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On the Analysis of Systems Described by Classes of Partial Differential Equations

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***Abstract*— We provide an algorithmic approach for the analysis of infinite dimensional systems described by Partial Differential Equations. In particular, we look at the stability properties of a class of strongly continuous semigroups generated by nonlinear parabolic partial differential equations with appropriate boundary conditions. Our approach is based on the application of semidefinite programming to the computation of Lyapunov-type certificates defined by polynomial functions. An illustrative example is given.**

I. INTRODUCTION

Many physical systems evolve not only as a function of time but also of space [1] and are many times described by Partial Differential Equations (PDEs); take for example the evolution of the temperature distribution inside a homogeneous rod that is being heated on its outer surface, which can be modeled as a linear parabolic Partial Differential Equation – a well-studied class of PDEs. If, however, the rod is inhomogeneous and has a spatially varying heat conductivity, itself dependent on the temperature, the describing equation becomes nonlinear. In this case, in order to evaluate certain properties of the solution one usually resorts to numerical simulation.

Numerical simulations are also typically used when the underlying PDE is linear but does not fall in one of the standard, well-documented classes, or when the spatial domain is complicated. In many cases, the simulation procedure can be very demanding indeed, depending on the fidelity that is required in the solution, the order of the PDE, etc. Many times we are interested in the *robust* properties of the solution, e.g., if the stability of the solution in a particular norm is retained as parameters or initial conditions change. Other times what is needed is not the solution to the PDE *per se*, given certain initial conditions, but rather what values some *output functional* of the solution takes. For example, we may be interested in the *mean* and *variance* of the temperature inside the rod.

In addressing the question of robust stability of nonlinear PDEs, numerical simulations alone cannot provide a definite answer, but only give counterexamples to assertions. However, definite answers to such questions can be obtained through the use of Lyapunov analysis in much the same way it is used in the analysis of nonlinear systems described by ODEs. Unfortunately, the relevant Lyapunov functionals may be difficult to construct by hand. Indeed, the fact that these functionals act on an infinite dimensional space makes the construction a particularly difficult task [2], [3], [4]. A comprehensive discussion on the methods for analysis of linear PDEs can be found in [5].

In this paper we use Lyapunov analysis and a polynomial programming methodology to answer stability questions for systems described by parabolic PDEs. In particular, we construct certificates algorithmically using semidefinite programming which ensure stability of systems of partial differential equations. A similar technique was applied in [6] and [7] to analyze systems described by differential equations with delay.

The paper is organized as follows. In section II we provide some background results on analysis of distributed parameter systems and within this section we discuss stability definitions. In Section III we pose the question of stability within the Lyapunov framework. In Section IV we present the algorithmic methodology we propose to use, and in section V we show how stability can be determined using this methodology. We conclude this paper in Section VI.

II. PRELIMINARIES

We will denote by $\mathbb{R}^n$ the $n-$dimensional Euclidean space, and by $\mathbb{N}$ the natural numbers. A domain $\Omega \subset \mathbb{R}^n$ is a connected, open subset of $\mathbb{R}^n$, and $\overline{\Omega}$ denotes its closure. The boundary $\partial\Omega$ of a domain Ω is defined by $\partial\Omega = \overline{\Omega} \backslash \Omega$ where '$\backslash$' is set subtraction. We denote the continuous functions on Ω by $C(\Omega)$, and those whose who have all derivatives up to order k continuous by $C^k(\Omega)$. In a similar fashion, $H(\Omega)$ will denote functions in a Hilbert space H on Ω. We also use the following Schwartz multi-index notation. We denote $\alpha = (\alpha_1, \ldots, \alpha_n)$ for $\alpha_i \in \mathbb{N}$, and define $|\alpha| = \sum_i \alpha_i$ and $\alpha! = (\alpha_1!) \cdots (\alpha_n!)$. Then the following abbreviations are allowed:

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

where $x = (x_1, \ldots, x_n)$. It is also customary to write $u_{x_1 x_2} := \frac{\partial^2 u}{\partial x_1 \partial x_2}$, and we will employ this notation. In this way, an $m-$th order PDE in $\mathbb{R}^n$ can be written as follows:

$$F(x, D^\alpha u) = 0, \quad |\alpha| \leq m.$$

If F is linear in u and all its derivatives, we write:

$$Lu = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u$$

where $a_\alpha(x) \in C^{|\alpha|}(\Omega)$ is a real valued function.

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We denote by $L^2(\Omega)$ the measureable real-valued functions u on Ω for which $\int_\Omega |u(x)|^2 dx < \infty$, and define $\|u\|_2 = \left(\int_\Omega |u(x)|^2 dx\right)^{1/2}$; $L^2(\Omega)$ is then a Hilbert space with inner product $\langle u, v\rangle = \int_\Omega u(x)\overline{v(x)}dx$. For further details the reader is referred to [8].

In this paper, we consider evolution equations which take the following form:

$$u_t = \dot{u} = Au, \quad u(0) = u_0 \in \mathcal{M} \subset H(\Omega)$$

where H is an infinite-dimensional Hilbert space and A is a nonlinear operator defined on $\mathcal{M}$, a closed subset of H.

Definition 1: Let $\mathcal{M}$ be closed in H. A semigroup of contractions on $\mathcal{M}$ is a function $T : [0, \infty) \to \mathcal{M}$ such that

1) $T(t+s)u = T(t)T(s)u, \quad \forall u \in \mathcal{M}, \quad t, s \geq 0$
2) $T(0)u = u, \quad \forall u \in \mathcal{M}$
3) $T(t)u$ is continuous in $t \geq 0, \quad \forall u \in \mathcal{M}$
4) $\|T(t)u - T(t)v\| \leq \|u - v\|, \quad \forall t > 0,\ \forall u, v \in \mathcal{M}$.

Furthermore, we say that T is a semigroup of type ω if 4) above is replaced by

$$\|T(t)u - T(t)v\| \leq e^{\omega t}\|u - v\|, \quad \forall t > 0,\ \forall u, v \in \mathcal{M}.$$

We will assume in this paper that A generates a nonlinear semigroup of contractions, i.e., continuous solutions to the PDE exist in $\mathcal{M}$ and are unique. Checking that A defines a nonlinear semigroup, i.e., that there exists some T such that

$$\lim_{t\to 0^+}\left\{\frac{T(t)u - u}{t}\right\} = Au, \quad u \in \mathcal{M} \subset H$$

may be quite hard. In this regard, the Cauchy-Kowalevski theorem plays an important role in that it ensures the existence and uniqueness of a real analytic solution under certain assumptions on the structure and form of F and the boundary conditions. Additionally, existence of such a semigroup is ensured if A is dissipative.

Definition 2: If H is a real Hilbert space, we say that a nonlinear operator A is dissipative if

$$\langle v_1 - v_2, u_1 - u_2\rangle \leq 0, \quad \forall u_1, u_2 \in \mathcal{M}, v_i = Au_i.$$

We know that A generates a nonlinear semigroup of type ω if $A - \omega I$ is maximal dissipative, i.e. if A has no proper dissipative extension in H. This is sometimes found under the name 'maximal monotone' — see [9], [10].

A. Stability Analysis: Lyapunov Functionals

Recall that if H is the Euclidean space $\mathbb{R}^n$, then Lyapunov's theorem states that the zero equilibrium of a system $\dot{u} = f(u)$ is globally asymptotically stable if there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0$, $V > 0$ in $\mathbb{R}^n\backslash\{0\}$, V is radially unbounded and $\dot{V} = \sum_{i=1}^n \frac{\partial V}{\partial u_i} f_i(u) < 0$ in $\mathbb{R}^n\backslash\{0\}$. In the linear case, an $n \times n$ matrix A is Hurwitz if and only if the matrix equation $A^*P + PA = -I$ has a unique solution $P \succ 0$, where A^* is the adjoint of A. If we view A as an operator on $\mathbb{R}^n$, then a Hurwitz matrix is a bounded operator; i.e. it generates a strongly continuous semigroup $\{T(t); t \geq 0\}$ and $\|T(t)\| \leq Me^{-\beta t}$ for $t \geq 0$ where M, β are positive constants.

The work by Datko [11] has extended the Lyapunov theorem to strongly continuous semigroups of operators on a Hilbert space, thus allowing the analysis of infinite dimensional systems in the time domain. For nonlinear semigroups, see [12]. In this subsection, we will present some stability notions, and state Lyapunov's theorem for nonlinear semigroups in a Hilbert space. A more complete discussion can be found in [2].

Definition 3: Let $\mathcal{C}$ be a closed subset of a complete metric space. A *dynamical system* on $\mathcal{C}$ is a family of maps $\{S(t) : t \geq 0\}$ such that:

- For each $t \geq 0$, $S(t)$ is continuous from $\mathcal{C}$ to $\mathcal{C}$,
- For each $u \in \mathcal{C}$, $t \to S(t)u$ is continuous,
- $S(0) = I$ on $\mathcal{C}$,
- $S(t)(S(\tau)u) = S(t+\tau)u$ for all $u \in \mathcal{C}$ and $\tau, t \geq 0$.

If $\{S(t) : t \geq 0\}$ is a dynamical system on $\mathcal{C}$, the *positive semi-orbit* through a point $u \in \mathcal{C}$ (i.e., the solution through u) is $\gamma(u) = \{S(t)u, t \geq 0\}$. We say u^* is a *steady-state* if $\gamma(u^*) = \{u^*\}$, and that $\gamma(u)$ is a *periodic orbit* if there exists a $p > 0$ such that $\gamma(u) = \{S(t)u, 0 \leq t \leq p\} \neq \{u\}$.

Properties of the orbit, such as stability, asymptotic stability etc., are defined as follows, where 'dist' denotes the metric on $\mathcal{C}$:

Definition 4: An orbit $\gamma(u)$ is *stable* if for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for all $t \geq 0$, $\text{dist}(S(t)u, S(t)v) < \epsilon$ whenever $\text{dist}(u, v) \leq \delta(\epsilon)$, $v \in \mathcal{C}$. An orbit $\gamma(u)$ is *unstable* if it is not stable. An orbit $\gamma(u)$ is *asymptotically stable* if it is stable and also there is a neighbourhood $B = \{v \in \mathcal{C} : \text{dist}(u, v) < r\}$ such that

$$\text{dist}(S(t)v, S(t)u) \to 0 \text{ as } t \to \infty$$

uniformly for $v \in B$.

Note that the definition of stability is different for different metrics 'dist'.

We now turn to the basic tool for stability analysis in the time-domain for these systems, what is known as a Lyapunov function. Given $\{S(t), t \geq 0\}$ a dynamical system on $\mathcal{C}$, a *Lyapunov function* is a continuous real-valued function V on $\mathcal{C}$ such that

$$\dot{V}(u) := \lim_{t\to 0^+} \sup \frac{1}{t}\{V(S(t)u) - V(u)\} \leq 0$$

for all $u \in \mathcal{C}$, not excluding the possibility $\dot{V}(u) = -\infty$. We then have the following theorem:

Theorem 5: Let $\{S(t), t \geq 0\}$ be a dynamical system on $\mathcal{C}$, and let 0 be a steady state in $\mathcal{C}$. Suppose V is a Lyapunov function which satisfies $V(0) = 0$, $V(u) \geq c(\|u\|)$ for $u \in \mathcal{C}$, $\|u\| = \text{dist}\{u, 0\}$, where $c(\cdot)$ is a continuous strictly increasing function, $c(0) = 0$ and $c(r) > 0$ for $r > 0$. Then 0 is stable. If in addition $\dot{V}(u) \leq -c_1(\|u\|)$ where $c_1(\cdot)$ is also continuous, increasing and positive with $c_1(0) = 0$, then 0 is asymptotically stable.

As in the case of nonlinear ODEs, finding a Lyapunov function in any particular case is usually difficult, especially finding one that satisfies all the above requirements.

Note that to actually show that $\{S(t), t \geq 0\}$ defines a dynamical system on $\mathcal{C}$, we may have to resort to dissipativity (or prove that the generator of the semigroup is sectorial, see [3] for more details), as discussed in the previous section. For closed linear operators A the Hille-Yosida theorem gives an answer to this question through the examination of the resolvent set of A.

In this paper we will present a technique that can be used to test whether a steady-state for a nonlinear PDE is stable, asymptotically stable etc., by constructing a Lyapunov function algorithmically. The same algorithm can be used to verify that a nonlinear operator A defined on a closed subset $\mathcal{C}$ of a Hilbert space H generates a nonlinear semigroup.

III. PROBLEM FORMULATION

We consider a class of partial differential equations of parabolic type. More specifically, suppose that $x \in \mathbb{R}^n$, a_i, b_i, and c are polynomials for $i = 1, \ldots, n$ and that the following holds on some connected region Ω with boundary Γ.

$$u_t = \sum_{i=1}^{n} a_i(x) u_{x_i x_i} + \sum_{i=1}^{n} b_i(x,u) u_{x_i} + c(x,u) \quad (1)$$

Furthermore, we assume that $u(x)$ is a known function of x along Γ so that one can calculate

$$\int_\Gamma h(x, u(x)) \nu_i dx \quad (2)$$

for an arbitrary polynomial function h where ν_i is the i^{th} component of the outward normal of Γ. We also assume that $u(0, x)$ is such that existence of solutions over some time interval is ensured.

The question of stability of dynamical systems defined by equations of the form given in Equation (1) can be addressed through the use of Lyapunov functionals. In particular, this paper is concerned with the search for Lyapunov functionals which take the following form.

$$V(t) = \int_\Omega p(x, u(t,x)) dx \quad (3)$$

In this case, the derivative of the Lyapunov functional along trajectories of the system is:

$$\dot{V}(t) = \int_\Omega \frac{\partial p(x,u)}{\partial u} u_t dx = \int_\Omega \sum_{i=1}^{n} \frac{\partial p(x,u)}{\partial u} a_i(x) u_{x_i x_i} dx + \int_\Omega \left(\sum_{i=1}^{n} \frac{\partial p(x,u)}{\partial u} b_i(x,u) u_{x_i} + \frac{\partial p(x,u)}{\partial u} c(x,u) \right) dx \quad (4)$$

Objective: The goal of this paper is to find ways of using convex optimization to construct polynomial functions p, such that the Lyapunov functional given by Equation (3) is positive for all $u \in \mathcal{C}[\mathbb{R}^+ \times \Omega]$ while the derivative of the functional, given by Equation (4) is negative for all $u \in \mathcal{C}[\mathbb{R}^+ \times \Omega]$.

Theorem 6: Suppose p is continuous and $p(x, 0) = 0$ for all $x \in I$. The following are equivalent.

(1) $\int_I p(x, u(x)) dx \geq 0$ for all $u \in L_2$, $x \in I$.
(2) $p(x, y) \geq 0$ for all $x \in I$,$y \in \mathbb{R}^n$.

Proof: Obviously $(2) \Rightarrow (1)$. Now suppose that (2) is false. Then there exists some x_0, y_0 such that $p(x_0, y_0) \leq -\epsilon$ for $\epsilon > 0$. Furthermore, since p is continuous in x, there exists some $\beta > 0$ such that $p(x, y_0) \leq -\epsilon/2$ for all $x \in \mathcal{B}(x_0, \beta)$. Now let $u(x) := \begin{cases} y_0 & x \in \mathcal{B}(x_0, \beta) \\ 0 & \text{otherwise} \end{cases}$. Then we have that

$$\int_I p(x, u(x)) dx = \int_{\mathcal{B}(x_0,\beta)} p(x, y_0) dx \leq - \int_{\mathcal{B}(x_0,\beta)} \epsilon/2 dx = -\beta\epsilon < 0$$

Thus by contradiction, we have that $p(x, y) \geq 0$ for all $x \in I$, $y \in \mathbb{R}^n$. ■

To begin, we will restrict our search to functionals of the form given by Equation (3) where p is a polynomial with homogeneous partial degree 2 or more in $u(t, x)$. In this case, the positivity of V for all possible $u \in \mathcal{C}(\Omega)$ is ensured from positivity of $p(x, u)$ on $\Omega \times \mathbb{R}$. But even then, the general question of polynomial non-negativity is an NP-hard problem. However, methods for enforcement of such polynomial programming constraints can be devised and are discussed in Section IV.

Ensuring negativity of the derivative of the functional is more difficult than positivity of the functional itself. The reason is the presence of partial derivatives of the form u_{x_i} and $u_{x_i x_i}$, which make it indefinite and thus treating these partial derivatives as independent variables will be fruitless. In general, a lot of intuition is required to manipulate the derivative form into something which can be shown to be negative, using mathematical equalities and inequalities, depending on the particular form of the PDE, the boundary conditions and the spatial domain Ω. In this paper we will show how one can construct such equalities and inequalities algorithmically, and how Lyapunov functions that satisfy the two Lyapunov conditions can be constructed.

Let us first motivate the discussion using an example. Consider the system

$$\begin{aligned} u_t &= u_{xx} + \mu u^3 + \lambda u \\ u(\pm 1, t) &= 0 \end{aligned} \quad (5)$$

with $\mu \leq 0$. We are interested in the stability of the zero solution, when λ is a parameter. The linear part of this system, i.e., the PDE described by

$$\begin{aligned} u_t &= u_{xx} + \lambda u \\ u(\pm 1, t) &= 0 \end{aligned}$$

is stable for $\lambda < \frac{4}{\pi}$. To see this, one can even verify that the following is a Lyapunov function for the linearized system:

$$V = \sum_{n \in \mathbb{N}} \frac{1}{2\left(\frac{n\pi}{2}\right)^2 - 2\lambda} \int_{-1}^{1} \int_{-1}^{1} u^*(x,t) \phi_n^*(\eta) \phi_n(x) u(\eta, t) d\eta dx$$

where
$$\{\phi_n\}_{n\geq 1} = \left\{\sin\left(\frac{n\pi(x+1)}{2}\right)\right\}$$
is a basis for $H_0^1[-1,1]$. One also needs to check that the PDE defines a dynamical system, but that is easy to see.

The analysis of the nonlinear PDE may be more difficult. For example, suppose one chooses the following Lyapunov candidate, to prove stability in L_2.

$$V(u) = \frac{1}{2}\int_{-1}^{1} u^2(\eta,t)d\eta$$

Obviously V is positive definite, and the time derivative of V is:

$$\begin{aligned}\frac{dV(u)}{dt} &= \int_{-1}^{1} uu_t(\eta,t)d\eta \\ &= \int_{-1}^{1} u(\eta,t)\left(u_{\eta\eta}(\eta,t) + \mu u^3(\eta,t) + \lambda u(\eta,t)\right)d\eta \\ &= u(\eta,t)u_\eta(\eta,t)|_{-1}^{1} + \int_{-1}^{1}\left(\lambda u^2 - u_\eta^2 + \mu u^4\right)d\eta \\ &= \int_{-1}^{1}\left(\lambda u^2 - u_\eta^2 + \mu u^4\right)d\eta\end{aligned}$$

At this point, unless one invokes an appropriate Poincaré inequality of the form

$$\int_\Omega (u(x))^2\,dx \leq K\int_\Omega\left(\frac{\partial u}{\partial x}\right)^2 dx$$

and constructs the constant $K > 0$, no conclusion can be drawn.

It is important to note that in the above procedure the term $uu_{\eta\eta}$ in the kernel of $\dot{V}$, is indefinite; through integration by parts and use of the boundary conditions, it is rendered definite. It is such conditions that we are after.

IV. PROPOSED METHODOLOGY

As mentioned in Section III, positivity of a Lyapunov functional (3) is ensured by uniform positivity of p on $\Omega\times\mathbb{R}$ when p is polynomial with minimum partial degree of 2 or more in u. Thus, we first address methods for the construction of polynomial functions p which are uniformly positive on $\Omega\times\mathbb{R}$ and for which $p(x,0) = 0$. Testing polynomial non-negativity is NP-hard if the polynomial is degree 4 and more; however, the Sum of Squares decomposition can be used to provide a conservative, yet computationally efficient method for testing non-negativity.

Denote by $\mathbb{R}[y]$ the ring of polynomials in $y = (y_1,\ldots,y_n)$ with real coefficients. Denote by Σ the cone of polynomials that admit a sum of squares decomposition, i.e., those $p \in \mathbb{R}[y]$ for which there exist $h_i \in \mathbb{R}[y]$, $i = 1,\ldots,M$ so that

$$p(y) = \sum_{i=1}^{M} h_i^2(y).$$

If $p(y) \in \Sigma$, then immediately $p(y) \geq 0$ for all y. The converse is not always true (but for univariate polynomials, quadratic forms and tertiary quartic forms). However, as mentioned above, testing if $p(y) \geq 0$ has been classified to be NP-hard, whereas testing if $p(y) \in \Sigma$ is equivalent to a semidefinite programme (SDP) [13], and hence is worst-case polynomial-time verifiable. The related SDP can be formulated efficiently and the solution can be retrieved using SOSTOOLS [14], which uses semidefinite solvers such as SeDuMi [15] or SDPT3 [16] to solve it.

Recall the derivative of V along trajectories generated by the dynamical system defined by Equation 1. As mentioned previously, it would be futile to use uniform polynomial positivity constraints to enforce negativity of the derivative of the functional. Instead, one typically uses various integral equalities and inequalities to transform the expression into one which has a uniformly positive form. Exactly which equalities and inequalities are useful will generally depend on the functions a, b and c. Here, we will propose a technique whereby one can search over various equality constraints using integration by parts. Once we have identified a set of equality constraints which hold for the system under consideration,

$$\int_\Omega h_j(x,u,u_{x_i},u_{x_ix_i})dx = 0 \text{ for } j = 1,\ldots,N,$$

we then proceed as follows. Suppose we are interested in proving positivity of a functional of the following form

$$V_D(u) = \int_\Omega f(x,u,u_{x_i},u_{x_ix_i})dx.$$

Now positivity of the functional V_D is equivalent to positivity of the following for $d_j \in \mathbb{R}$.

$$\tilde{V}_D(u) = \int_\Omega\left(f(x,u,u_{x_i},u_{x_ix_i}) + \sum_{j=1}^{N} d_j h_j(x,u,u_{x_i},u_{x_ix_i})\right)dx$$

Since the h_j are given and since the d_j enter the expression in a linear manner, one can search for a set of d_j such that the kernel of $\tilde{V}_D$ is uniformly positive. As mentioned previously, once the h_j have been identified, this search can be performed using semidefinite programming.

A. Integration by Parts

In this paper, we will restrict ourselves to the use of equalities generated through the use of the technique known as integration by parts. In its most general form, this equality is defined by the following, where ν_i is the i^{th} component of the unit outward normal to Γ direction.

$$\int_\Omega \frac{\partial u}{\partial x_i}v dx = \int_\Gamma uv\nu_i dx - \int_\Omega u\frac{\partial v}{\partial x_i}dx$$

For example, in the case where Ω is simply the interval $[a,b]$, we have that

$$\int_a^b u(x,t)u_{xx}(x,t)dx = u(x,t)u_x(x,t)|_a^b - \int_a^b u_x^2(x,t)dx.$$

In this case, h is given by the following.

$$h = u(x,t)u_{xx}(x,t) + u_x^2(x,t) - (b-a)\,u(x,t)u_x(x,t)|_a^b$$

Of course, in general, there are many variations on the use of integration by parts and to include even a substantial number of them would be impractical. In this paper we restrict our attention to a few such transformations. Recall that polynomial functions are actually the linear combination of a finite number of monomials. Furthermore, due to the restriction we have placed on the form of the partial differential equations under consideration, as indicated in Equation (1), and the structure of the functional derivative indicated in Equation (4), these monomials will take one of the two forms listed below.

$$\begin{aligned} m(\alpha,\beta,\gamma) &= \Pi_{j\neq i} x_j^{\alpha_j} x_i^{\gamma} u^{\beta} u_{x_i} \\ n(\alpha,\beta,\gamma) &= \Pi_{j\neq i} x_j^{\alpha_j} x_i^{\gamma} u^{\beta} u_{x_i x_i} \end{aligned}$$

Here $\alpha \in \mathbb{N}^n$ and $\beta,\gamma \in \mathbb{N}$ with $\sum_{j\neq i}\alpha_i + \beta + \gamma \leq d$ where d is the degree of the polynomial. We now make the additional restriction that admissible transformations must tend to eliminate terms such as $u_{x_i x_i}$ and u_{x_i}. Thus, in the integration by parts formula, the term $u_{x_i x_i}$ and u_{x_i} will appear as $\frac{\partial u}{\partial x_i}$ and the terms u and x will be grouped into v. The reason for this is that the goal of the transformations is to eliminate the terms $u_{x_i x_i}$ and u_{x_i} from the final expression. Under these restrictions, the valid set of transformations is listed below which hold for $\alpha_i, \beta = 0,\ldots,d$, $\gamma = 1,\ldots,d$, $\sum_{j\neq i}\alpha_i + \beta + \gamma \leq d$ and $i = 1,\ldots,n$.

$$\begin{aligned} h_i(\beta,\gamma) =&(\beta+1)\Pi_{j\neq i} x_j^{\alpha_j} x_i^{\gamma} u^{\beta} u_{x_i} + \gamma \Pi_{j\neq i} x_j^{\alpha_j} x_i^{\gamma-1} u^{\beta+1} \\ &- g_1(\alpha,\beta,\gamma) \\ k_i(\beta,\gamma) =&\Pi_{j\neq i} x_j^{\alpha_j} x_i^{\gamma} u^{\beta} u_{x_i x_i} + \beta \Pi_{j\neq i} x_j^{\alpha_j} x_i^{\gamma} u^{\beta-1} u_{x_i}^2 \\ &+ \gamma \Pi_{j\neq i} x_j^{\alpha_j} x_i^{\gamma-1} u^{\beta} u_{x_i} - g_2(\alpha,\beta,\gamma) \end{aligned}$$

Here $g_1(\alpha,\beta,\gamma) = V(\Omega)^{-1}\int_\Gamma \Pi_{j\neq i} x_j^{\alpha_j} x_i^{\gamma} u^{\beta+1} dx$, $g_2(\alpha,\beta,\gamma) = V(\Omega)^{-1}\int_\Gamma \Pi_{j\neq i} x_j^{\alpha_j} x_i^{\gamma} u^{\beta} u_{x_i} dx$, and $V(\Omega)$ is the volume of the region Ω. For $\gamma = 0$, $\alpha_i = 0,\ldots,d$, $\beta = 1,\ldots,d$ and $\sum_{j\neq i}\alpha_i + \beta \leq d$, we also have the following.

$$\begin{aligned} h_i(\alpha,\beta,0) =&(\beta+1)\Pi_{j\neq i} x_j^{\alpha_j} u^{\beta} u_{x_i} - g_1(\alpha,\beta,0) \\ k_i(\alpha,\beta,0) =&\Pi_{j\neq i} x_j^{\alpha_j} u^{\beta} u_{x_i x_i} + \beta \Pi_{j\neq i} x_j^{\alpha_j} u^{\beta-1} u_{x_i}^2 \\ &- g_2(\alpha,\beta,0) \end{aligned}$$

Finally, of course, we have that $h_i(\alpha,0,0) = \Pi_{j\neq i} x_j^{\alpha_j} u_{x_i} - g_1(\alpha,0,0)$ and $k_i(\alpha,0,0) = \Pi_{j\neq i} x_j^{\alpha_j} u_{x_i x_i} - g_2(\alpha,0,0)$.

Complexity Issues: Naturally, with so many 'library' functions, the issue of computationally complexity becomes important. Fortunately, the number of functions does not increase significantly faster than the number of monomials in the polynomial variables. To be precise, the number of expressions is $2nO(n,d)$, where $O(n,d) = \binom{n+d}{d}$ is the number of monomials in an n-dimensional expression of degree d.

V. STABILITY ANALYSIS

In this section we demonstrate how to use the above results to prove stability of certain parabolic partial differential equations and give an illustrative example.

Recall that we are interested in partial differential equations of the form (1) and we seek to construct Lyapunov structures of the form (3) with time-derivatives (4). Let $\{g_i\}_{i=1}^{2nO(n,d)}$ be the set of library functions generated by integration by parts where d is the maximum degree of the polynomial terms in $\dot{V}$.

Proposition 7: Suppose there exists a polynomial $p(x,u)$, $s_i \in \mathbb{R}$ for $i = 1,\ldots,2nO(n,d)$ and $\epsilon > 0$ such that the following hold:

$$\begin{aligned} &p(x,u) - \epsilon u^2 \geq 0 \qquad \text{for } x \in \Omega \\ &\sum_{i=1}^{n} \frac{\partial p(x,u)}{\partial u} a_i(x) u_{x_i x_i} + \sum_{i=1}^{n} \frac{\partial p(x,u)}{\partial u} b_i(x,u) u_{x_i} \\ &\quad + \frac{\partial p(x,u)}{\partial u} c(x,u) + \sum_{i=1}^{2nO(n,d)} s_i g_i \leq 0 \qquad \text{for } x \in \Omega \end{aligned}$$

Then the steady-state of system (1) is stable.

Let us now illustrate this methodology by a numerical example. Recall system (5) and consider the following functional structure:

$$V(u) = \int_{-1}^{1} p(u(\eta),\eta) d\eta \tag{6}$$

where p is a polynomial of degree at least 2 in u, and any degree in η. As discussed previously, there are a number of conditions that result from integrating by parts u against test functions. In particular, we note that

$$\int_{-1}^{1} \left\{ \frac{\partial p}{\partial u} u_{\eta\eta} + u_\eta^2 \frac{\partial^2 p}{\partial u^2} - u \left[\frac{\partial^3 p}{\partial u^2 \partial \eta} u_\eta + \frac{\partial^3 p}{\partial u_\eta \partial \eta^2} \right] \right\} d\eta = 0$$

is satisfied. Here we have used the fact that $u \frac{\partial^2 p}{\partial u \partial \eta}\Big|_{-1}^{1} = 0$ and the fact that $u_\eta \frac{\partial p}{\partial u}\Big|_{-1}^{1} = 0$ since p is a polynomial of degree at least 2 in u. Notice that the following is also true:

$$\int_{-1}^{1} \eta^n u^2 d\eta = -\int_{-1}^{1} n\eta^n u^2 d\eta - 2\int_{-1}^{1} \eta^{n+1} u u_\eta d\eta$$

i.e.,

$$\int_{-1}^{1} \left[(1+n)\eta^n u^2 + 2\eta^{n+1} u u_\eta\right] d\eta = 0 \tag{7}$$

Such constraints can, of course, be constructed systematically but here we have illustrated them in a concrete fashion. We now apply the results of Proposition 7. To measure accuracy, we introduce a parameter, ρ, as follows:

$$\begin{aligned} &\rho(\text{degree of } p \text{ wrt } \eta) = \\ &\qquad 1 - \left(\frac{4\lambda}{\pi^2} \text{ for which a } V \text{ can be constructed} \right) \end{aligned}$$

Then for the functional shown in Equation (6) ρ varies as shown in Table I, using the construction in [17]. We see that as the order of the polynomial $p(u,\eta)$ is increased with respect to η, the conservativeness decreases.

Degree of p wrt η	0	2	4	6	8	10	12	14	16
ρ	1	0.5	0.32	0.20	0.15	0.11	0.08	0.07	0.06

TABLE I

ρ FOR DIFFERENT DEGREE POLYNOMIALS a IN η (EQUATION (6)) FOR SYSTEM 5

VI. CONCLUSIONS

In this paper we have presented an algorithmic methodology for the analysis of systems described by certain types of parabolic partial differential equations. It was shown how certain Lyapunov structures could be constructed to prove stability using transformations defined through integration by parts. The method was illustrated through the use of a numerical example.

Although the methodology described in this paper can be used in many cases, there are types of PDEs for which different structures and transformations may be necessary. Here we note some immediate extensions that are currently being explored. First, it has been noted that in many cases, the use of a structure of the following form is required:

$$V(u) = \int_\Omega \int_\Omega p(u(\eta), u(\theta), \eta, \theta) d\theta d\eta$$

Although the transformations presented in this paper from the integration by parts technique still applies, the restriction of positivity of p will be conservative. In this case, more complicated conditions such as those outlined in a related paper on time-delay systems [7] will be useful.

In addition to more complicated structures, more complicated transformations may also be necessary. For example we can use Poincaré inequalities of the following form:

$$\int_\Omega (u(x))^2 \, dx \leq K \int_\Omega \left(\frac{\partial u}{\partial x}\right)^2 dx$$

A generalization of such transformations will be included in our library of functions mentioned previously.

Many times we may not be interested in the stability properties of the steady-state of a PDE, but we may want estimates of the functional output such as the one given by (2). This is usually done through the numerical solution of the PDEs followed by a computation of the output functional, and it can be computationally demanding and difficult to implement. Having an *a priori* bound on the functional output may be beneficial for several reasons: for one, the numerical solution of the PDE is evaded. More importantly, if we wanted to estimate the solution itself, such functional estimates can help increase the numerical accuracy of the solution. Indeed, methods developed in the past draw information from the functional output estimates to adapt the discretization mesh so as to obtain better numerical accuracy [18]. The methods developed so far for estimating functional outputs of PDEs are twofold. In [19] the authors concentrate on an augmented Lagrangian approach to obtain upper bounds on the functional output, using a dual max-min relaxation. In [20], an approach based on the generalized moment problem on semialgebraic sets and semidefinite programming is proposed. We will investigate the possibility that the methodology presented in this paper can be used in estimating functional outputs for PDEs.

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