξ. As the order of the kernels is increased, the delay-dependent stability conditions obtained analytically (i.e. using frequency domain methods) can be approached. In order to solve the resulting infinite dimensional LMIs, we propose a method using the SOS decomposition of multivariate polynomials.

III. The Sum of Squares Decomposition

In this section we give a brief introduction to sum of squares (SOS) polynomials, their use, and how the existence of a SOS decomposition can be verified algorithmically. A more detailed description can be found in [13].

Definition 3: A multivariate polynomial $p(x)$, $x \in \mathbb{R}^n$ is a *Sum of Squares* if there exist polynomials $f_i(x)$, $i = 1, \dots, M$ such that $p(x) = \sum_{i=1}^{M} f_i^2(x)$.
An equivalent characterization of SOS polynomials is given in the following proposition.

Proposition 4: [13] A polynomial $p(x)$ of degree $2d$ is SOS if and only if there exists a positive semidefinite matrix Q and a vector $Z(x)$ containing monomials in x of degree $\leq d$ so that

$$p = Z(x)^T Q Z(x)$$

In general, the monomials in $Z(x)$ are not algebraically independent. Expanding $Z(x)^T Q Z(x)$ and equating the coefficients of the resulting monomials to the ones in $p(x)$, we obtain a set of affine relations in the elements of Q. Since $p(x)$ being SOS is equivalent to $Q \geq 0$, the problem of finding a Q which proves that $p(x)$ is an SOS can be cast as a semidefinite program (SDP) [13]. Therefore the problem of seeking a Q such that p is a SOS can be formulated as an LMI. Note that if a polynomial $p(x)$ is a SOS, then it is globally nonnegative. The converse is not always true: not all positive semi-definite polynomials can be written as SOS, apart from 3 special cases (see [13]) - in fact, testing global non-negativity of a polynomial $p(x)$ is known to be NP-hard when the degree of $p(x)$ is greater than 4 [7], whereas checking whether p can be written as a SOS is computationally tractable - it can be formulated as an SDP which has a worst-case polynomial-time complexity. The construction of the SDP related to the SOS conditions can be performed efficiently using SOSTOOLS [15], a software that formulates general SOS programmes as SDPs and calls semidefinite programming solvers to solve them.

If the monomials in the polynomial $p(x)$ have *unknown* coefficients then the search for feasible values of those coefficients such that $p(x)$ is nonnegative is also an SDP, a fact that is important for the construction of Lyapunov functions and other S-procedure type multipliers.

IV. Main Results

In this section we consider the system (1) with $k = 1$:

$$\dot{x} = A_0 x(t) + A_1 x(t - \tau) \triangleq f(x_t) \tag{5}$$

where $x(t) \in \mathbb{R}^n$ and $x_t = x(t + \theta)$, $\theta \in [-\tau, 0]$. The results we present here can be easily extended to the multiple delay case. We are interested in delay-dependent conditions for stability of this system.

Here we consider structures similar to the complete quadratic Lyapunov-Krasovskii functional (4) for which we construct certificates in which the kernels (matrices P_1, P_2 etc) are polynomials in the variables (θ, ξ). To proceed, we use Theorem 1 and consider the following candidate Lyapunov functional, which is a slight generalization of the complete Lyapunov functional (4):

$$\begin{aligned} V(x_t) &= a_0(x_t(0)) + \int_{-\tau}^{0} \int_{\theta}^{0} a_2(x_t(\zeta)) d\zeta d\theta \\ &+ \int_{-\tau}^{0} \int_{-\tau}^{0} a_1(\theta, \xi, x_t(0), x_t(\theta), x_t(\xi)) d\theta d\xi \\ &+ \int_{-\tau}^{0} \int_{\xi}^{0} a_3(x_t(\zeta)) d\zeta d\xi \end{aligned} \tag{6}$$

where the a_i are polynomials in the indicated variables with bound on the degree. The polynomials are restricted to be quadratic with respect to $x_t(0), x_t(\theta)$ and $x_t(\xi)$ and allowed to be any order with respect to variables θ and ξ. Such polynomials are called *bipartite* and their SOS decomposition has a special structure [3]. Writing the expression with a_1 as an infinite dimensional LMI, we have

$$\int_{-\tau}^{0}\int_{-\tau}^{0} a_1(\theta,\xi,x_t(0),x_t(\theta),x_t(\xi))d\theta d\xi$$
$$=\int_{-\tau}^{0}\int_{-\tau}^{0} x_t(\theta)^T\bar{a}_1(\theta,\xi)x_t(\xi)d\theta d\xi$$

where $\bar{a}_1$ may contain δ-functions. Positivity of this expression is an LMI in the state x_t with the integral taking the place of the summation and where $\bar{a}_1(i,j)$ would correspond to $\bar{A}_{i,j}$ for a finite-dimensional matrix A. In order to express sufficient conditions for the positivity of this LMI in terms of the polynomial a_1, we can rewrite a_1 as follows.

$$a_1(\theta,\xi,x_t(0),x_t(\theta),x_t(\xi))$$
$$=\begin{bmatrix} x_t(0)\\ x_t(\theta)\\ x_t(\xi)\end{bmatrix}^T \begin{bmatrix} & & \\ & \tilde{a}_1(\theta,\xi) & \\ & & \end{bmatrix}\begin{bmatrix} x_t(0)\\ x_t(\theta)\\ x_t(\xi)\end{bmatrix}$$

where $\tilde{a}_1(\theta,\xi)$ is a polynomial matrix in (θ,ξ). Since a_1 is now expressed as a quadratic form with kernel $\tilde{a}_1$, pointwise positivity of $\tilde{a}_1$ will now be sufficient for positivity of the expression. Now denote $x_d = x_t(-\tau)$, $y = x_t(\theta)$, $z = x_t(\xi)$ and $x = x_t(0)$ for brevity. The time derivative of $V(x_t)$ along f given by (5) is:

$$\tau^2\dot{V}(x_t)=\int_{-\tau}^{0}\int_{-\tau}^{0}\begin{pmatrix} \frac{da_0}{dx}+\tau^2\left(\frac{\partial a_1}{\partial x}f-\frac{\partial a_1}{\partial\theta}-\frac{\partial a_1}{\partial\xi}\right)\\ +\tau a_1(0,\xi,x,x,z)+\tau a_2(x)\\ -\tau a_1(-\tau,\xi,x,x_d,z)-\tau a_2(y)\\ +\tau a_1(\theta,0,x,y,x)+\tau a_3(x)\\ -\tau a_1(\theta,-\tau,x,y,x_d)-\tau a_3(z)\end{pmatrix}d\theta d\xi$$

The kernel of this expression is also quadratic in variables x, x_d, y and z and can be written similarly to a_1. The conditions of positive definiteness of V and negative definiteness of $\dot{V}$ are infinite dimensional LMIs. To create sufficient conditions for feasibility, we express the LMIs using a quadratic form with kernel similar to $\tilde{a}_1$. One can impose positivity and negativity conditions on these *kernels* for all θ and ξ. Positivity of this kernel implies positivity of the quadratic form which implies positivity of the integral. However, enforcing this pointwise positivity condition can be conservative. Later on, we will show how to reduce the conservativeness through the use of special functions.

By structuring the polynomials and testing positivity of V and negativity of $\dot{V}$ as explained above, it is easy to see that the resulting sufficient conditions will be parameterized finite dimensional LMIs in (θ,ξ). However, for notational simplicity we will be working at the polynomial level - we multiply out the quadratic form and search for a polynomial certificate using the bipartite structure of the resulting expression to simplify the search. Sufficient conditions for stability of the system can be found in the following proposition:

Proposition 5: Consider the system given by Equation (5). Suppose we can find polynomials $a_0(x)$ $a_1(\theta,\xi,x,y,z)$, $a_2(x_t(\zeta))$ and $a_3(x_t(\zeta))$ and a positive constant ϵ such that the following conditions hold:

1) $a_0(x)-\epsilon\|x\|^2\geq 0$,
2) $a_1(\theta,\xi,x,y,z)\geq 0$, $\forall\theta,\xi\in[-\tau,0]$,
3) $a_2(x_t(\zeta))\geq 0$, $a_3(x_t(\zeta))\geq 0$,
4) $\frac{da_0}{dx}f+\tau^2\frac{\partial a_1}{\partial x}f-\tau^2\frac{\partial a_1}{\partial\theta}-\tau^2\frac{\partial a_1}{\partial\xi}+\tau a_2(x)-\tau a_2(y)+\tau a_3(x)-\tau a_3(z)+\tau a_1(0,\xi,x,x,z)-\tau a_1(-\tau,\xi,x,x_d,z)+\tau a_1(\theta,0,x,y,x)-\tau a_1(\theta,-\tau,x,y,x_d)\leq-\epsilon\|x\|^2$, $\forall\theta,\xi\in[-\tau,0]$.

Then the system described by Equation (5) is asymptotically stable.

Proof: The first three conditions impose that:

$$V(x_t)\geq\epsilon\|x_t(0)\|^2.$$

Similarly, the fourth condition, and the discussion before the statement of the proposition imply that

$$\dot{V}(x_t)\leq-\tilde{\epsilon}\|x_t(0)\|^2$$

for some $\tilde{\epsilon}>0$. Therefore from the statement of Theorem 1, the system (5) is asymptotically stable. ■

Condition (2) in the above proposition asks for a_1 to be non-negative only on a certain interval of θ and ξ, as does condition (4). To restrict ourselves to the intervals $\theta\in[-\tau,0]$ and $\xi\in[-\tau,0]$, we use a process similar to the S-procedure. The polynomial a_1 is required to be non-negative only when $g_1\triangleq\theta(\theta+\tau)\leq 0$ and $g_2\triangleq\xi(\xi+\tau)\leq 0$ are satisfied, which can be tested as follows:

$$a_1+p_1g_1+p_2g_2\geq 0 \qquad (7)$$

where p_1 and p_2 are sums of squares of degree 2 in x, y and z and of bounded degree in θ and ξ — this will retain the bipartite structure of the whole expression, which will be taken advantage of in the computation. The same can be done with Constraint (4) in the above proposition.

In order to reduce conservativeness of the positivity condition on the kernel of the quadratic form, we can now add polynomial terms which integrate to zero to the kernels. For example, we may test pointwise positivity of $\tilde{a}_1+b$ with the constraint $\int_{-\tau}^{0}\int_{-\tau}^{0}b(\theta,\xi)d\theta d\xi=0$ where b has the following structure

$$b=\begin{bmatrix} b_{11}(\theta,\xi) & b_{12}(\xi) & b_{13}(\theta)\\ b_{12}(\xi) & b_{22}(\xi) & 0\\ b_{13}(\theta) & 0 & b_{33}(\theta)\end{bmatrix} \qquad (8)$$

Likewise, for $\dot{V}$, we may use c where $\int_{-\tau}^{0}\int_{-\tau}^{0}c(\theta,\xi)d\theta d\xi=0$ and where

$$c = \begin{bmatrix} c_{11}(\theta,\xi) & c_{12}(\theta,\xi) & c_{13}(\xi) & c_{14}(\theta) \\ c_{12}(\theta,\xi) & c_{22}(\theta,\xi) & c_{23}(\xi) & c_{24}(\theta) \\ c_{13}(\xi) & c_{23}(\xi) & c_{33}(\xi) & 0 \\ c_{14}(\theta) & c_{24}(\theta) & 0 & c_{44}(\theta) \end{bmatrix}. \tag{9}$$

The computational complexity of this method increases as the order of the polynomials a_1 with respect to ξ and θ is increased. Although the SOS algorithm uses polynomials in x, y and z as well as ξ and θ, since the order of the variables x, y and z is fixed at 2 one can take advantage of the bipartite structure of the expression. For more details on how sparsity can be exploited, see [13] and the references therein. SOSTOOLS has routines that exploit the sparse bipartite structure to reduce the size of the resulting LMIs.

The system with delay τ can therefore be tested for stability by solving a SOS program (10). Note that the above procedure can be extended to handle systems with more than one delay. A different Lyapunov structure may then be required which allows the polynomials to be discontinuous at discrete points. We now present an example which investigates stability of a linear time-delay system.

Example 6: The following two dimensional system has been analyzed extensively in the past, and various LMI tests developed by other researchers have been tested against this example.

$$\begin{aligned} \dot{x}_1(t) &= -2x_1(t) - x_1(t-\tau) \triangleq f_1 \\ \dot{x}_2(t) &= -0.9x_2(t) - x_1(t-\tau) - x_2(t-\tau) \triangleq f_2 \end{aligned}$$

The system has been shown analytically to be asymptotically stable for $\tau \in [0, 6.17]$. The best bound on τ that can be obtained without using the discretisation method in [4] was $\tau_{\max} = 4.3588$ in [12]. Using $V(x_t)$ given by (6) we can test the maximum delays given in Table I. We see that as the order of the polynomials is increased, the LMI conditions we can obtain are better, but this is at the expense of computational complexity.

V. Robust stability under parametric uncertainty

An important issue in control theory is robust stability, i.e. ensuring stability under uncertainty. In this section we consider robust stability of linear time-delay systems under parametric uncertainty. Consider a time-delay system of the form (5) with an uncertain parameter p:

$$\dot{x}(t) = A_0(p)x(t) + A_1(p)x(t-\tau) = f(x_t, p) \tag{11}$$

This kind of uncertainty can be handled directly using the above tools as follows. Suppose one is interested in the parameter set

$$P = \{p \in \mathbb{R}^m \,|\, q_i(p) \geq 0,\ i = 1, \ldots, N\}$$

i..e the parametric uncertainty is captured by certain inequalities. We will be proving robust stability for the above system, by constructing a *Parameter Dependent* Lyapunov functional, as follows:

$$\begin{aligned} V(x_t,p) &= a_0(x_t(0),p) \\ &+ \int_{-\tau}^{0}\int_{-\tau}^{0} a_1(\theta,\xi,x_t(0),x_t(\theta),x_t(\xi),p)\,d\theta d\xi + \\ &+ \int_{-\tau}^{0}\int_{\theta}^{0} a_2(x_t(\zeta),p)\,d\zeta d\theta + \int_{-\tau}^{0}\int_{\xi}^{0} a_3(x_t(\zeta),p)\,d\zeta d\xi. \end{aligned} \tag{12}$$

Then the conditions for stability are similar to the ones in Proposition 5:

Proposition 7: Suppose there exist polynomials $a_0(x,p)$, $a_1(\theta,\xi,x,y,z,p)$, $a_2(x_t(\zeta),p)$ and $a_3(x_t(\zeta),p)$, and a positive constant ϵ such that the following hold for all $p \in P$:

1) $a_0(x,p) - \epsilon\|x\|^2 \geq 0$,
2) $a_1(\theta,\xi,x,y,z,p) \geq 0\ \forall\theta \in [-\tau,0]$, $\xi \in [-\tau,0]$,
3) $a_2(x_t(\zeta),p) \geq 0, \quad a_3(x_t(\zeta),p) \geq 0$
4) $\tau a_1(0,\xi,x,x,z,p) - \tau a_1(-\tau,\xi,x,x_d,z,p) + \tau a_1(\theta,0,x,y,x,p) - \tau a_1(\theta,-\tau,x,y,x_d,p) + \frac{da_0}{dx(t)}f + \tau a_2(x,p) + \tau a_3(x,p) - \tau a_2(y,p) - \tau a_3(z,p)) + \tau^2\frac{\partial a_1}{\partial x}f - \tau^2\frac{\partial a_1}{\partial\theta} - \tau^2\frac{\partial a_1}{\partial\xi} \leq -\epsilon\|x\|^2$ $\forall\theta \in [-\tau,0]$, $\xi \in [-\tau,0]$.

Then the system given by Equation (5) is stable for all $p \in P$.

The conditions $p \in P$, which are captured by the inequality constraints $q_i(p) \geq 0$ can be adjoined to the above inequality constraints using SOS multipliers in the similar manner that the inequalities $\theta(\theta+\tau) \leq 0$ are adjoined. A similar SOS program to (10) can easily be constructed.

VI. Conclusion

In this paper we have developed a new methodology for constructing the complete Lyapunov-Krasovskii functionals necessary for stability of linear time-delay systems based on the SOS decomposition. The results presented in this paper complement the results already obtained for nonlinear delay systems using SOS [10], [14]. A significant contribution of this paper is that the infinite dimensional LMIs obtained when considering complete L-K functionals for delay-dependent stability of linear time delay systems can be solved algorithmically using the SOS decomposition. The bipartite structure of the the resulting SOS conditions, and the use of special techniques discussed allows the solution of the infinite dimensional LMIs with little conservatism. This same idea was extended to the case of proving stability under parametric uncertainty.

SOS program to test stability of system (5) for a delay of size τ

Find polynomials $a_0(x), a_1(\theta, \xi, x, y, z), a_2(x(\zeta)), a_3(x(\zeta)), \epsilon > 0$ and SOS $q_{i,j}(\theta, \xi, x, y, z)$ for $i, j = 1, 2$ and polynomials $b(\theta, \xi, x, y, z)$ and $c(\theta, \xi, x, x_d, y, z)$ of the form (8) and (9) such that

$$
\begin{aligned}
& a_0(x) - \epsilon\|x\|^2 \text{ is SOS}, \quad a_1(\theta, \xi, x, y, z) + \sum_{j=1}^{2} q_{1,j} g_i(\theta, \xi) + b(\theta, \xi, x, y, z) \text{ is SOS}, \\
& a_2(x_t(\zeta)) \text{ is SOS}, \quad a_3(x_t(\zeta)) \text{ is SOS}, \\
& \left\{ \begin{array}{c} -\frac{da_0}{dx} f(x_t) - \tau^2 \frac{\partial a_1}{\partial x} f(x_t) + \tau^2 \frac{\partial a_1}{\partial \theta} + \tau^2 \frac{\partial a_1}{\partial \xi} - \tau a_1(0, \xi, x, x, z) \\ +\tau a_1(-\tau, \xi, x, x_d, z) - \tau a_1(\theta, 0, x, y, x) \\ +\tau a_1(\theta, -\tau, x, y, x_d) - \tau a_2(x) - \tau a_3(x) + \tau a_2(y) + \tau a_3(z) \\ -\epsilon\|x\|^2 + \sum_{j=1}^{2} q_{2,j} g_i(\theta, \xi) + c(\theta, \xi, x, x_d, y, z) \end{array} \right\} \text{ is SOS}, \\
& \int_{-\tau}^{0}\int_{-\tau}^{0} b(\theta, \xi, x, y, z) d\theta d\xi = 0, \quad \int_{-\tau}^{0}\int_{-\tau}^{0} c(\theta, \xi, x, x_d, y, z) d\theta d\xi = 0.
\end{aligned}
\tag{10}
$$

Order of a_1 in θ and ξ	0	1	2	3	4	5	6
max τ	4.472	5.17	5.75	6.02	6.09	6.15	6.16

TABLE I

τ_{max} FOR DIFFERENT DEGREE POLYNOMIALS a_1 IN θ AND ξ IN $V(x_t)$ FOR EXAMPLE 6

The complexity of this method increases as higher order polynomials and delays closer to the analytical ones are tested. However, one can take advantage of sparsity in the resulting polynomials therefore reducing significantly the computational complexity. Also, if the presence of certain functions (exponential, trigonometric, etc) in these functionals is known, then it is sometimes possible to search certificates that contain them [11]. Moreover, the procedure presented in this paper can be extended to many other cases that appear for time-delay systems: stability of neutral systems, systems with distributed delays, synthesis etc. More importantly, it can be extended to the analysis of nonlinear time delay systems, which was the subject of earlier work [10], [14].

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