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# On Global Stability of Internet Congestion Control

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**Abstract**

In this paper, we address the question of stability of TCP/AQM congestion control protocols described by nonlinear, discontinuous differential equations with delay. We analyze a well-known model, whose dynamics were previously shown to be locally stable for arbitrary delay via analysis of its linearization. We use a generalization of passivity theory to show that a reformulation of the original nonlinear, discontinuous model is input-output stable for arbitrary delay. We further show that input-output stability of this model implies global asymptotic stability of the original formulation. These results apply to the case of a single link with sources of identical fixed delay and demonstrate that in certain cases, stability of the nonlinear model holds under the same conditions as the linearized model.

## 1 Introduction and Prior Work

The analysis of Internet congestion control protocols has received much attention recently. Explicit mathematical modeling of the Internet has allowed analysis of existing protocols from a number of different theoretical perspectives and has generated some suggestions for improvement to current protocols. This work has been motivated by concern about the ability of current protocols to ensure stability and performance of the Internet as the number of users and amount of bandwidth continues to increase. Although the protocols that have been used in the past have performed remarkably well as the Internet has increased in size, analysis [11] indicates that as capacities and delays increase, instability will become a problem.

Many algorithms have been proposed for Internet congestion control, some of which have been shown to be globally stable in the presence of delays, nonlinearities and discontinuities. These proofs can be grouped into several categories according to methodology. In particular, Lyapunov-Razumhikin theory has been used to show global stability in [21, 3, 7, 19, 20], Lyapunov-Krasovskii functionals have been used to show global stability in [1, 13, 16, 15] and an input-output approach was taken in [19, 5]. In all of these cases, stability has been proven with varying degrees of conservatism with respect to restrictions on system parameters or delays.

In the paper by Kelly et al. [10], the Internet congestion control problem was first cast as a decentralized optimization problem. In Low and Lapsley [12], it was shown

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that the dynamics of the Internet with a certain class of control algorithms could be interpreted as a decentralized implementation of the gradient projection algorithm to solve the dual to the network optimization problem, thus showing global convergence to optimality for sufficiently small step size. In Paganini et al. [14] it was shown that with a certain set of pricing functions, a bound of $\alpha < \pi/2$ on a certain parameter $\alpha$ at the source allows a proof of local stability for arbitrary topology and heterogeneous time delays. Global asymptotic stability with time-delay of the protocols by Paganini et al. was discussed by Wang and Paganini [19] for a nonlinear implementation. In the case of a single source with a single link, this paper gave a proof of global asymptotic stability with time delay for all $\alpha \le \alpha_{\max}$, where $\alpha_{\max} = \ln(x_{\max}/c)/\big((x_{\max}/c) - 1\big)$, $c$ is the capacity of the link and $x_{\max}$ is a maximum data rate parameter. In the subsequent paper by Papachrisodoulou [16], a global stability result was given in the same case for $\alpha \le c/x_{\max}$. Although more conservative, the work by Papachristodoulou was extended to arbitrary network topology. In this paper, we show that for all $0 < \alpha < \pi\alpha_{\max}/2$, the congestion control algorithm with time-delays is globally asymptotically stable. If $x_{\max} = c$, then $\alpha_{\max} = 1$ and this is the same bound used in [14] for the linearized case.

Stability analysis of nonlinear, discontinuous differential equations with delay is quite difficult. Although frequency domain techniques have been shown to be effective when applied to linear systems with delay, these tools fail in the presence of nonlinearity. In addition, although time-domain analysis of nonlinear finite dimensional systems has had some success, analysis of the infinite dimensional systems associated with delay has been more problematic. In this paper, we are able to obtain improved results by decomposing the nonlinear, discontinuous, delayed system into an interconnection of a linear system with delay and a nonlinear, discontinuous system without delay. We analyze the subsystems separately and prove a passivity result for each. One benefit of such an approach is that it allows us to use frequency-domain arguments in addressing the infinite dimensional linear system. We can then use simple time-domain arguments in the analysis of the single state nonlinear system. These individual results can then be combined using the newly developed generalized passivity framework of Rantzer and Megretski known as Integral Quadratic Constaints. This approach yields improved results by allowing us to decompose the original difficult problem into simpler subproblems, each of which may be solved with less conservatism.

This paper is organized as follows. In Section 2.2, we derive the TCP model and mention previous results concerning stability. In Section 2.4, we discuss a generalization of classical passivity known as the theory of Integral Quadratic Constraints. In Sections 3.1 and 3.1.1, we reformulate the TCP stability problem in the input-output framework as the interconnection of a linear system with delay and a nonlinear, discontinuous system without delay. In Sections 3.2 and 3.3, we analyze each subsystem separately using the generalized passivity framework. We use frequency domain methods to prove passivity of the linear subsystem and time domain techniques to prove passivity of the nonlinear, discontinuous subsystem. In Section 3.4, we use the generalized passivity theorem which was presented in Section 2.4 to prove input-output stability of the interconnection problem. In Section 3.5, we show that the input-output stability result also implies asymptotic stability of the original formulation. Finally, in Sections 4 and 5, we briefly discuss implementation and give a conclusion.

## 2 Background Material

### 2.1 Notation

Let $\mathbb{R}_+ = \{\, x \in \mathbb{R} \mid x \ge 0 \,\}$, and let $\mathcal{C}$ denote the set of continuous functions. We use $\mathcal{C}_\tau$ to denote the Banach space of continuous functions $u : [-\tau, 0] \to \mathbb{R}^n$ with norm

$\|u\| = \sup_{t\in\mathbb{R}_+} \|u(t)\|_2$. A function $x : \mathbb{R} \to \mathbb{R}$ is said to be ***absolutely continuous*** if for any integer $N$ and any sequence $t_1, \ldots, t_N$, we have $\sum_{k=1}^{N-1} |x(t_k) - x(t_{k+1})| \to 0$ whenever $\sum_{k=1}^{N-1} |t_k - t_{k+1}| \to 0$.

$L_2(-\infty, \infty)$ is the Hilbert space of Lebesgue measurable real vector-valued functions $x : \mathbb{R} \to \mathbb{R}^n$ with inner-product $\langle u, v\rangle_2 = \int_{-\infty}^{\infty} u(t)^T v(t)dt$. $L_2$ denotes $L_2[0,\infty) = \{\, x \in L_2(-\infty,\infty) \mid x(t) = 0 \text{ for all } t < 0 \,\}$ and is a Hilbert subspace of $L_2(-\infty, \infty)$. $P_T$ is the truncation operator such that if $y = P_T z$, then $y(t) = z(t)$ for all $t \le T$ and $y(t) = 0$ otherwise. $L_{2e}$ denotes the space of functions such that for any $T > 0$ and $y \in L_2e$, we have $P_T y \in L_2$. We also make use of the space $W_2 = \{y : y, \dot{y} \in L_2\}$ with inner product $\langle x, y\rangle_{W_2} = \langle x, y\rangle_{L_2} + \langle \dot{x}, \dot{y}\rangle_{L_2}$ and extended space $W_{2e} = \{y : y, \dot{y} \in L_{2e}\}$. The dimensions of the various $L_2$ and $W_2$ spaces used should be clear from context and are not explicitly stated.

A causal operator $H : L_{2e} \to L_{2e}$ is bounded if $H(0) = 0$ and if it has finite gain, defined as

$$\|H\| = \sup_{u\in L_2 \neq 0} \frac{\|Hu\|}{\|u\|}$$

$\hat{L}_2$ denotes the Hilbert space of complex vector-valued functions on the imaginary axis, $x : j\mathbb{R} \to \mathbb{C}^n$ with inner-product $\langle \hat{u}, \hat{v}\rangle_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(j\omega)^* \hat{v}(j\omega)d\omega$. $\hat{L}_\infty$ denotes the Banach space of matrix-valued functions on the imaginary axis, $\hat{G} : j\mathbb{R} \to \mathbb{C}^{m\times n}$ with norm $\|\hat{G}\|_\infty = \operatorname{ess\,sup}_{\omega\in\mathbb{R}} \bar{\sigma}\left(\hat{G}(j\omega)\right)$ where $\bar{\sigma}(\hat{G}(j\omega))$ denotes the maximum singular value of $\hat{G}(j\omega)$. $\hat{u}$ denotes the either the Fourier or Laplace transform of $u$, depending on $u$. We will also make use of the following specialized set of transfer functions which define bounded linear operators on $L_2$. $\mathcal{A}$ is defined to be those transfer functions which are the Laplace transform of functions of the form

$$g(t) = \begin{cases} h(t) + \sum_{i=1}^{N} g_i \delta(t - t_i) & \text{if } t \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

where $h \in L_1$, $g_i \in \mathbb{R}$ and $t_i \ge 0$.

### 2.2 The Internet Optimization Problem

We view the Internet as an abstract collection of sources and links. The term ***source*** refers to a connection between a user and a particular destination. The source transmits data in packets. The rate at which a source $i$ transmits packets is dictated by the source's round-trip time, $\tau_i$, as well as window size, $w_i$. The *round-trip time* is defined as the time between transmission of a packet and receipt of an acknowledgement for that packet from the destination. The *window size* is defined as the number of packets which are allowed to be simultaneously unacknowledged. In this paper, we assume that packet losses do not affect the source transmission rates, since any lost packet will presumably be detected by the user after one round-trip time and resent. We assume that acknowledgements contribute to delay but do not contribute to congestion at the links. We assume a fixed bit size for all packets and that $\tau_i$ is known at least for the purposes of determining data rate. The packet transmission rate, $x_i$, at source $i$ can be controlled by the window size according to

$$w_i = x_i \tau_i. \tag{1}$$

The term ***link*** refers to a single congested resource such as a router. Packets arriving at a link enter an entrance queue. A link can process packets in the queue at some rate capacity $c_j$. If too many data packets arrive in a given period of time, the size of the queue may grow and some packets may experience a queueing delay while in the queue. In this paper, we assume that the dynamics from this variable queueing delay are negligible and we only model the delay due to the fixed propagation time. Links must also be able to feed back information. This can be done either through the ECN bit in the packet header, through packet dropping schemes or through measurement of variations in queueing delay. The value of the congestion indicator at link $j$ is denoted $p_j$. We also assume that the congestion indicator received at each source is the summation of the indicators of all links in the source's route. This value is denoted $q_i$.

Sources and links are related by routing tables which specify the route or set of links, $J_i$ through which the packets from source $i$ to its destination must pass. The rate of packets received at a link $j$ is then the sum of the rates of all sources using that link and is denoted by $y_j$. The set of users for link $j$ is denoted $I_j$. Ignoring delay for the moment, we have the following equations.

$$y = Rx, \quad q = R^T p,$$

where

$$R_{ji} = \begin{cases} 1 & \text{if source } i \text{ uses link } j \\ 0 & \text{otherwise} \end{cases}$$

#### 2.2.1 Optimization Model

The following model for optimizing flow rates in a network was proposed by Kelly et al. [10].

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_i^N U_i(x_i) \\ \text{subject to} & x \geq 0, \quad Rx \leq c \end{array}$$

Assume that the $U_i$ are continuously differentiable strictly concave non-decreasing functions. If all sources utilize at least one link, then the problem has a unique optimum. Note that, as $N$ increases, the problem becomes progressively more difficult to solve using a centralized algorithm. We now consider the dual problem with dual variable $p \in \mathbb{R}^M$, where $M$ is the number of links, which is given by

$$\begin{array}{ll} \text{minimize} & h(p) \\ \text{subject to} & p \geq 0 \end{array}$$

where the dual function $h$ is given by

$$\begin{aligned} h(p) &= \max_{x \geq 0} \sum_i \big(U_i(x_i)\big) - p^T(Rx - c) \\ &= \sum_i \Big(U_i\big(x_{\text{opt},i}(p)\big)\Big) - p^T(Rx_{\text{opt}}(p) - c) \\ x_{\text{opt},i}(p) &= \max\{0, U_i'^{-1}(\sum_j R_{j,i} p_j)\} \\ &= \max\{0, U_i'^{-1}(q_i(p))\} \\ q(p) &= R^T p \end{aligned}$$

The map $U_i'^{-1} : \mathbb{R}^+ \to \{\mathbb{R} \cup \infty\}$ is well defined since $U_i' \in \mathcal{C}$ and $U_i$ is strictly concave. We would like to construct a dynamical system which converges to the solution of the dual problem. One such system is given by the gradient projection algorithm. In discrete-time, this is

$$p_j(t+1) = \max\{0, p_j(t) - \gamma_j D_j h(p(t))\},$$

where $D_j$ denotes the partial derivative with respect to the $j$'th argument and $\gamma_j$ is the step size. Since the $U_i$ are strictly concave, $h(p)$ is continuously differentiable with the following derivatives [2].

$$\begin{aligned} D_j h(p) &= c_j - \sum_{i \in I_j} x_{\text{opt},i} \\ &= c_j - y_{\text{opt},j}(p) \\ y_{\text{opt}}(p) &= R x_{\text{opt}}(p). \end{aligned}$$

If $\gamma$ is sufficiently small, the discrete-time gradient projection algorithm will converge to the solution of the dual problem [12]. Because of convexity of the problem, strong duality implies that $x_{\text{opt}}(p)$ will converge to the unique optimum of the primal problem. A continuous-time implementation of this algorithm in the network framework is as follows.

$$\begin{aligned} \dot{p}_j(t) &= \begin{cases} \gamma_j(y_j(t) - c_j) & p_j(t) > 0 \\ \max\{0, \gamma_j(y_j(t) - c_j)\} & p_j(t) \leq 0 \end{cases} \\ x_i(t) &= \max\{0, U_i'^{-1}(q_i(t))\} \\ y(t) &= Rx(t), \quad q(t) = R^T p(t) \end{aligned}$$

$\gamma_j$ now denotes a gain parameter, corresponding to step-size in discrete time. This algorithm has the remarkable property that it is decentralized, corresponding to the separable structure of the constraints. $p_j$ is computed at each of $M$ links. Link $j$ requires only knowledge of $y_j$ to compute this value. $x_i$ is computed at each of $N$ sources. Source $i$ requires only knowledge of $q_i$ to compute this value.

### 2.2.2 Stability Properties

To ensure that the continuous-time gradient projection algorithm will converge when implemented with the current internet framework, we must also consider the delay in transmitting packets from the source to the link and then receiving acknowledgements at the source. The delay from source $i$ to link $j$ is denoted $\tau_{ij}^f$ and the delay from link $j$ to source $i$ is denoted $\tau_{ij}^b$. For any source $i$, the total round trip time is fixed, i.e. $\tau_i = \tau_{ij}^f + \tau_{ij}^b$ for all $j \in J_i$. We express these delays in the frequency domain by replacing the entries of the routing matrix $R$ with forward and backward delay transfer functions $\hat{R}^f$ and $\hat{R}^b$, giving

$$\begin{aligned} \hat{y}(s) &= \hat{R}^f(s)\hat{x}(s), \quad \hat{q}(s) = \hat{R}^b(s)^T \hat{p}(s) \\ \hat{R}_{ji}^f(s) &= \begin{cases} e^{-\tau_{ij}^f s} & \text{if source } i \text{ uses link } j \\ 0 & \text{otherwise} \end{cases} \\ \hat{R}_{ji}^b(s) &= \begin{cases} e^{-\tau_{ij}^b s} & \text{if source } i \text{ uses link } j \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The work by Paganini et al. [14] introduced a class of utility functions under which this system was shown to have a stable linearization about its positive equilibrium point for a fixed gain parameter $\gamma_j = 1/c_j$. This class was given by the set of $U_i$ such that

$$\frac{d}{dq_i} U_i'^{-1}(q_i) = -\frac{\alpha_i}{M_i \tau_i} U_i'^{-1}(q_i),$$

where $M_i$ is a bound on the number of links in the path of source $i$ and $\alpha_i < \pi/2$. In particular, the choice of

$$U_i(x) = \frac{M_i \tau_i}{\alpha_i} x \left(1 - \ln \frac{x}{x_{\max,i}}\right),$$

with restricted domain $x \le x_{\max,i}$ was suggested in [14] as a strictly concave utility function such that the function $U_i'^{-1}(q) = x_{\max,i} e^{-\frac{\alpha_i}{M_i \tau_i} q} \ge 0$ has the necessary derivative.

Some efforts have been made to extend this local stability result to the global case. For a single source and a single link, the paper by Wang and Paganini [19] has shown this implementation to be globally stable for $\alpha \le f_1(x_{\max}/c)$, where

$$f_1(x) = \frac{\ln x}{x - 1}.$$

In addition, the paper by Papachristodoulou [16] has shown this implementation to be stable when $\alpha \le f_2(x_{\max}/c)$, where

$$f_2(x) = \frac{1}{x}.$$

$f_1$ and $f_2$ are illustrated in Figure 1. Note that when $x_{\max} = c$, both these conditions become $\alpha \le 1$ which is more restrictive than the local stability bound of $\alpha < \pi/2$. This paper attempts to eliminate the gap between local and global stability results by showing global stability for $\alpha < \pi/2 f_1(x_{\max}/c)$.

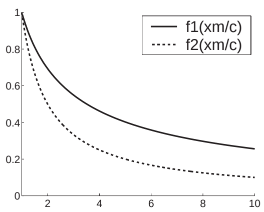


Figure 1: Plot of $f_1(x)$ and $f_2(x)$ vs. $x$

### 2.3 Stability Framework

Two concepts of stability will be used in this paper. The first, input-output stability, is used to define stability of an operator and describes the relationship between inputs and outputs. The second, internal stability, defines stability of a differential equation and is a property of the behavior of the state given initial conditions.

**Input-output stability** Consider an operator, $\Psi$.

**Definition 1.** *For normed spaces $X, Y$, the operator $\Psi$ is $Y$ stable on $X$ if it defines a single-valued map from $X$ to $Y$ and there exists some $\beta$ such that $\|\Psi u\|_Y \le \beta \|u\|_X$ for all $u \in X$.*

**Internal stability** Now consider a delay-differential equation of the following form

$$\dot{x}(t) = \begin{cases} f\big(x(t), x(t-\tau)\big) & \text{for all } t \ge \tau \\ f\big(x(t), x_0(t-\tau)\big) & \text{for all } t \in [0, \tau) \end{cases} \tag{2}$$

where $f(0,0) = 0$. The system is well-posed if for each $x_0 \in \mathcal{C}_\tau$ there exists a unique $x \in \mathcal{C}$ such that $x$ satisfies (2) for $t \geq 0$ and $x(0) = x_0(0)$. In this case, the function $f$ defines a map

$$\Phi_f : \mathcal{C}_\tau \to \mathcal{C}$$

**Definition 2.** *The solution map $\Phi$ defined by $f$ is* ***globally stable*** *on $X \subset \mathcal{C}_\tau$ if*

*(i) $\Phi x$ is bounded for any $x \in X$*

*(ii) $\Phi$ is continuous at $0$ with respect to the supremum norm on $\mathcal{C}$ and $\mathcal{C}_\tau$.*

**Note:** This is the usual notion of Lyapunov stability, which states that for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|y\| < \delta$ implies $\|\Phi y\| < \varepsilon$.

**Definition 3.** *The solution map $\Phi$ defined by $f$ is* ***globally asymptotically stable*** *on $X \subset \mathcal{C}_\tau$ if $y = \Phi x_0$ implies $\lim_{t\to\infty} y(t) = 0$ for any $x_0 \in X$.*

### 2.4 Theory of Integral-Quadratic Constraints

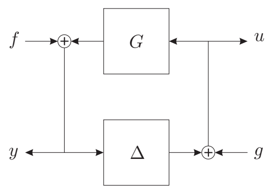


Figure 2: Interconnection of systems

Let $G$ be a linear operator with transfer function $\hat{G} \in \mathcal{A}$ and let the operator $\Delta : L_2 \to L_2$ be causal and bounded. Define inputs $f \in W_2$ and $g \in L_2$. The ***interconnection*** of $G$ and $\Delta$ is defined by the following equations.

$$y = Gu + f$$
$$u = \Delta y + g$$

**Definition 4 (Jönsson [8], p71).** *The interconnection of $G$ and $\Delta$ is* ***well-posed*** *if for every pair $(f,g)$ with $f \in W_2$ and $g \in L_2$, there exists a solution $u \in L_{2e}$, $y \in W_{2e}$ and the map $(f,g) \to (y,u)$ is causal.*

If the interconnection of $\Delta$ and $G$ is well posed, then the interconnection defines an operator $\Phi : W_2 \times L_2 \to W_{2e} \times L_{2e}$.

In this paper, we use a result by Rantzer and Megretski [17] which can be viewed as generalization of the classical notion of passivity. Recall the the following classical passivity theorem from e.g. Desoer and Vidyasagar(p. 182, [4])

**Theorem 5.** *The interconnection of $\Delta$ and $G$ is $L_2$ stable on $L_2$ if there exists some $\epsilon > 0$ such that for any $x \in L_2$,*

$$\langle \Delta x, x \rangle \geq 0$$
$$\langle x, Gx \rangle \leq -\epsilon \|x\|$$

Now given bounded linear transformations $\Pi_1, \Pi_2$, define the following functional

$$\langle x, y \rangle_\Pi = \left\langle \Pi_1 \begin{bmatrix} x \\ y \end{bmatrix}, \Pi_2 \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle$$

Ignoring technical details for the moment, the result by Rantzer and Megretski states that the interconnection of $\Delta$ and $G$ is stable if there exists an $\epsilon > 0$ such that for any $x \in L_2$,

$$\begin{aligned} \langle x, \Delta x \rangle_\Pi &\geq 0 \\ \langle Gx, x \rangle_\Pi &\leq -\epsilon \|x\| \end{aligned}$$

This idea of generalized passivity is motivated from a geometric standpoint by the topological separation argument, introduced by Safonov [18]. Consider the following definition of an operator graph.

**Definition 6.** *For an operator, $\rho : X \to X$, the **graph** of $\rho$ is the set of points $\Phi(\rho) = \{(x, y) : y = \rho(x), x \in X\}$. The **inverse graph** of $\rho$ is the set $\Phi_i(\rho) = \{(x, y) : x = \rho(y), y \in X\}$.*

Two graphs are separate if they intersect only at the origin. Consider graphs $g_1$ and $g_2$. If, for some functional $\sigma : X \to \mathbb{R}$ and $\epsilon > 0$, we have $\sigma(x) \geq 0$ for all $x \in g_1$ and $\sigma(y) \leq -\epsilon\|y\|$ for all $y \in g_2$, then the graphs $g_1$ and $g_2$ can only intersect at the origin and are said to be separated by the functional $\sigma$. Now consider the interconnection of operators $G$ and $\Delta$ when we let the input $g = 0$. We have the equation $f = (I - G\Delta)y$. The $L_2$ stability question becomes when does $(I - G\Delta)$ have a well defined bounded inverse on $L_2$. Now suppose there exists a nonzero $y \in L_2$ such that $f = (I - G\Delta)y = 0$, then $(I - G\Delta)$ has a nontrivial kernel and can not have a bounded inverse. To ensure that $(I - G\Delta)$ has a trivial kernel, we will consider the graph of $G$ and the inverse graph of $\Delta$. The following well-known result shows that separation of the graph and inverse graph of interconnected operators is necessary for stability.

**Theorem 7.** *Let $f = (I - G\Delta)y$. The following are equivalent*

- $(I - G\Delta)y = 0 \quad \Rightarrow \quad y = 0$
- $\Phi(G) \cap \Phi_i(\Delta) = 0$

*Proof.* ($\Leftarrow$) If there exists a $y \neq 0$ such that $(I - G\Delta)y = 0$ then let $x = \Delta y$. Thus

$$\begin{bmatrix} x \\ Gx \end{bmatrix} = \begin{bmatrix} \Delta y \\ y \end{bmatrix} \in \Phi(G) \cap \Phi_i(\Delta)$$

($\Rightarrow$) If there exists $y, x \neq 0$ such that

$$\begin{bmatrix} x \\ Gx \end{bmatrix} = \begin{bmatrix} \Delta y \\ y \end{bmatrix}$$

then $f = y - G\Delta y = Gx - G\Delta y = G\Delta y - G\Delta y = 0$ □

Although the separation of graphs is clearly necessary for most definitions of stability, it is unclear under what additional conditions separation may also be sufficient for $L_2$ stability. The result mentioned above by Rantzer and Megretiski gives a particular class of functionals for which graph separation may be considered sufficient for stability on $L_2$. Many classical theorems concerning the stability of the interconnection of operators can be viewed as proving the separation of graphs and inverse graphs by using these type of separating functionals. For example, the small gain theorem can be expressed using the separating functional $\sigma((x,y)) = \|x\| - k\|y\|$ for some $k > 0$. Similarly, classical passivity can be expressed using the functional $\sigma((x,y)) = \langle x, y\rangle$. The complete class of functionals shown by [17] to be sufficient for $L_2$ stability is given by Definition 8.

**Definition 8 (Rantzer [17]).** *The mapping $\sigma : L_2 \to \mathbb{R}$ is **quadratically continuous** if for every $\delta > 0$, there exists a $\eta_\delta$ such that the following holds.*

$$|\sigma(x_1) - \sigma(x_2)| \le \eta_\delta \|x_1 - x_2\|^2 + \delta\|x_2\|^2$$

*for all $x_1, x_2 \in L_2$*

This class includes the small gain and passivity functions. Furthermore, for any bounded linear transformations $\Pi_1, \Pi_2$, the function $\Sigma(w) = \langle \Pi_1 w, \Pi_2 w\rangle$ is quadratically continuous. In this paper we use the following generalization of the work by Rantzer and Megretski as presented in the thesis work by Jönsson [8].

**Definition 9.** *Let $\Pi_B : j\mathbb{R} \to \mathbb{C}^{n\times n}$ be a bounded and measurable function that takes Hermitian values and $\lambda \in \mathbb{R}$. We say that $\Delta$ **satisfies the IQC** defined by $\Pi_B, \lambda$, if there exists a positive constant $\gamma$ such that for all $y \in W_2$ and $v = \Delta y \in L_2$,*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \begin{bmatrix}\hat{y}(j\omega)\\ \hat{v}(j\omega)\end{bmatrix}^* \Pi_B(j\omega) \begin{bmatrix}\hat{y}(j\omega)\\ \hat{v}(j\omega)\end{bmatrix} d\omega + 2\langle v, \lambda\dot{y}\rangle \ge -\gamma|y(0)|^2$$

**Theorem 10.** *Assume that*

1. *$G$ is a linear causal bounded operator with $s\hat{G}(s), \hat{G}(s) \in \mathcal{A}$*
2. *For all $\kappa \in [0,1]$, the interconnection of $\kappa\Delta$ and $G$ is well-posed*
3. *For all $\kappa \in [0,1]$, $\kappa\Delta$ satisfies the IQC defined by $\Pi_B, \lambda$*
4. *There exists $\eta > 0$ such that for all $\omega \in \mathbb{R}$*

$$\begin{bmatrix}\hat{G}(j\omega)\\ I\end{bmatrix}^* \left(\Pi_B(j\omega) + \begin{bmatrix}0 & \lambda j\omega^*\\ \lambda j\omega & 0\end{bmatrix}\right) \begin{bmatrix}\hat{G}(j\omega)\\ I\end{bmatrix} \le -\eta I$$

*Then the interconnection of $G$ and $\Delta$ is $W_2 \times L_2$ stable on $W_2 \times L_2$.*

# 3 Results

In this section we reformulate the proposed congestion control algorithm as the interconnection of a linear system with delay and a nonlinear system without delay. This approach was motivated by the work of Wang[19] and Jönsson[9]. We then use the IQC defined by $\Pi_B$, $\lambda = \frac{2}{\pi}$, where

$$\Pi_B = \begin{bmatrix}0 & \beta\\ \beta & -\frac{4}{\pi} - 2\end{bmatrix}$$

and $\beta = \alpha/(\alpha_{\max}\tau)$, to show that we can establish $W_2 \times L_2$ stability on $W_2$ of the interconnection for any $\tau \ge 0$, $0 < \alpha < \pi/2\alpha_{\max}$. We also show that $W_2 \times L_2$ stability on $W_2$ of the interconnection implies asymptotic stability of the original formulation of the congestion control protocol.

### 3.1 Preliminary Results

If we consider the problem of a single link and a single source, then from the development in Section 2.2 we have that $y(t) = x(t - \tau^f)$ and $q(t) = p(t - \tau^b)$ where $\tau^f + \tau^b = \tau$. Given an initial condition $x_0 \in \mathcal{C}_\tau$, the dynamics can now be summarized as $p(t) = x_0(t)$ for $t \in [-\tau, 0]$ and the following for $t \geq 0$

$$\dot{p}(t) = \begin{cases} \frac{x_{\max}}{c} e^{-\frac{\alpha}{\tau} p(t-\tau)} - 1 & p(t) > 0 \\ \max\{0, \frac{x_{\max}}{c} e^{-\frac{\alpha}{\tau} p(t-\tau)} - 1\} & p(t) \leq 0 \end{cases} \tag{3}$$

$$x(t) = x_{\max} e^{-\frac{\alpha}{\tau} p(t-\tau^b)} \tag{4}$$

Since the dynamics of Equation (3) are decoupled from those of (4) and stability of $x$ follows from that of $p$, we need only consider stability of Equation (3). Now consider the equilibrium point of Equation (3), $p_0 = \frac{\tau}{\alpha} \ln \frac{x_{\max}}{c}$. As is customary, we change to variable $z$, where $z(t) = p(t) - p_0$ so that the origin is an equilibrium point. Now we have $z(t) = x_0(t) - p_0$ for $t \in [-\tau, 0]$ and the following for $t \geq 0$

$$\dot{z}(t) = \begin{cases} e^{-\frac{\alpha}{\tau} z(t-\tau)} - 1 & z(t) > -p_0 \\ \max\{0, e^{-\frac{\alpha}{\tau} z(t-\tau)} - 1\} & z(t) \leq -p_0 \end{cases} \tag{5}$$

For convenience and efficiency of presentation, we will refer to the solution map defined by Equation (5) as $A : \mathcal{C}_\tau \to \mathcal{C}$. Implicit in these dynamics is the constraint $z(t) \geq -p_0$. If we assume that any initial condition will satisfy this constraint, we can include the constraint in the dynamics without altering the solution map. For convenience, we define the following bounded continuous functions

$$f_1(y) = \min\left\{e^{\frac{\alpha}{\tau} y} - 1, e^{\frac{\alpha}{\tau} p_0} - 1\right\}$$
$$f_2(y) = \max\left\{0, f_1(y)\right\}$$
$$f_c(x, y) = \begin{cases} f_1(y) & \text{if } x > -p_0 \\ f_2(y) & \text{otherwise} \end{cases}$$

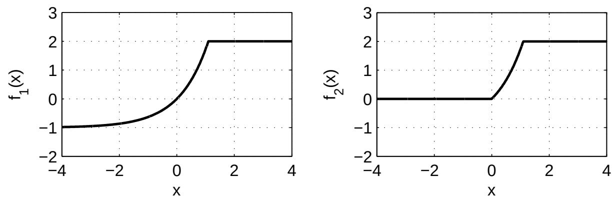


Figure 3: $f_1$ and $f_2$

These functions are illustrated in Figure 3. We now have the following equation for $t \geq 0$.

$$\dot{z}(t) = f_c(z(t), -z(t - \tau)) \tag{6}$$

For completeness, we briefly discuss well-posedness of the solution map, $A$. Given any absolutely continuous solution $z(t)$ on some interval $[T_1, T_1 + \tau]$, we observe that $\dot{z}(t) = f_c(z(t), z(t - \tau)) = f_c(z(t), t)$ is a function only of time and state $z(t)$ for the interval $[T_1 + \tau, T_1 + 2\tau]$. From boundedness of the $f_c$, continuity with respect to $z(t - \tau)$ and noting upper semi-continuity of the associated differential inclusion with respect to $z(t)$, we can now establish via Fillipov [6][p77] the existence and uniqueness of a continuous solution $z(t)$ over the interval $[T_1 + \tau, T_1 + 2\tau]$. Assuming a continuous initial condition, this implies the existence and uniqueness of the solution map $A : \mathcal{C}_\tau \to \mathcal{C}$.

### 3.1.1 Separation into subsystems

Equation (6) is a delay-differential equation defined by a nonlinear, discontinuous function. To aid in the analysis, we will reformulate the problem as the interconnection of two subsystems where the $W_2$ stability on $W_2 \times L_2$ stability of this interconnection implies asymptotic stability on $X$ of the original formulation for some set $X$. Define the map $G$ by $w = Gu$ if

$$w(t) = \int_{t-\tau}^{t} u(\theta)d\theta$$

Note that $G$ is a linear operator which can be represented by the convolution with $g(t) = \text{step}(t) - \text{step}(t-\tau) \in L_1$. This implies that $\hat{G} \in \mathcal{A}$. Moreover, $G$ can be represented in the frequency domain by $\hat{G}(s) = \frac{1-e^{-\tau s}}{s}$ which implies $G$ is a bounded operator on $L_2$ since $\|\hat{G}(j\omega)\|_\infty = \tau$. In addition, $s\hat{G}(s) \in \mathcal{A}$ since it can be represented by convolution with $\delta(t) - \delta(t-\tau)$.

Define the map $\Delta_z$ by $z = \Delta_z y$ if $z(0) = 0$ and

$$\dot{z}(t) = f_c\big(z(t), y(t) - z(t)\big)$$

We define the map $\Delta$ by $v = \Delta y$ if $v(t) = \dot{z}(t)$ where $z = \Delta_z y$. Addressing well-posedness, if $y \in W_2$, then $y$ is absolutely continuous on any finite interval(See p. 25 in Jönsson [8]). From boundedness of $f_c$, continuity with respect to $y(t)$ and upper semi-continuity of the associated differential inclusion with respect to $z(t)$, we can again establish the existence and uniqueness of an absolutely continuous solution $z$ and thus of the map $\Delta_z$. Well-posedness of $\Delta$ follows immediately. Further properties of $\Delta$ will be derived in later sections.

If we now form the interconnection of $G$ and $\kappa\Delta$ as defined above with a single input $f \in W_2$, we can construct the map from input $f$ to outputs $y, u$. For convenience and efficiency of presentation, we will denote the interconnection map for $\kappa = 1$ by $B : W_2 \to W_{2e} \times L_{2e}$. Furthermore, for $\kappa = 1$, we denote the map from input $f$ to internal variable $z$ by $B_z$. For $t \leq 0$, $u(t) = y(t) = z(t) = f(t) = 0$ and for $t \geq 0$, the interconnection dynamics combine as follows.

$$\begin{aligned}
u(t) &= \kappa\dot{z}(t) \\
y(t) &= \int_{t-\tau}^{t} u(t)dt + f(t) \\
&= \kappa\big(z(t) - z(t-\tau)\big) + f(t) \\
\dot{z}(t) &= f_c(z(t), y(t) - z(t)) \\
&= f_c(z(t), f(t) - \kappa z(t-\tau) - (1-\kappa)z(t))
\end{aligned}$$

As before, from continuity with respect to $f(t)$ and $z(t-\tau)$, upper semi-continuity with respect to $z(t)$ and boundedness of $f_c$, we can conclude existence and uniqueness of $z \in L_{2e}$. Since $z$ has bounded derivative and $z(t) = 0$ for $t \leq 0$, we now have existence and uniqueness of the map $B_z : W_2 \to W_{2e}$. Furthermore, this implies that $y \in W_{2e}$ and $u \in L_{2e}$ which yields well-posedness of the interconnection for any $\kappa \in [0, 1]$ and specifically of the map $B : W_2 \to W_{2e} \times L_{2e}$.

## 3.2 $\Delta$ satisfies the IQC

In this section we show that if $\alpha > 0$, then $\Delta$ and consequently $\kappa\Delta$ are bounded and satisfy the IQC defined by $\Pi_B, \lambda = \frac{2}{\pi}$ for all $\kappa \in [0, 1]$. The methods used in this section were motivated by those in Jönnson [9] and Wang [19]. For $\gamma = 4\beta/\pi > 0$, we prove the following for all $y \in W_2$, $v = \Delta y$.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \begin{bmatrix} \hat{y}(j\omega) \\ \hat{v}(j\omega) \end{bmatrix}^* \begin{bmatrix} 0 & \beta \\ \beta & -\frac{4}{\pi} - 2 \end{bmatrix} \begin{bmatrix} \hat{y}(j\omega) \\ \hat{v}(j\omega) \end{bmatrix} + \frac{4}{\pi}\langle v, \dot{y}\rangle \\ \geq -\gamma |y(0)|^2$$

By Parseval's equality, this is equivalent to

$$\frac{2}{\pi}\langle v, \dot{y} - v\rangle + \langle v, \beta y - v\rangle \geq -\frac{\gamma}{2}|y(0)|^2$$

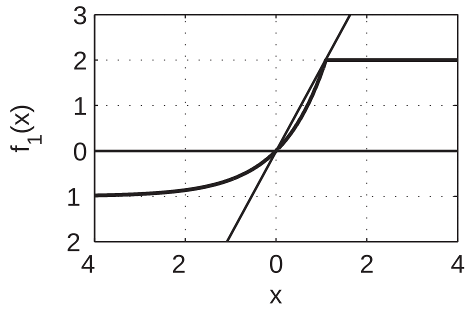


Figure 4: The nonlinearity $f_1$ satisfies a sector bound

A critical result used in the analysis of this section is the existence of a sector bound on the nonlinearity $f_1$ and consequently on $f_2$, i.e. $0 \leq f_i(x)x \leq \beta x^2$ where $\beta = \frac{e^{\frac{\alpha}{\tau}p_0} - 1}{p_0}$, denoted $f_i \in \text{sector}[0, \beta]$. This key feature is illustrated in Figure 4. Also notice that

$$f_c(x, y) = \begin{cases} f_1(y) & \text{if } x > -p_0 \text{ or } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

**Lemma 11.** *If $v = \Delta y$ with $y \in W_2$, then*

1. *$v \in L_2$ with norm bound $\beta\|y\|$,*

2. *$\langle v, \beta y - v\rangle \geq 0$*

*Proof.* As a consequence of the above sector bounds, we have

$$f_c(x, y)^2 \leq \beta y f_c(x, y).$$

Let $z = \Delta_z y$, then this implies

$$\dot{z}(t)^2 = f_c\big(z(t), y(t) - z(t)\big)\dot{z}(t) \\ \leq \beta\big(y(t) - z(t)\big)\dot{z}(t)$$

Now for any $T \geq 0$, we have

$$\begin{aligned}
\|P_T v\|^2 &= \int_0^T v(t)^2 dt = \int_0^T \dot{z}(t)^2 dt \\
&\leq \beta \int_0^T \dot{z}(t)(y(t) - z(t))dt \\
&= \beta \int_0^T \dot{z}(t)y(t)dt - \frac{\beta}{2}(z(T)^2 - z(0)^2) \\
&\leq \beta \langle P_T \dot{z}, y\rangle \qquad (7) \\
&\leq \beta \|P_T \dot{z}\| \|y\| = \beta \|P_T v\| \|y\|
\end{aligned}$$

Therefore, $\|P_T v\| \leq \beta\|y\|$ for all $T \geq 0$. Thus $v \in L_2$ with norm bounded by $\beta\|y\|$. Statement 2 follows from line 7 by letting $T \to \infty$. □

**Lemma 12.** *Let $z = \Delta_z y$ with $y \in W_2$, then $\lim_{t\to\infty} z(t) = 0$.*

*Proof.* Let $v = \dot{z} = \Delta y$. Suppose that $T_2 > T_1 > 0$ and let $H = P_{T_2} - P_{T_1}$. Then

$$\begin{aligned}
\|Hv\|_2^2 &= \int_{T_1}^{T_2} \dot{z}(t)^2 dt \\
&\leq \beta \int_{T_1}^{T_2} \dot{z}(t)y(t)\,dt - \beta \int_{T_1}^{T_2} \dot{z}(t)z(t)\,dt \\
&= \beta\langle Hv, Hy\rangle - \frac{\beta}{2}(z(T_2)^2 - z(T_1)^2) \\
&\leq \beta\|Hv\|_2\|Hy\|_2 - \frac{\beta}{2}\big(z(T_2)^2 - z(T_1)^2\big)
\end{aligned}$$

Hence

$$\begin{aligned}
z(T_2)^2 - z(T_1)^2 &\leq 2\|Hv\|_2\|Hy\|_2 - \frac{2}{\beta}\|Hv\|_2^2 \\
&\leq 2\|Hv\|_2\|Hy\|_2
\end{aligned}$$

By Lemma 11, $v \in L_2$. Since $\|v\|$ and $\|y\|$ exist, we can use the Cauchy criterion and the above inequality to establish that for any $\delta > 0$, there exists a $T_\delta$ such that $T_2 > T_1 > T_\delta$ implies $(z(T_2)^2 - z(T_1)^2) < \delta$. It is shown in Lemma 17 in the Appendix that this implies that for any infinite increasing sequence $\{T_i\}$, $\{z(T_i)^2\}$ is a Cauchy sequence and therefore $z(t)^2$ converges to a limit. Since $z$ is continuous, this implies that $z(t)$ also converges to a limit, $z_\infty$. Since $y \in W_2$, we have $\lim_{t\to\infty} y(t) = y_\infty = 0$. Recall $f_c(a, b) = f_1(b)$ at points such that $a > -p_0$ or $b > 0$. Suppose $z_\infty \neq 0$. If $z_\infty < 0$, then $y_\infty - z_\infty > 0$ and $f_c(a, b) = f_1(b)$ in some neighborhood of $(z_\infty, y_\infty - z_\infty)$. If $z_\infty > 0$, then $z_\infty > 0 \geq -p_0$ and $f_c(a, b) = f_1(b)$ in some neighborhood of $(z_\infty, y_\infty - z_\infty)$. Since $f_1$ is continuous, we have $\lim_{t\to\infty} \dot{z}(t) = \lim_{t\to\infty} f_c(z(t), y(t) - z(t)) = f_1(y_\infty - z_\infty) = f_1(-z_\infty)$. By inspection of the function $f_1$, we see that $z_\infty \neq 0$ implies that $\dot{z}$ has a nonzero limit. However, since $\dot{z} \in L_2$, it cannot have a nonzero limit. Thus we conclude by condradiction that $z_\infty = 0$. □

**Lemma 13.** *If $v = \Delta y$ with $y \in W_2$, then $\langle v, \dot{y} - v\rangle \geq -\beta|y(0)|^2$.*

*Proof.* Let $z = \Delta_z v$ and define the variable $r(t) = y(t) - z(t)$ and the set $M = \{t : z(t) > -p_0 \text{ or } r(t) \geq 0\}$, then

$$\begin{aligned}\langle v, \dot{y} - v\rangle = \langle \dot{z}, \dot{y} - \dot{z}\rangle &= \int_0^\infty \dot{z}(t)\dot{r}(t)dt \\ &= \int_M f_1(r(t))\dot{r}(t)dt \leq \beta\|y\|\|\dot{y}\| + \beta^2\|v\|^2\end{aligned}$$

Since $y \in W_2$, we have that $y$ is absolutely continuous and thus $r$ is absolutely continuous. Since $r, z$ are absolutely continuous functions and since by Lemma 12, we have $z(t) \to 0$, we can partition the set $M$ into the countable union of sequential disjoint intervals $\bigcup_i I_i \bigcup I_f$ where $I_i = [T_{a,i}, T_{b,i})$ with $\{T_{a,i}\}, \{T_{b,i}\} \subset \mathbb{R}^+$ and $I_f = [T_{a,f}, \infty)$. To see that the intervals are closed on the left, suppose $I_i$ were open on the left. Then, since $T_{a,i} \notin M$, $z(T_{a,i}) = -p_0$ and $r(T_{a,i}) < 0$. However, since $r$ is continuous, $r(T_{a,i} + \eta) < 0$ for $\eta$ sufficiently small. Since $r(t) < 0$ implies $\dot{z}(t) \leq 0$, we have that $z(T_{a,i} + \eta) \leq -p_0$ and thus $T_{a,i} + \eta \notin M$ for $\eta$ sufficiently small, which is a contradiction. Thus all the intervals are closed on the left. Similarly, one can show that all the intervals are open on the right.

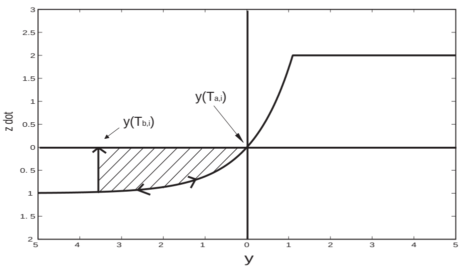


Figure 5: Value of $y, \dot{z}$ at times $T_{a,i}$ and $T_{b,i}$

Now, consider time $T_a > 0$, where $T_a \in M$ defines the start of one of the intervals described above. If $z(T_a) > -p_0$, then since $z$ is continuous, $z(T_a - \eta) > -p_0$ for all $\eta$ sufficiently small. Therefore $T_a - \eta \in M$ for all $\eta$ sufficiently small. This contradicts the statement that the intervals are disjoint. We thus conclude $z(T_a) = -p_0$ and consequently $r(T_a) \geq 0$ by definition of $M$. Now suppose $r(T_a) > 0$. Since $r$ is continuous, $r(T_a - \epsilon) > 0$ and consequently $T_a - \epsilon \in M$ for all $\epsilon$ sufficiently small, which contradicts the statement that the intervals are disjoint. Therefore we conclude $r(T_a) = 0$ if $T_a \neq 0$. Then

$$\begin{aligned}\langle v, \dot{y} - v\rangle &= \sum_i \int_{I_i} f_1(r(t))\dot{r}(t)dt + \int_{T_{a,f}}^\infty f_1(r(t))\dot{r}(t)dt \\ &= \sum_i \int_{T_{a,i}}^{T_{b,i}} f_1(r(t))\dot{r}(t)dt + \int_{T_{a,f}}^\infty f_1(r(t))\dot{r}(t)dt\end{aligned}$$

We will assume that $T_{a,1} = 0$. If $T_{a,1} \neq 0$, we have $r(T_{a,1}) = 0$ and the proof becomes simpler. Since $f_1(r)$ is continuous in $r$ and $r(t)$ is absolutely continuous in time, by the substitution rule we have

$$\begin{aligned}\langle v, \dot{y} - v\rangle &= \sum_i \int_{r(T_{a,i})}^{r(T_{b,i})} f_1(r)dr \\ &= \int_{r(0)}^{r(T_{b,1})} f_1(r)dr + \sum_{i\neq 1} \int_0^{r(T_{b,i})} f_1(r)dr \\ &= \int_{r(0)}^{0} f_1(r)dr + \sum_i \int_0^{r(T_{b,i})} f_1(r)dr\end{aligned}$$

Since $f_1 \in$ sector$[0,\beta]$, $\int_0^{r(T_{b,i})} f_1(r)dr \geq 0$ for any $r(T_{b,i}) \in \mathbb{R}$. The summation converges since it is bounded, increasing. Furthermore, since $r(0) = y(0) - z(0) = y(0)$ and $|\int_0^y f_1(r)dr| \leq f_1(y)y \leq \beta y^2$ for any $y$, we have

$$\langle v, \dot{y} - v\rangle = \int_{y(0)}^{0} f_1(r)dr + \sum_i \int_0^{r(T_{b,i})} f_1(r)dr \geq -\beta|y(0)|^2$$

□

To summarize this section, we have shown that $\Delta$ is bounded and that for any $y \in W_2$ and $v = \Delta y$, we have $\langle v, \beta y - v\rangle \geq 0$ and $\langle v, \dot{y} - v\rangle \geq -\beta|y(0)|^2$. Therefore, we conclude that $\Delta$ satisfies the IQC defined by $\Pi_B, \lambda = \frac{2}{\pi}$, since

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} \begin{bmatrix}\hat{y}(j\omega)\\ \hat{v}(j\omega)\end{bmatrix}^* \begin{bmatrix}0 & \beta \\ \beta & -\frac{4}{\pi} - 2\end{bmatrix} \begin{bmatrix}\hat{y}(j\omega)\\ \hat{v}(j\omega)\end{bmatrix} + \frac{4}{\pi}\langle v, \dot{y}\rangle \\ \geq -\frac{4\beta}{\pi}|y(0)|^2\end{aligned}$$

We conclude as a consequence that $\kappa\Delta$ satisfies the IQC defined by $\Pi_B, \lambda = \frac{2}{\pi}$ for any $\kappa \in [0,1]$, since

$$\begin{aligned}&\frac{2}{\pi}\langle \kappa v, \dot{y} - \kappa v\rangle + \langle \kappa v, \beta y - \kappa v\rangle \\ &\geq \kappa\big(\frac{2}{\pi}\langle v, \dot{y} - v\rangle + \langle v, \beta y - v\rangle\big) \\ &\geq -\kappa\frac{2\beta}{\pi}|y(0)|^2 \geq -\frac{2\beta}{\pi}|y(0)|^2\end{aligned}$$

### 3.3 Properties of $G$

Recall that we define the map $G$ as follows. $w = Gu$ if

$$w(t) = \int_{t-\tau}^{t} u(\theta)d\theta.$$

Recall that $G$ can be represented as a transfer function in the frequency domain by

$\hat{G} = \frac{1-e^{-j\omega\tau}}{j\omega}$. Now, examine the term

$$\begin{aligned}
&\begin{bmatrix} \hat{G}(j\omega) \\ I \end{bmatrix}^* \left( \Pi_B + \begin{bmatrix} 0 & \lambda j\omega^* \\ \lambda j\omega & 0 \end{bmatrix} \right) \begin{bmatrix} \hat{G}(j\omega) \\ I \end{bmatrix} \\
&= \begin{bmatrix} \frac{1-e^{-j\omega\tau}}{j\omega} \\ 1 \end{bmatrix}^* \begin{bmatrix} 0 & \beta + \frac{2}{\pi} j\omega^* \\ \beta + \frac{2}{\pi} j\omega & -\frac{4}{\pi} - 2 \end{bmatrix} \begin{bmatrix} \frac{1-e^{-j\omega\tau}}{j\omega} \\ 1 \end{bmatrix} \\
&= 2 \cdot \text{Real} \left( \beta \frac{1-e^{-j\omega\tau}}{j\omega} - \frac{2}{\pi} e^{-j\omega\tau} - 1 \right) \\
&= 2 \left( \beta\tau \frac{\sin(\omega\tau)}{\omega\tau} - \frac{2}{\pi} \cos(\omega\tau) - 1 \right) = 2p(\omega\tau)
\end{aligned}$$

Define $p_0(\omega) = p(\omega)$ for $\beta\tau = \pi/2$. The plot of $p_0(\omega) = \frac{\pi}{2} \frac{\sin(\omega)}{\omega} - \frac{2}{\pi} \cos(\omega) - 1$ is given

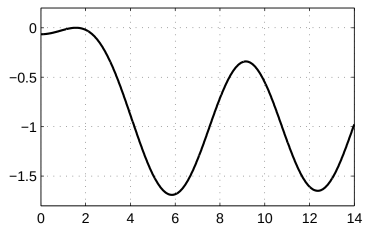


Figure 6: Plot of $p_0(\omega) = \frac{\pi}{2} \frac{\sin(\omega)}{\omega} - \frac{2}{\pi} \cos(\omega) - 1$ vs. $\omega$

in Figure 6. As one can see, the function is non-positive near the origin. In fact, for any $\beta\tau < \pi/2$, there exists an $\eta > 0$ such that $p(\omega) < -\eta$ for all $\omega$. To see this, consider the following domains

$\omega > 2\pi$ We have that $|\frac{\sin(\omega)}{\omega}| \leq 1/2\pi$ for $|\omega| > 2\pi$ and $-1 - \frac{2}{\pi}\cos(\omega)) \leq -1 + 2/\pi$ for all $\omega$. Therefore, $p(\omega) \leq \frac{1}{4} - (1 + \frac{2}{\pi}\cos(\omega)) \leq \frac{2-3\pi/4}{\pi} < -.1$.

$\pi - .1 \leq \omega \leq 2\pi$ For the interval $[\pi - .1, 2\pi]$, we have $\frac{\sin(\omega)}{\omega} \leq .04$. Therefore $p(\omega) \leq -1 + 2/\pi + .04\frac{\pi}{2} < -.3$.

$0 \leq \omega < \pi - .1$ One can see from plot of $p_0$ in Figure 6 that $p_0(\omega) \leq 0$ on the interval $[0, \pi - .1]$. Since $\beta\tau < \pi/2$, we can let $\epsilon = \pi/2 - \beta\tau > 0$. Then $p(\omega) = p_0(\omega) - \epsilon\frac{\sin(\omega)}{\omega} \leq -\epsilon\frac{\sin(\omega)}{\omega} < -.2\epsilon$ on $[0, \pi - .1]$.

Therefore, for any $\beta\tau < \frac{\pi}{2}$, let $\eta = \min\{.1, .2(\pi/2 - \beta\tau)\}$. Then $p(\omega) < -\eta$ for all $\omega \in \mathbb{R}$. Since $\beta\tau = \tau(e^{\frac{\alpha}{\tau}p_0} - 1)/p_0 = \alpha/\alpha_{\max}$, if $\alpha < \pi\alpha_{\max}/2$, we have that $\beta\tau < \pi/2$, and hence if $0 < \alpha < \pi\alpha_{\max}/2$, condition 4 of Theorem 10 is satisfied.

### 3.4 Stability of the Interconnection

We conclude our discussion of input-output stability with the following Theorem concerning the stability of the the interconnection of $\Delta$ and $G$.

**Theorem 14.** *Suppose $\alpha \in (0, \pi\alpha_{max}/2)$. Then the map $B$, defining the interconnection of $\Delta$ and $G$ is $W_2 \times L_2$ stable on $W_2$.*

*Proof.* We have shown that $G$ is a linear causal bounded operator with $\hat{G}(s), s\hat{G}(s) \in \mathcal{A}$. We have also shown that the interconnection of $G$ and $\kappa\Delta$ is well-posed for all $\kappa \in [0,1]$ and that $\kappa\Delta$ satisfies the IQC defined by $\Pi_B, \lambda = \frac{2}{\pi}$ for all $\kappa \in [0,1]$. Finally, we have that for all $\alpha \in (0, \pi\alpha_{\max}/2)$, condition 4 of Theorem 10 is satisfied. We can therefore use Theorem 10 to prove $W_2 \times L_2$ stability on $W_2$ of the interconnection for any $\alpha \in (0, \pi\alpha_{\max}/2)$. □

### 3.5 Asymptotic Stability

Recall that the original solution map $A$ is defined by the following.

$$\dot{z}(t) = f_c(z(t), -z(t-\tau)) \qquad t \geq 0 \tag{8}$$

$$z(t) = x_0(t) \qquad t \in [-\tau, 0] \tag{9}$$

The interconnection maps $B$ and $B_z$, however, are defined by the following differential equation.

$$\dot{z}(t) = f_c(z(t), f(t) - z(t-\tau)) \qquad t \geq 0 \tag{10}$$

$$z(t) = 0 \qquad t \leq 0 \tag{11}$$

In the previous section, we have proven $W_2 \times L_2$ stability on $W_2$ of the map $B$ where $B$ represents a reformulation of the problem in the input-output framework. We would like to show, however, that for some $X \subset \mathcal{C}_\tau$ this result also implies asymptotic stability on $X$ of the solution map $A$, where $A$ represents the original formulation of the problem. This is done in the following Theorem.

**Theorem 15.** *Suppose $\alpha \in (0, \pi\alpha_{max}/2)$. Then the delay-differential equation* (3) *describing the algorithm proposed by Paganini et al. [14] is asymptotically stable on $X = \{x : x \in W_2 \cap \mathcal{C}_\tau, x(t) \geq -p_0, x(-\tau) > -p_0\}$.*

*Proof.* We have already shown $W_2 \times L_2$ stability on $W_2$ of the map $B$. Let $x_0 \in X$ be an arbitrary initial condition. Theorem 16 in the Appendix states that for any initial condition $x_0 \in X$, there exists some $f \in W_2$ and $T > 0$ such that $A(x_0, t) = B_z(f, t+T)$ for all $t \geq 0$. Let $(y, u) = B(f)$, then $y \in W_2$. Furthermore, recall $B_z f = \Delta_z y$ where $B_z$ is the map to internal variable $z$. By Lemma 12, if $y \in W_2$, then $\lim_{t\to\infty} \Delta_z(y,t) = 0$. Therefore, $\lim_{t\to\infty} A(x_0,t) = \lim_{t\to\infty} B_z(f, t+T) = \lim_{t\to\infty} \Delta_z(y, t+T) = 0$. □

## 4 Implementation

To implement the proposed algorithm in the Internet framework, the window size is adjusted to deliver the required packet rate as given by equation (1). In implementation, the delay is unlikely to be known. In this case, a bound on the expected delay size can be used. Overestimation of the delay will result in an increased stability margin.

Modification of the link can take many forms. Price information from the link must be fed back to the source. Since queues themselves integrate excess rate, price of a congested resource can be computed directly using the queueing delay. However, this approach results in non-empty equilibrium queues and the possibility of unmodeled dynamics due to variable queueing delays. If a link instead uses a virtual capacity to avoid non-empty equilibrium queues, then explicit integration of incoming packets would be required and another mechanism must be used to feed back price information. An example of direct feedback of price information using packet marking is given by ECN. In one of the proposed implementations, packets are randomly marked at each link with probability

$1 - \phi^{-p_j(t)}$ for some fixed $\phi > 1$. Thus, assuming no duplications, if $\nu$ is the percentage of marked packets received at the source, then the aggregate price can be measured as $q_i(t) = -\frac{\log(1-\nu)}{\log(\phi)}$. This variant is known as random exponential marking.

## 5 Conclusion

To summarize, for the case of a single source with a single link, we have demonstrated both input-output and asymptotic stability. We have used a generalized passivity framework to decompose a difficult nonlinear, discontinuous, infinite-dimensional problem into separate linear, infinite-dimensional and nonlinear, finite-dimensional subproblems, each of which is amenable to existing analysis techniques. A key feature of the analysis of the nonlinear subsystem was the existence of a sector bound on the nonlinearity.

In addition to the case of a single source with a single link, the result presented in this paper applies directly to the case of multiple sources with identical fixed delay. The proof may also be easily adapted to alternate implementations of the proposed linearized protocols so long as they admit a similar sector bound on the nonlinearity. Although we would like to have proven a result in the case of multiple heterogeneous delays, we have as yet not been able to construct a sector bound similar to that in the case of a single delay. The root of the problem seems to lie in the discontinuity due to the projection used in the gradient projection algorithm. In addition to analysis of multiple heterogeneous delays, directions for future research include analysis of other proposed congestion control protocols with an aim toward reducing conservative of existing nonlinear stability results.

## 6 Appendix

**Theorem 16.** *For any initial condition $x_0 \in W_2$ with $x_0(-\tau) > -p_0$, $x_0(\theta) \geq -p_0$, there exists a $f \in W_2$ and $T > 0$ such that $A(x_0, t) = B_z(f, t+T)$ for $t \geq 0$.*

*Proof.* Recall the map $B_z$ is defined by the following dynamics for $z = B_z y$. $z(t) = 0$ for $t \leq 0$ and for $t \geq 0$

$$\dot{z}(t) = f_c\big(z(t), y(t) - z(t-\tau)\big)$$

Now suppose we let $y(t) = f(t) + z(t-\tau)$ on a finite interval $[0, T']$ for some $f \in W_2$. The system is still well-posed and $y \in W_2$ if $f \in W_2$ since the derivative of $z$ is bounded. The dynamics are now given by $z(t) = 0$ for $t \leq 0$ and the following for $t \geq 0$

$$\dot{z}(t) = f_c\big(z(t), f(t)\big)$$

**Part 1:** The first part of the proof is to construct a $f \in W_2$ that drives the state $z(t)$ to $z(T') = x_0(0)$ for some $T' \geq 0$. Furthermore, for continuity with Part 2, we require that $f(T') = -x_0(-\tau)$. If $x_0(-\tau) = 0$ and $x_0(0) = 0$, then we are done. Otherwise, there are 4 cases to consider.

$x_0(-\tau) < 0, x_0(0) > 0$ First, let $f(t) = \epsilon t$ until $T_1 = -\frac{x_0(-\tau)}{\epsilon}$ for $\epsilon > 0$. Let $\epsilon$ be sufficiently large so that $z(T_1) < x_0(0)$. Such an $\epsilon$ exists since $\dot{z}$ is bounded. Let $f(t) = -x_0(-\tau) > 0$ until time $T'$ such that $z(T') = x_0(0)$. Such $T'$ exists since $z(t)$ is now linearly increasing.

$x_0(-\tau) > 0, x_0(0) < 0$ This case is handled similarly to the previous one.

$x_0(-\tau) < 0, x_0(0) < 0$ For this case, let $f(t) = -\epsilon t$ until time $T_1 = \frac{1}{\epsilon}$. Then let $f(t) = -1$ until time $T_2$ such that $z(T_2) = x_0(0) - \gamma$ for some $\gamma > 0$. Such time exists for any $\gamma < p_0 - x_0(0)$ since $z(t)$ is linearly decreasing for $z(t) \geq -p_0$. Then let $f(t) = x_0(0) - \gamma + \lambda t$ for time $\Delta t = \frac{1}{\lambda}(x_0(-\tau) + 1)$. Make $\lambda$ sufficiently large so that $z(T_2 + \Delta t) < x_0(0)$. This is possible since $\dot{z}$ is bounded. Finally let $f(t) = -x_0(-\tau) > 0$ until time $T'$ when $z(T') = x_0(0)$. Such $T'$ exists since $z(t)$ is now linearly increasing.

$x_0(-\tau) > 0, x_0(0) > 0$ **or** $x_0(0) = 0$ **or** $x_0(-\tau) = 0$ These cases are handled similarly to the previous one.

**Part 2:** For time $t \in [T', T' + \tau]$, let $f(t) = -x_0(t - T')$. We then have $y(T' + \tau) = x_0(t - T') - z(T') = x_0(-\tau) - x_0(-\tau) = 0$. Let $y(t) = 0$ for $t \geq T' + \tau$. Therefore $y \in W_2$ and we conclude that the dynamics of the interconnection for time $t \in [T', T' + \tau]$ are given by $z(T') = x_0(0)$

$$\begin{aligned} \dot{z}(t) &= f_c(z(t), y(t) + z(t - \tau)) && (12) \\ &= f_c(z(t), x_0(t - \tau)) && (13) \end{aligned}$$

And for $t \geq T' + \tau$, we have

$$\begin{aligned} \dot{z}(t) &= f_c(z(t), y(t) + z(t - \tau)) && (14) \\ &= f_c(z(t), -z(t - \tau)) && (15) \end{aligned}$$

Therefore, we have that $A(y, t + T') = B(x_0, t)$.

□

**Lemma 17.** *Suppose that for any $\delta > 0$, there exists a $T_\delta$ such that $T_2 > T_1 > T_\delta$ implies $z(T_2)^2 - z(T_1)^2 < \delta$. Then for any infinite, increasing sequence, $\{T_i\}$, $\{z(T_i)^2\}$ is a Cauchy sequence.*

*Proof.* Proof by contradiction. Suppose that $\{z(T_i)^2\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ such that for any $N > 0$, there exists $i, j > N$ such that $|z(T_i)^2 - z(T_j)^2| > \epsilon$.

Let $\delta = \frac{\epsilon}{2}$. By assumption there exists a $T_\delta$ such that for all $T_2 > T_1 > T_\delta$,

$$z(T_2)^2 - z(T_1)^2 < \frac{\epsilon}{2} \tag{16}$$

Since $\{T_i\}$ is strictly increasing, infinite, there exists a $N > 0$ such that $T_N > T_\delta$. Since $\{z(T_i)^2\}$ is not Cauchy, there exists $i_1, j_1 > N$ such that $|z(T_{i_1})^2 - z(T_{j_1})^2| > \epsilon$. Without loss of generality, we may assume $i_1 > j_1$. We now have one of the two possibilities.

- $z(T_{i_1})^2 - z(T_{j_1})^2 > \epsilon$
- $z(T_{i_1})^2 - z(T_{j_1})^2 < -\epsilon$

However, by Equation (16), the first option is not possible since $T_i > T_j > T_N > T_\delta$. Therefore $z(T_{i_1})^2 < z(T_{j_1})^2 - \epsilon$. Thus for all $k > i_1$, by (16) and since $T_k > T_{i_1} > T_{j_1} > T_N \geq T_\delta$, we have

$$z(T_k)^2 < z(T_{i_1})^2 + \frac{\epsilon}{2} < z(T_{j_1})^2 - \epsilon + \frac{\epsilon}{2} = z(T_{j_1})^2 - \frac{\epsilon}{2}$$

Now, let $N = i_1$. Again, since $\{z(T_i)^2\}$ is not Cauchy, there exists $i_2, j_2 > i_1$ such that $|z(T_{i_2})^2 - z(T_{j_2})^2| > \epsilon$. Using the method above, for all $k > i_2$ since $T_k > T_{i_2} > T_{j_2} > T_{i_1} > T_\delta$, by (16) we have

$$z(T_k)^2 < z(T_{i_2})^2 + \frac{\epsilon}{2} < z(T_{j_2})^2 - \frac{\epsilon}{2} < z(T_{j_1})^2 - 2\frac{\epsilon}{2}$$

Repeating $n$ times, we find a $i_n$ such that for all $k > i_n$, $z(T_k)^2 < z(T_{j_1})^2 - n\frac{\epsilon}{2}$. Since we can assume $z(T_{j_1})^2$ is finite, let $n$ be sufficiently large and we have the existence of an $i$ such that that $z(T_i)^2 < 0$, which is a contradiction. Thus we have that $\{z(T_i)^2\}$ is a Cauchy sequence. □

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