Proceedings of the 45th IEEE Conference on Decision & Control
Manchester Grand Hyatt Hotel
San Diego, CA, USA, December 13-15, 2006

Positive Forms and Stability of Linear Time-Delay Systems

Matthew Peet[1,4] Antonis Papachristodoulou[2,5] Sanjay Lall[3,4]

Abstract

We consider the problem of constructing Lyapunov functions for linear differential equations with delays. For such systems it is known that stability implies that there exists a quadratic Lyapunov function on the state space, although this is in general infinite dimensional. We give an explicit parametrization of a finite-dimensional subset of the cone of Lyapunov functions. Positivity of this class of functions is enforced using sum-of-squares polynomial matrices. This allows the computation to be formulated as a semidefinite program.

1 Introduction

In this paper we present an approach for construction of polynomial Lyapunov functions for systems with time-delays. Specifically, we are interested in systems of the form

$$\dot{x}(t) = \sum_{i=0}^{k} A_i x(t - h_i)$$

where $x(t) \in \mathbb{R}^n$. In the simplest case we are given the delays $h_0, \dots, h_k$ and the matrices $A_0, \dots, A_k$ and we would like to determine whether the system is stable.

For such systems it is known that if the system is stable then there will exist a Lyapunov function of the form

$$V(\phi) = \int_{-h}^{0} \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix}^T M(s) \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix} ds + \int_{-h}^{0}\int_{-h}^{0} \phi(s)^T N(s,t)\phi(t)\, ds\, dt$$

where M and N are piecewise continuous matrix-valued functions. Here $\phi : [-h, 0] \to \mathbb{R}^n$ is an element of the state space, which is the infinite-dimensional space of continuous functions mapping $[-h, 0]$ to $\mathbb{R}^n$. We use semidefinite programming to construct functions $M(s)$ and $N(s,t)$ which are piecewise polynomial in s and t.

Such Lyapunov functions must be positive and have negative derivative, and the main contribution of this paper is a method for parameterizing functions M and N for which the positivity conditions on V hold. Roughly speaking, we show that

$$V_1(\phi) = \int_{-h}^{0} \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix}^T M(s) \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix} ds$$

is positive for all ϕ if and only if there exists a piecewise continuous matrix-valued function T such that

$$M(t) + \begin{bmatrix}T(t) & 0\\ 0 & 0\end{bmatrix} \geq 0 \quad \text{for all } t$$

$$\int_{-h}^{0} T(t)\, dt = 0$$

That is, we convert positivity of the integral to *pointwise* positivity. This result is stated precisely in Theorem 2. Pointwise positivity may then be easily enforced, and in particular in this case is equivalent to a sum-of-squares constraint. The constraint that T integrates to zero is a simple linear constraint on the coefficients. The condition that the derivative of the Lyapunov function be negative is similarly enforced. Notice that the sufficient condition that $M(s)$ be pointwise nonnegative is conservative, and as the equivalence above shows it is easy to generate examples where V_1 is nonnegative even though $M(s)$ is not pointwise nonnegative.

1.1 Prior Work

The use of Lyapunov functionals on an infinite dimensional space to analyze differential equations with delay originates with the work of Krasovskii [8]. For linear systems, quadratic Lyapunov functions were first considered by Repin [11]. The book of Gu, Kharitonov, and Chen [2] presents many useful results in this area, and further references may be found there as well as in Hale and Lunel [3], Kolmanovskii and Myshkis [7] and Niculescu [9]. The idea of using sum-of-squares polynomials together with semidefinite programming to construct Lyapunov functions originates in Parrilo [10].

1.2 Notation

Let $\mathbb{N}$ denote the set of nonnegative integers. Let $\mathbb{S}^n$ be the set of $n \times n$ real symmetric matrices, and for $X \in \mathbb{S}^n$ we write $S \succeq 0$ to mean that S is positive semidefinite. For X any Banach space and $I \subset \mathbb{R}$ any interval let $\Omega(I, X)$ be the space of all functions

$$\Omega(I, X) = \{\, f : I \to X \,\}$$

[1]Email: mmpeet@stanford.edu
[2]Email: antonis@eng.ox.ac.uk
[3]Email: lall@stanford.edu
[4]Department of Aeronautics and Astronautics, Stanford University, Stanford CA 94305-4035, U.S.A.
[5]Department of Engineering Science, University of Oxford, Oxford, OX1 3PJ, U.K.

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and let $C(I, X)$ be the Banach space of bounded continuous functions

$$C(I, X) = \{\, f : I \to X \mid f \text{ is continuous and bounded}\,\}$$

equipped with the norm

$$\|f\| = \sup_{t\in I} \|f(t)\|_X$$

We will omit the range space when it is clear from the context; for example we will simply write $C[a, b]$ to mean $C([a, b], X)$. A function $f \in \Omega[a, b]$ is called piecewise C^n if there exists a finite number of points $a \le h_1 < \cdots < h_k \le b$ such that f is continuous at all $x \in [a, b]\backslash\{h_1, \ldots , h_k\}$. We will write, as usual

$$f(a+) = \lim_{t\to a+} f(t)$$

Define also the projection $H_t : \Omega[-h, \infty) \to \Omega[-h, 0]$ for $t \ge 0$ and $h > 0$ by

$$(H_t x)(s) = x(t - s) \quad \text{for all } s \in [-h, 0]$$

We follow the usual convention and denote $H_t x$ by x_t.

We consider polynomials in n variables. For $\alpha \in \mathbb{N}^n$ define the monomial in n variables x^α by $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. We say M is a real matrix polynomial in n variables if for some finite set $W \subset \mathbb{N}$ we have

$$M(x) = \sum_{\alpha\in W} A_\alpha x^\alpha$$

where A_α is a matrix for each $\alpha \in W$.

2 System Formulation

Suppose $0 = h_0 < h_1 < \cdots < h_k = h$, and let $H = \{-h_0, \ldots , -h_k\}$, and let $H^c = [-h, 0]\backslash H$. Suppose and $A_0, \ldots , A_k \in \mathbb{R}^{n\times n}$. We consider linear differential equations with delay, of the form

$$\dot{x}(t) = \sum_{i=0}^{k} A_i x(t - h_i) \quad \text{for all } t \ge 0 \tag{1}$$

where the trajectory $x : [-h, \infty) \to \mathbb{R}^n$. The boundary conditions are specified by a given function $\phi : [-h, 0] \to \mathbb{R}^n$ and the constraint

$$x(t) = \phi(t) \quad \text{for all } t \in [-h, 0] \tag{2}$$

If $\phi \in C[-h, 0]$, then there exists a unique solution x satisfying (1) and (2). The system is called ***exponentially stable*** if there exists $\sigma > 0$ and $a \in \mathbb{R}$ such that for every initial condition $\phi \in C[-h, 0]$ the corresponding solution x satisfies

$$\|x(t)\| \le a e^{-\sigma t}\|\phi\| \quad \text{for all } t \ge 0$$

We write the solution as an explicit function of the initial conditions using the map $G : C[-h, 0] \to \Omega[-h, \infty)$, defined by

$$(G\phi)(t) = x(t) \quad \text{for all } t \ge -h$$

where x is the unique solution of (1) and (2) corresponding to initial condition ϕ. Also for $s \ge 0$ define the *flow map* $\Gamma_s : C[-h, 0] \to C[-h, 0]$ by

$$\Gamma_s\phi = H_s G\phi$$

which maps the state of the system x_t to the state at a later time $x_{t+s} = \Gamma_s x_t$.

2.1 Lyapunov Functions

Suppose $V : C[-h, 0] \to \mathbb{R}$. We use the notion of derivative as follows. Define the ***Lie derivative*** of V with respect to Γ by

$$\dot{V}(\phi) = \limsup_{r\to 0+} \frac{1}{r}\big(V(\Gamma_r\phi) - V(\phi)\big)$$

We will use the notation $\dot{V}$ for both the Lie derivative and the usual derivative, and state explicitly which is meant if it is not clear from context. We will consider the set X of quadratic functions, where $V \in X$ if there exists bounded piecewise C^1 functions $M : [-h, 0) \to \mathbb{S}^{2n}$ and $N : [-h, 0) \times [-h, 0) \to \mathbb{R}^{n\times n}$ such that

$$\begin{aligned} V(\phi) = \int_{-h}^{0} \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix}^T M(s) \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix} ds \\ + \int_{-h}^{0}\int_{-h}^{0} \phi(s)^T N(s, t)\phi(t)\, ds\, dt \end{aligned} \tag{3}$$

The following important result shows that for linear systems with delay, the system is exponentially stable if and only if there exists a quadratic Lyapunov function.

Theorem 1. *The linear system defined by equations (1) and (2) is exponentially stable if and only if there exists a Lie-differentiable function $V \in X$ and $\varepsilon > 0$ such that for all $\phi \in C[-h, 0]$*

$$\begin{aligned} V(\phi) &\ge \varepsilon\|\phi(0)\|^2 \\ \dot{V}(\phi) &\le -\varepsilon\|\phi(0)\|^2 \end{aligned} \tag{4}$$

Further $V \in X$ may be chosen such that the corresponding functions M and N of equation (3) have the following smoothness property; $N(s)$ and $M(s, t)$ are C^1 at all s, t such that $s \ne h_i$ and $t \ne h_i$ for all i.

Proof. See Kharitonov and Hinrichsen [5] for a recent proof. ∎

3 Positivity of Integrals

We would like to be able to computationally find functions $V \in X$ which satisfy the positivity conditions (4). We now develop some results to enable this.

Lemma 1. *Suppose $f\colon [-h,0] \to \mathbb{R}$ is piecewise continuous. Then the following are equivalent.*

(i) $\displaystyle\int_{-h}^{0} f(t)\,dt \geq 0$

(ii) there exists a function $g\colon [-h,0] \to \mathbb{R}$ which is piecewise continuous and satisfies

$$f(t) + g(t) \geq 0 \quad \textit{for all } t$$
$$\int_{-h}^{0} g(t)\,dt = 0$$

Proof. The direction (ii) $\implies$ (i) is immediate. To show the other direction, suppose (i) holds, and let g be

$$g(t) = -f(t) + \int_{-h}^{0} f(s)\,ds \quad \text{for all } t$$

Then g satisfies (ii). ∎

Lemma 2. *Suppose $H = \{-h_0, \dots, -h_k\}$ and let $H^c = [-h,0]\backslash H$. Let $f\colon [-h,0] \times \mathbb{R}^n \to \mathbb{R}$ be continuous on $H^c \times \mathbb{R}^n$, and suppose there exists a bounded function $z\colon [-h,0] \to \mathbb{R}$, continuous on H^c, such that for all $t \in [-h,0]$*

$$f\big(t, z(t)\big) = \inf_x f(t,x)$$

Further suppose for each bounded set $X \subset \mathbb{R}^n$ the set

$$\{\, f(t,x) \mid x \in X, t \in [-h,0] \,\}$$

is bounded. Then

$$\inf_{y \in C[-h,0]} \int_{-h}^{0} f\big(t, y(t)\big)\,dt = \int_{-h}^{0} \inf_x f(t,x)\,dt \qquad (5)$$

Proof. Let

$$K = \int_{-h}^{0} \inf_x f(t,x)\,dt$$

It is easy to see that

$$\inf_{y \in C[-h,0]} \int_{-h}^{0} f\big(t, y(t)\big)\,dt \geq K$$

since if not there would exist some continuous function y and some interval on which

$$f\big(t, y(t)\big) < \inf_x f(t,x)$$

which is clearly impossible.

We now show that the left-hand side of (5) is also less than or equal to K, and hence equals K. We need to show that for any $\varepsilon > 0$ there exists $y \in C[-h,0]$ such that

$$\int_{-h}^{0} f\big(t, y(t)\big)\,dt < K + \varepsilon$$

To do this, for each $n \in \mathbb{N}$ define the set $H_n \subset \mathbb{R}$ by

$$H_n = \bigcup_{i=1}^{k-1} (h_i - \alpha/n, h_i + \alpha/n)$$

and choose $\alpha > 0$ sufficiently small so that $H_1 \subset (-h, 0)$. Let z be as in the hypothesis of the lemma, and pick M and R so that

$$M > \sup_t \|z(t)\|$$
$$R = \sup\big\{\, |f(t,x)| \;\big|\; t \in [-h,0], \|x\| \leq M \,\big\}$$

For each n choose a continuous function $x_n : [-h,0] \to \mathbb{R}^n$ such that $x_n(t) = z(t)$ for all $t \notin H_n$ and

$$\sup_{t \in [-h,0]} \|x_n(t)\| < M$$

This is possible, for example, by linear interpolation. Now we have, for the continuous function x_n

$$\begin{aligned}\int_{-h}^{0} f\big(t, x_n(t)\big)\,dt &= K + \int_{-h}^{0} \Big(f\big(t, x_n(t)\big) - f\big(t, z(t)\big)\Big)\,dt \\ &= K + \int_{H_n} \Big(f\big(t, x_n(t)\big) - f\big(t, z(t)\big)\Big)\,dt \\ &\leq K + 4R\alpha/n\end{aligned}$$

This proves the desired result. ∎

Lemma 3. *Suppose $f\colon [-h,0] \times \mathbb{R}^n \to \mathbb{R}$ and the hypotheses of Lemma 2 hold. Then the following are equivalent.*

(i) For all $y \in C[-h,0]$

$$\int_{-h}^{0} f\big(t, y(t)\big)\,dt \geq 0$$

(ii) There exists $g : [-h,0] \to \mathbb{R}$ which is piecewise continuous and satisfies

$$f(t,z) + g(t) \geq 0 \quad \textit{for all } t, z$$
$$\int_{-h}^{0} g(t)\,dt = 0$$

Proof. Again we only need to show that (i) implies (ii). Suppose (i) holds, then

$$\inf_{y \in C[-h,0]} \int_{-h}^{0} f\big(t, y(t)\big)\,dt \geq 0$$

and hence by Lemma 2 we have

$$\int_{-h}^{0} r(t)\,dt \geq 0$$

where $r : [-h, 0] \to \mathbb{R}^n$ is given by

$$r(t) = \inf_x f(t, x) \quad \text{for all } t$$

The function r is continuous on H^c since f is continuous on $H^c \times \mathbb{R}^n$. Hence by Lemma 1, there exists g such that condition (ii) holds, as desired. ■

Theorem 2. *Suppose $M : [-h, 0] \to \mathbb{S}^{m+n}$ is piecewise continuous, and there exists $\varepsilon > 0$ such that for all $t \in [-h, 0]$ we have*

$$\begin{aligned} M_{22}(t) &\geq \varepsilon I \\ M(t) &\leq \varepsilon^{-1} I \end{aligned}$$

Then the following are equivalent.

(i) For all $x \in \mathbb{R}^m$ and continuous $y : [-h, 0] \to \mathbb{R}^n$

$$\int_{-h}^{0} \begin{bmatrix} x \\ y(t) \end{bmatrix}^T M \begin{bmatrix} x \\ y(t) \end{bmatrix} dt \geq 0 \tag{6}$$

(ii) There exists a function $T : [-h, 0] \to \mathbb{S}^m$ which is piecewise continuous and satisfies

$$M(t) + \begin{bmatrix} T(t) & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \quad \textit{for all } t \in [-h, 0]$$

$$\int_{-h}^{0} T(t)\,dt = 0$$

Proof. Again we only need to show (i) implies (ii). Suppose $x \in \mathbb{R}^n$, and define

$$f(t, z) = \begin{bmatrix} x \\ z \end{bmatrix}^T M(t) \begin{bmatrix} x \\ z \end{bmatrix} \quad \text{for all } t, z$$

Since by the hypothesis M_{22} has a lower bound, it is invertible for all t and its inverse is piecewise continuous. Therefore $z(t) = -M_{22}(t)^{-1} M_{21}(t) x$ is the unique minimizer of $f(t, z)$ with respect to z. By the hypothesis (i), we have that for all $y \in C[-h, 0]$

$$\int_{-h}^{0} f\big(t, y(t)\big)\,dt \geq 0$$

Hence by Lemma 3 there exists a function g such that

$$\begin{aligned} g(t) + f(t, z) \geq 0 \quad \text{for all } t, z \\ \int_{-h}^{0} g(t)\,dt = 0 \end{aligned} \tag{7}$$

The proof of Lemma 1 gives one such function as

$$g(t) = -f\big(t, z(t)\big) + \int_{-h}^{0} f(s, z(s))\,dt$$

We have

$$f\big(t, z(t)\big) = x^T \big(M_{11}(t) - M_{12}(t) M_{22}^{-1}(t) M_{21}(t)\big) x$$

and therefore $g(t)$ is a quadratic function of x, say $g(t) = x^T T(t) x$, and $T : [-h, 0] \to \mathbb{S}^m$ is continuous on H^c. Then equation (7) implies

$$x^T T(t) x + \begin{bmatrix} x \\ z \end{bmatrix}^T M(t) \begin{bmatrix} x \\ z \end{bmatrix} \geq 0 \quad \text{for all } t, z, x$$

as required. ■

We have now shown that the convex cone of functions M such that the first term of (3) is nonnegative is exactly equal to the sum of the cone of pointwise nonnegative functions and the linear space of functions whose integral is zero. Note that in (6) the vectors x and y are allowed to vary independently, whereas (3) requires that $x = y(0)$. It is however straightforward to show that this additional constraint does not change the result, using the technique in the proof of Lemma 2.

The key benefit of this is that it is easy to parametrize the latter class of functions, and in particular when M is a polynomial these constraints are semidefinite representable constraints on the coefficients of M.

4 Lie Derivatives

4.1 Single Delay Case

We first present the single delay case, as it will illustrate the formulation in the more complicated case. Suppose that $V \in X$ is given by (3), where $M : [-h, 0] \to \mathbb{S}^{2n}$ and $N : [-h, 0] \times [-h, 0] \to \mathbb{R}^{n \times n}$. Since there is only one delay, if the system is exponentially stable then there always exists a Lyapunov function of this form with C^1 functions M and N. Then the Lie derivative of V is

$$\dot{V}(\phi) = \int_{-h}^{0} \begin{bmatrix} \phi(0) \\ \phi(-h) \\ \phi(s) \end{bmatrix}^T D(s) \begin{bmatrix} \phi(0) \\ \phi(-h) \\ \phi(s) \end{bmatrix} ds + \int_{-h}^{0}\int_{-h}^{0} \phi(s)^T E(s, t) \phi(t)\,ds\,dt \tag{8}$$

Partition D and M as

$$M(t) = \begin{bmatrix} M_{11} & M_{12}(t) \\ M_{21}(t) & M_{22}(t) \end{bmatrix} \qquad D(t) = \begin{bmatrix} D_{11} & D_{12}(t) \\ D_{21}(t) & D_{22}(t) \end{bmatrix}$$

so that $M_{11} \in \mathbb{S}^n$ and $D_{11} \in \mathbb{S}^{2n}$. Without loss of generality we can assume M_{11} and D_{11} are constant. The functions D and E are linearly related to M and N by

$$\begin{aligned} D_{11} = &\begin{bmatrix} A_0^T M_{11} + M_{11} A_0 & M_{11} A_1 \\ A_1^T M_{11} & 0 \end{bmatrix} \\ &+ \frac{1}{h} \begin{bmatrix} M_{12}(0) + M_{21}(0) & -M_{12}(-h) \\ -M_{21}(-h) & 0 \end{bmatrix} \\ &+ \frac{1}{h} \begin{bmatrix} M_{22}(0) & 0 \\ 0 & -M_{22}(-h) \end{bmatrix} \end{aligned}$$

$$D_{12}(t) = \begin{bmatrix} A_0^T M_{12}(t) - \dot{M}_{12}(t) + N(0,t) \\ A_1^T M_{12}(t) - N(-h,t) \end{bmatrix}$$

$$D_{22}(t) = -\dot{M}_{22}(t)$$

$$E(s,t) = \frac{\partial N(s,t)}{\partial s} + \frac{\partial N(s,t)}{\partial t}$$

4.2 Multiple-delay case

We now define the class of functions under consideration for the Lyapunov functions. Define the intervals

$$H_i = \begin{cases} [-h_1, 0] & \text{if } i = 1 \\ [-h_i, -h_{i-1}) & \text{if } i = 2, \dots, k \end{cases}$$

For the Lyapunov function V, define the sets of functions

$$\begin{aligned} Y_1 = \Big\{ & M : [-h,0] \to \mathbb{S}^{2n} \mid \\ & M_{11}(t) = M_{11}(s) && \text{for all } s,t \in [-h,0] \\ & M \text{ is } C^1 \text{ on } H_i && \text{for all } i = 1,\dots,k \Big\} \end{aligned}$$

$$\begin{aligned} Y_2 = \Big\{ & N : [-h,0] \to \mathbb{S}^{n} \mid \\ & N(s,t) = N(t,s)^T && \text{for all } s,t \in [-h,0] \\ & N \text{ is } C^1 \text{ on } H_i \times H_j && \text{for all } i,j = 1,\dots,k \Big\} \end{aligned}$$

and for its derivative, define

$$\begin{aligned} Z_1 = \Big\{ & D : [-h,0] \to \mathbb{S}^{(k+2)n} \mid \\ & D_{ij}(t) = D_{ij}(s) && \text{for all } s,t \in [-h,0] \\ & && \text{for } i,j = 1,\dots,3 \\ & D \text{ is } C^0 \text{ on } H_i && \text{for all } i = 1,\dots,k \Big\} \end{aligned}$$

$$\begin{aligned} Z_2 = \Big\{ & E : [-h,0] \to \mathbb{S}^{n} \mid \\ & E(s,t) = E(t,s)^T && \text{for all } s,t \in [-h,0] \\ & E \text{ is } C^0 \text{ on } H_i \times H_j && \text{for all } i,j = 1,\dots,k \Big\} \end{aligned}$$

Here $D \in Z_1$ is partitioned according to

$$D(t) = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14}(t) \\ D_{21} & D_{22} & D_{23} & D_{24}(t) \\ D_{31} & D_{32} & D_{33} & D_{34}(t) \\ D_{41}(t) & D_{42}(t) & D_{43}(t) & D_{44}(t) \end{bmatrix} \tag{9}$$

where $D_{11}, D_{33}, D_{44} \in \mathbb{S}^n$ and $D_{22} \in S^{(k-1)n}$. Let $Y = Y_1 \times Y_2$ and $Z = Z_1 \times Z_2$. Notice that if $M \in Y_1$, then M need not be continuous at h_i for $1 \le i \le k-1$, however, it must be right continuous at these points. We also define the derivative $\dot{M}(t)$ at these points to be the right-hand derivative of M. We define the continuity and derivatives of functions in Y_2, Z_1 and Z_2 similarly.

We define the jump values of M and N at the discontinuities as follows.

$$\Delta M(h_i) = M(-h_i+) - M(-h_i-)$$

for each $i = 1, \dots, k-1$, and similarly define

$$\Delta N(h_i, t) = N(-h_i+, t) - N(-h_i-, t)$$

Definition 4. *Define the map $L : Y \to Z$ by $(D,E) = L(M,N)$ if for all $t,s \in [-h,0]$ we have*

$$\begin{aligned} D_{11} &= A_0^T M_{11} + M_{11} A_0 \\ &\quad + \frac{1}{h}\big(M_{12}(0) + M_{21}(0) + M_{22}(0)\big) \\ D_{12} &= \begin{bmatrix} M_{11}A_1 & \dots & M_{11}A_{k-1} \end{bmatrix} \\ &\quad - \begin{bmatrix} \Delta M_{12}(h_1) & \dots & \Delta M_{12}(h_{k-1}) \end{bmatrix} \\ D_{13} &= \frac{1}{h}\big(M_{11}A_k - M_{12}(-h)\big) \\ D_{22} &= \frac{1}{h} \operatorname{diag}\big(-\Delta M_{22}(h_1), \dots, -\Delta M_{22}(h_{k-1})\big) \\ D_{23} &= 0 \\ D_{33} &= -\frac{1}{h} M_{22}(-h) \end{aligned}$$

$$D_{14}(t) = N(0,t) + A_0^T M_{12}(t) - \dot{M}_{12}(t)$$

$$D_{24}(t) = \begin{bmatrix} \Delta N(-h_1,t) + A_1^T M_{12}(t) \\ \vdots \\ \Delta N(-h_{k-1},t) + A_{k-1}^T M_{12}(t) \end{bmatrix}$$

$$D_{34}(t) = A_k^T M_{12}(t) - N(-h,t)$$

$$D_{44}(t) = -\dot{M}_{22}(t)$$

and

$$E(s,t) = \frac{\partial N(s,t)}{\partial s} + \frac{\partial N(s,t)}{\partial t}$$

Here D is partitioned as in (9) and the remaining entries are defined by symmetry.

The map L is the Lie derivative operator applied to the set of functions specified by (3); this is stated precisely below. Notice that this implies that L is a linear map.

Lemma 5. *Suppose $M \in Y_1$ and $N \in Y_2$ and V is given by (3). Let $(D,E) = L(M,N)$. Then the Lie derivative of V is given by*

$$\dot{V}(\phi) = \int_{-h}^{0} \begin{bmatrix} \phi(-h_0) \\ \vdots \\ \phi(-h_k) \\ \phi(s) \end{bmatrix}^T D(s) \begin{bmatrix} \phi(-h_0) \\ \vdots \\ \phi(-h_k) \\ \phi(s) \end{bmatrix} ds + \int_{-h}^{0}\int_{-h}^{0} \phi(s)^T E(s,t) \phi(t) \, ds \, dt \tag{10}$$

Proof. The proof is straightforward by differentiation and integration by parts of (3). ∎

5 Matrix Sums of Squares

In this paper we use polynomial matrices as a conveniently parametrized class of functions to represent the functions M and N defining the Lyapunov function (3). Theorem 2 has reduced nonnegativity of the first term of (3) to pointwise nonnegativity of a matrix polynomial in one variable. A matrix polynomial in one variable is pointwise nonnegative semidefinite if and only if it is a sum of squares; see Choi, Lam, and Reznick [1]. We now briefly describe the matrix sum-of-squares construction. Suppose z is a vector of N monomials in variables y, for example

$$z(y)^T = \begin{bmatrix} 1 & y_1 & y_1 y_2^2 & y_3^4 \end{bmatrix}$$

For each y, we can define $K(y) : \mathbb{R}^n \to \mathbb{R}^{Nn}$ by

$$\big(K(y)\big)x = x \otimes z(y)$$

This means $K(y) = I \otimes z(y)$, and the entries of K are monomials in y. Now suppose $U \in \mathbb{S}^{Nn}$ is a symmetric matrix, and let

$$M(y) = K^T(y) U K(y)$$

Then each entry of the matrix M is a polynomial in y. The matrix M is called a *matrix sum-of-squares* if U is positive semidefinite, in which case $M(y) \succeq 0$ for all y. Given a matrix polynomial $G(y)$, we can test whether it is a sum-of-squares by finding a matrix U such that

$$G(y) = K^T(y) U K(y) \tag{11}$$

$$U \succeq 0 \tag{12}$$

Equation (11) is interpreted as equating the two polynomials G and $K^T U K$. Equating their coefficients gives a family of linear constraints on the matrix U. Therefore to find such a U we need to find a positive semidefinite matrix subject to linear constraints, and this is therefore a semidefinite program. See Hol and Scherer [4] and Kojima [6] for more details on matrix sums-of-squares and their properties, and Vandenberghe and Boyd [12] for background on semidefinite programming.

Piecewise matrix sums-of-squares. Define the vector of indicator functions $g : [-h, 0] \to \mathbb{R}^k$ by

$$g_i(t) = \begin{cases} 1 & \text{if } t \in H_i \\ 0 & \text{otherwise} \end{cases}$$

for all $i = 1, \dots, k$ and all $t \in [-h, 0]$. Now let $z(t)$ be the vector of monomials

$$z(t)^T = \begin{bmatrix} 1 & t & t^2 & \dots & z^d \end{bmatrix}$$

Define the function $Z_{n,d} : [-h, 0] \to \mathbb{R}^{(d+1)kn \times n}$ by

$$Z_{n,d}(t) = I \otimes z(t) \otimes g(t)$$

Then, define the sets $P_{n,d}$ and $\Sigma_{n,d}$ by

$$P_{n,d} = \left\{ Z_{n,d}^T(t) U Z_{n,d}(t) \mid U \in \mathbb{S}^{(d+1)nk} \right\}$$
$$\Sigma_{n,d} = \left\{ Z_{n,d}^T(t) U Z_{n,d}(t) \mid U \in \mathbb{S}^{(d+1)nk}, U \succeq 0 \right\}$$

These sets are defined so that if $M \in P_{n,d}$ then each entry of the $n \times n$ matrix M is a piecewise polynomial in t. If $M \in \Sigma_{n,d}$, then it is called a *piecewise matrix sum-of-squares*. We will omit the subscripts n, d where possible in order to lighten the notation. If we are given a function $G : [-h, 0] \to \mathbb{R}^n$ which is piecewise polynomial, and polynomial on each interval H_i, then it is a piecewise sum of squares of degree $2d$ if and only if there exists a matrix U such that

$$G(t) = Z_{n,d}^T(t) U Z_{n,d}(t) \tag{13}$$

$$U \succeq 0 \tag{14}$$

which, as for the non-piecewise case discussed above is a semidefinite program.

Kernel functions. We now consider functionals of the form of the second term of the Lyapunov function (3). The following class of functions N satisfy the condition that this be positive. We define the set of piecewise polynomial positive kernels in two variables s, t by

$$\Gamma_{n,d} = \left\{ Z_{n,d}^T(s) U Z_{n,d}(t) \mid U \in \mathbb{S}^{(d+1)nk}, U \succeq 0 \right\}$$

For any function $N \in \Gamma_{n,d}$ we have

$$\int_{-h}^{0} \int_{-h}^{0} \phi(s)^T N(s,t) \phi(t) \, ds \, dt \geq 0$$

for all $\phi \in C([-h, 0], \mathbb{R}^n)$.

Theorem 3. *Suppose there exist $d \in \mathbb{N}$ and piecewise matrix polynomials M, T, N, D, U, E such that*

$$M + \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} \in \Sigma_{2n,d}$$
$$-D + \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} \in \Sigma_{(k+2)n,d}$$
$$N \in \Gamma_{n,d}$$
$$-E \in \Gamma_{n,d}$$
$$(D, E) = L(M, N)$$
$$\int_{-h}^{0} T(s) \, ds = 0$$
$$\int_{-h}^{0} U(s) \, ds = 0$$
$$M_{11} \succ 0$$
$$D_{11} \prec 0$$

Then the system defined by equations (1) and (2) is exponentially stable.

Proof. Assume M, T, N, D, U, E satisfy the above conditions, and define the function V by (3). Then Lemma 5 implies that $\dot{V}$ is given by (10). The function V is the sum of two terms, each of which is nonnegative. The first is nonnegative by Theorem 2 and the second is nonnegative since $N \in \Gamma_{n,d}$. The same is true for $\dot{V}$. The strict positivity conditions of equations (4) hold since $M_{11} > 0$ and $D_{11} < 0$, and Theorem 1 then implies stability. ■

Remark 6. *The feasibility conditions of Theorem 3 are semidefinite-representable. In particular the condition that a piecewise polynomial matrix lie in Σ is a set of linear and positive semidefiniteness constraints on its coefficients. Similarly, the condition that T and U integrate to zero is simply a linear equality constraint on its coefficients. Standard semidefinite programming codes may therefore be used to efficiently find such piecewise polynomial matrices. Most such codes will also return a dual certificate of infeasibility if no such polynomials exist.*

As in the Lyapunov analysis of nonlinear systems using sum-of-squares polynomials, the set of candidate Lyapunov functions is parametrized by the degree d. This allows one to search first over polynomials of low degree, and increase the degree if that search fails.

There are various natural extensions of this result. The first is to the case of uncertain systems, where we would like to prove stability for all matrices A_i in some given semialgebraic set. This is possible by extending Theorem 3 to allow Lyapunov functions which depend polynomially on unknown parameters. A similar approach may be used to check stability for systems with uncertain delays. It is also straightforward to extend the class of Lyapunov functions, since it is not necessary that each piece of the piecewise sums-of-squares functions be nonnegative on the whole real line. To do this, one can use techniques for parameterizing polynomials nonnegative on an interval; for example, every polynomial $p(x) = f(x) - (x-1)(x-2)g(x)$ where f and g are sums of squares is nonnegative on the interval $[1, 2]$.

Numerical Example. The following system of equations was analyzed in Gu et al. [2].

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & \frac{1}{10} \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x(t - h/2) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t - h)$$

Using the above approach we show stability when $h = 1.3722$ and when $h = 0.20247$. These results are consistent with those of Gu et al. [2], where the system is analyzed using piecewise linear Lyapunov functions.

6 Summary

In this paper we developed an approach for computing Lyapunov functions for linear systems with delay. The approach relies on being able to parametrize a class of positive quadratic functionals, and the main result here is Theorem 2. Combining this with the well-known approach using sum-of-squares polynomials allows one to use standard semidefinite programming software to compute Lyapunov functions. It is possible that Theorem 2 and its proof techniques are applicable more widely, specifically to stability analysis of nonlinear systems with delay, as well as to controller synthesis for such systems.

References

[1] M.D. Choi, T.Y. Lam, and B. Reznick. Real zeros of positive semidefinite forms I. *Mathematische Zeitschrift*, 171(1):1–26, 1980.

[2] K. Gu, V. L. Kharitonov, and J. Chen. *Stability of Time-Delay Systems*. Birkhauser, 2003.

[3] J. K. Hale and S. M. Verduyn Lunel. *Introduction to Functional Differential Equations*, volume 99 of *Applied Mathematical Sciences*. Springer-Verlag, 1993.

[4] C. W. J. Hol and C. W. Scherer. Sum of squares relaxations for polynomial semi-definite programming. In *Proceedings of the International Symposium on Mathematical Theory of Networks and Systems (MTNS)*, 2004.

[5] V. L. Kharitonov and D. Hinrichsen. Exponential estimates for time delay systems. *Systems and control letters*, 53(5):395–405, 2004.

[6] M. Kojima. Sums of squares relaxations of polynomial semidefinite programs. Research Report B-397, Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, 2003.

[7] V. Kolmanovskii and A. Myshkis. *Introduction to the Theory and Applications of Functional Differential Equations*. Kluwer Academic Publishers, 1999.

[8] N. N. Krasovskii. *Stability of Motion*. Stanford University Press, 1963.

[9] S.-I. Niculescu. *Delay Effects on Stability: A Robust Control Approach*, volume 269 of *Lecture Notes in Control and Information Science*. Springer-Verlag, May 2001.

[10] P. A. Parrilo. *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*. PhD thesis, California Institute of Technology, 2000.

[11] I. M. Repin. Quadratic Liapunov functionals for systems with delay. *Journal of Applied Mathematics and Mechanics*, 29:669–672, 1965.

[12] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38(1):49–95, 1996.