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Stability Analysis of Linear Systems with Time-Varying Delays: Delay Uncertainty and Quenching

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Abstract— **This paper addresses the problem of stability analysis of a class of linear systems with time-varying delays. We develop conditions for robust stability that can be tested using Semidefinite Programming using the Sum of Squares decomposition of multivariate polynomials and the Lyapunov-Krasovskii theorem. We show how appropriate Lyapunov-Krasovskii functionals can be constructed algorithmically to prove stability of linear systems with a variation in delay, by using bounds on the size and rate of change of the delay. We also explore the quenching phenomenon, a term used to describe the difference in behaviour between a system with fixed delay and one whose delay varies with time. Numerical examples illustrate changes in the stability window as a function of the bound on the rate of change of delay.**

I. Introduction

The stability of time-delay systems is a problem of recurrent interest due to a number of applications from communication networks to biology and population dynamics (see, for instance, [6], [12], [17] and the references therein). While the study of linear systems with constant delays has received considerable attention in the last decade, the case of *time-varying delay* has received less, possibly due to the perceived difficulty of the problem – changes in delay make the system's state-space vary with time, which complicates the use of standard analysis tools. See, for instance, the construction of Lyapunov-Krasovskii functionals proposed by Louisell [14]. Among the existing results, we can cite: delay-independent/delay-dependent sufficient conditions by using comparison systems [19], estimation of delay bounds of time-varying uncertainty around some nominal delay value by using Lyapunov-Krasovskii functionals [10] (inspired by the construction in [11]) or frequency-domain approaches [18]. Finally, first estimates of the stability domain in the time-varying delay case have been considered quite recently in the context of some extension of the Lyapunov-Poincaré theorem [16]. All the approaches above make use of some particular interpretations of the time-varying character of the delay combined with appropriate model transformation of the original systems. Such bounds become conservative if the delays are assumed to vary in some large range. Next, the case of fast-varying delay was considered by [3] (by using some of the improvements suggested by [2]). Finally, the robust stability of uncertain systems with 'non-small' delay has been analyzed also via input-output approach to stability by [5] and [9].

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Based on recent advances in Semidefinite Programming (SDP) and the Sum of Squares (SOS) decomposition, and extending our work on the stability analysis of autonomous time-delay systems through the construction of the complete Lyapunov-Krasovskii functional, in the first part of this paper we address the problem of robust stability analysis of linear systems with time-varying delays. The stability conditions we obtain are in the form of matrix polynomial inequalities, which can be tested algorithmically using SDP. It is important to note that the delay function enters explicitly in the corresponding matrix polynomial conditions. As a byproduct of this analysis, we will emphasize the connection between the delay range and its corresponding variation in a general setting.

In the second half of the paper, we will use our algorithm to address the *quenching phenomenon* in time-delay systems. Roughly speaking, quenching occurs when stability (resp. instability) of a time-delay system with fixed delays within a certain interval is lost when the delay is assumed to be time-varying inside the same interval and vice-versa. In other words, the problem can be interpreted as a robustness/fragility problem by changing the delay uncertainty character from constant to time-varying. Quenching in delay systems was first mentioned in [15], where the author developed theory for the existence of this phenomenon in delay systems and discussed in detail some examples by applying frequency-domain techniques in the case in which the delay varies periodically with time. Using the SOS algorithmic framework, we will emphasize the connection between the delay range and its corresponding variation. It will be seen that our algorithm produces tight bounds for *uncertain* (time-varying) *delays* around some prescribed nominal value.

The paper is organized as follows. In Section II we formulate the problem and recall some background material. The main results on the stability analysis of linear systems with variations in delay are derived in Section III. In Section IV we present the algorithmic tools for stability analysis. This is followed by some illustrative examples in Section V. In section VI we discuss how the results in Section III can be used to explore the quenching phenomenon, giving illustrative examples. Concluding remarks end the paper.

II. Time-Delay Systems with Time-Varying Delays

In this paper we consider delayed linear systems, in which the delay is time-varying. For simplicity we will concentrate on systems with a single delay. However, our results are easily applied to the case of distributed delay or multiple time-varying delays. Additionally, in the case of multiple delays, the size of the associated semidefinite programming problem is affine with respect to the number of delays.

Consider a linear system of the form:

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t-\tau(t)) \tag{1}$$

Here $\tau : \mathbb{R}_+ \to [\underline{\tau}, \overline{\tau}]$ is a continuous bounded function, $\underline{\tau} \geq 0$, and the matrices $A_0(t)$ and $A_1(t)$ are of bounded variation. The initial condition is given by the mapping $\phi : [-\overline{\tau}, 0] \mapsto \mathbb{R}^n$. Denote by C the space $C([-\overline{\tau},0],\mathbb{R}^n)$ of continuous functions mapping the interval $[-\overline{\tau},0]$ to $\mathbb{R}^n$; the norm of an element $\phi \in C$ is designated by $\|\phi\|_c = \sup_{-\overline{\tau}\leq\theta\leq 0} |\phi(\theta)|$. We assume that all functions are sufficiently regular so as to allow a unique solution for any initial condition $\phi \in C([-\overline{\tau},0],\mathbb{R}^n)$.

In this paper we are interested in whether the zero equilibrium is asymptotically stable for given conditions on $\tau(t)$. These conditions are given as the upper and lower bounds on the delay size ($\overline{\tau}$ and $\underline{\tau}$) and an upper bound on the rate of variation:

$$\underline{\tau} \leq \tau(t) \leq \overline{\tau}, \quad |\dot{\tau}(t)| \leq \mu, \quad \forall\ t \geq 0$$

where $\mu > 0$.

a) Lyapunov Functionals: We consider Lyapunov Krasovskii functionals $V : \mathbb{R} \times C \to \mathbb{R}^n$ that are completely continuous functions with upper right Dini derivative

$$\dot{V} = \overline{\lim}_{h\to 0^+} \frac{1}{h}\left[V(t+h, x_{t+h}(t,\phi)) - V(t,\phi)\right].$$

Here $x_t = x(t+\theta)$, $\theta \in [-\overline{\tau}, 0]$ denotes the state of the system of differential equations at time t. The following general theorem imposes conditions on V which guarantee stability of the solutions of a nonlinear functional differential equation of the form:

$$\dot{x}(t) = f(t, x_t) \tag{2}$$

Theorem 1: Suppose $f : \mathbb{R}\times C \to \mathbb{R}^n$ maps $\mathbb{R}\times$ (bounded sets in C) into bounded sets in $\mathbb{R}^n$ and that $u, v, w : \overline{R}_+ \to \overline{R}_+$ are continuous nondecreasing functions, where additionally $u(s)$ and $v(s)$ are positive for $s > 0$ and $u(0) = v(0) = 0$. If there exists a continuous differentiable functional $V : \mathbb{R} \times C \to \mathbb{R}$ such that

$$u(|\phi(0)|) \leq V(t,\phi) \leq v(\|\phi\|_c)$$

and

$$\dot{V}(t,\phi) \leq -w(|\phi(0)|),$$

then the trivial solution of (2) is uniformly stable. If $w(s) > 0$ for $s > 0$ then it is uniformly asymptotically stable. If, in addition, $\lim_{s\to\infty} u(s) = \infty$, then it is globally uniformly asymptotically stable.

b) A Structure with Necessity: Consider the autonomous linear system with fixed delay.

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau), \tag{3}$$

In this case converse Lyapunov theorems are known which, for an asymptotically stable system, guarantee the existence of a quadratic functional which decreases with time and which takes the form:

$$V(x_t) = \frac{1}{\tau}\int_{-\tau}^{0} \begin{bmatrix} x(t) \\ x(t+\theta) \end{bmatrix}^T \begin{bmatrix} P_{11} & \tau P_{12}(\theta) \\ \tau P_{12}^T(\theta) & \tau P_{22}(\theta) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t+\theta) \end{bmatrix} d\theta + \int_{-\tau}^{0}\int_{-\tau}^{0} x^T(t+\theta)R(\theta,\xi)x(t+\xi)d\theta d\xi \tag{4}$$

Methodologies for constructing functionals of the above type have been developed based on solving related semidefinite programmes. Particular examples include the piecewise linear approach [6] and the sum of squares approach [21], [24]. See also [20], [25].

III. An Algorithm for Linear Systems with Variations in Delay

Consider a delay differential system of the form (1). We consider a form of quadratic functional based on the structure given by (4). Proceeding as was done for the case of fixed delay, we give conditions for stability in the following proposition.

Proposition 2: Consider system (1) and suppose there exist matrices $P(\theta,t)$, $S_1(\theta,t)$, $S_2(\theta,t)$, $T_1(\theta,t)$ and $T_2(\theta,t)$ of appropriate dimensions and a positive constant ϵ such that the following conditions are satisfied:

$$\begin{bmatrix} P_{11}(t) - \epsilon I + T_1(\theta,t) & \tau(t)P_{12}(\theta,t) \\ \tau(t)P_{12}^T(\theta,t) & \tau(t)P_{22}(\theta,t) \end{bmatrix} \geq 0 \quad \text{for } \theta \in [-\tau(t),0] \text{ and } t \geq 0, \tag{5}$$

$$R(\theta,\xi,t) = S_1^T(\theta,t)S_1(\xi,t) \tag{6}$$

$$\left[\begin{array}{c|c} \begin{matrix} Q_{11}(t)+\epsilon I & Q_{12}(t) \\ Q_{12}^T(t) & Q_{22}(t) \end{matrix} + T_2(\theta,t) & \begin{matrix} \tau(t)Q_{13}(\theta,t) \\ \tau(t)Q_{23}(\theta,t) \end{matrix} \\ \hline \begin{matrix} \tau(t)Q_{13}^T(\theta,t) & \tau(t)Q_{23}^T(\theta,t) \end{matrix} & \tau(t)Q_{33}(\theta,t) \end{array}\right] \leq 0 \quad \text{for } \theta \in [-\tau(t),0] \text{ and } t \geq 0, \tag{7}$$

$$\frac{\partial R}{\partial \theta} + \frac{\partial R}{\partial \xi} - \frac{\partial R}{\partial t} = S_2^T(\theta,t)S_2(\xi,t) \tag{8}$$

$$\int_{-\tau(t)}^{0} T_1(\theta,t)d\theta = 0 \tag{9}$$

$$\int_{-\tau(t)}^{0} T_2(\theta,t)d\theta = 0 \tag{10}$$

where

$$Q_{11} = A_0^T(t)P_{11}(t) + P_{11}(t)A_0(t) + P_{12}(0,t) + P_{12}^T(0,t) + P_{22}(0,t) + \frac{\partial P_{11}(t)}{\partial t}$$

$$Q_{12} = P_{11}(t)A_1(t) + P_{12}(-\tau(t),t)(\dot{\tau}(t)-1)$$

$$Q_{13} = A_0^T(t)P_{12}(\theta,t) - \frac{\partial P_{12}(\theta,t)}{\partial\theta} + \frac{\partial P_{11}(\theta,t)}{\partial t} + R(0,\theta)$$

$$Q_{22} = (\dot{\tau}(t)-1)P_{22}(-\tau(t),t)$$

$$Q_{23} = A_1^T(t)P_{12}(\theta,t) + (\dot{\tau}(t)-1)R(-\tau(t),\theta,t)$$

$$Q_{33} = -\frac{\partial P_{22}(\theta,t)}{\partial\theta} + \frac{\partial P_{22}(\theta,t)}{\partial t}$$

Then the system (1) is asymptotically stable.

Proof: Consider a Lyapunov-Krasovskii functional of the form shown in Equation (11). This Lyapunov functional, $V(t,x_t)$, as given by Equation (11) is positive definite due to positivity of the first part and the non-negativity of the

$$V(t,x_t) = \frac{1}{\tau(t)}\int_{-\tau(t)}^{0} \begin{bmatrix} x(t) \\ x(t+\theta) \end{bmatrix}^T \begin{bmatrix} P_{11}(t) & \tau(t)P_{12}(\theta,t) \\ \tau(t)P_{12}^T(\theta,t) & \tau(t)P_{22}(\theta,t) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t+\theta) \end{bmatrix} d\theta + \int_{-\tau(t)}^{0}\int_{-\tau(t)}^{0} x^T(t+\theta)R(\theta,\xi,t)x(t+\xi)d\theta d\xi \quad (11)$$

second part. Positivity of the first part is due to Conditions (5) and (9) since

$$\frac{1}{\tau(t)}\int_{-\tau(t)}^{0} \begin{bmatrix} x(t) \\ x(t+\theta) \end{bmatrix}^T \begin{bmatrix} P_{11}(t)-\epsilon I + T_1(\theta,t) & \tau(t)P_{12}(\theta,t) \\ \tau(t)P_{12}^T(\theta,t) & \tau(t)P_{22}(\theta,t) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t+\theta) \end{bmatrix} d\theta$$

$$= \frac{1}{\tau(t)}\int_{-\tau(t)}^{0} \begin{bmatrix} x(t) \\ x(t+\theta) \end{bmatrix}^T \begin{bmatrix} P_{11}(t) & \tau(t)P_{12}(\theta,t) \\ \tau(t)P_{12}^T(\theta,t) & \tau(t)P_{22}(\theta,t) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t+\theta) \end{bmatrix} d\theta + \frac{1}{\tau(t)}x(t)^T\left(\int_{-\tau(t)}^{0} T_1(\theta,t)\,d\theta\right)x(t) - \epsilon x(t)^T x(t),$$

and along with Condition (9) we obtain

$$\frac{1}{\tau(t)}\int_{-\tau(t)}^{0} \begin{bmatrix} x(t) \\ x(t+\theta) \end{bmatrix}^T \begin{bmatrix} P_{11}(t) & \tau(t)P_{12}(\theta,t) \\ \tau(t)P_{12}^T(\theta,t) & \tau(t)P_{22}(\theta,t) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t+\theta) \end{bmatrix} d\theta \geq \epsilon x(t)^T x(t) > 0.$$

Moreover, the second term in the functional (11) is positive due to condition (6) since

$$\int_{-\tau(t)}^{0}\int_{-\tau(t)}^{0} x^T(t+\theta)R(\theta,\xi,t)x(t+\xi)d\theta d\xi = \left(\int_{-\tau(t)}^{0} S_1(\theta,t)x(t+\theta)d\theta\right)^T \left(\int_{-\tau(t)}^{0} S_1(\xi,t)x(t+\xi)d\xi\right) \geq 0.$$

Therefore the first Lyapunov condition in Theorem 1 is satisfied.

The time-derivative of this functional along solutions of (1) is given by Equation (12). Here, the Q_{ij} are as in the statement of the Proposition and the $\star$ denotes entries so that the matrices are symmetric. Using an argument similar to the one we used to show that the Lyapunov functional (11) satisfies the first condition in the Lyapunov-Krasovskii theorem we can show that under (7–8) and (10) then the second Lyapunov condition is also satisfied. Thus system (1) is asymptotically stable. ■

Obtaining matrix functions P, S_1 and S_2 that satisfy the conditions in the above proposition is not an easy task. This was the main reason for using simplified Lyapunov-Krasovskii functionals in the past, yielding many times *conservative bounds* on both the delay bound and its variation. The conditions in Proposition 2 are expressed as functional constraints and cannot be implemented without additional manipulation. In this paper we will be able to construct numerical algorithms based on the SOS decomposition by assuming the functions to be bounded degree polynomials. The next section provides a brief overview of some tools for optimization of polynomials (see also [23], [20], [25] for more details).

IV. Algorithmic Tools

The methodology we use to implement the conditions of Proposition 2 is partly based on the sum-of-squares decomposition of positive polynomials. When applying this methodology we assume that all matrix functions are polynomial, can be approximated by polynomials, or there is a change of coordinates that renders them polynomial.

Denote by $\mathbb{R}[y]$ the ring of polynomials in $y = (y_1,\ldots,y_n)$ with real coefficients. Denote by Σ the cone of polynomials that admits a SOS decomposition, i.e., those $p \in \mathbb{R}[y]$ for which there exist $h_i \in \mathbb{R}[y]$, $i = 1,\ldots,M$ so that

$$p(y) = \sum_{i=1}^{M} h_i^2(y).$$

If $p(y) \in \Sigma$, then immediately $p(y) \geq 0$ for all y. The converse is not always true (but for univariate polynomials, quadratic forms and tertiary quartic forms). However, testing if $p(y) \geq 0$ has been classified to be NP-hard, whereas testing if $p(y) \in \Sigma$ is equivalent to an SDP [23], and hence is worst-case polynomial-time verifiable. The related SDP can be formulated efficiently and the solution can be retrieved using SOSTOOLS [26], which uses semidefinite solvers such as SeDuMi [27] or SDPT3 [28] to solve it.

Consider now the conditions in Proposition 2. If we make the assumption stated earlier on, that all the matrix functions are in fact matrix polynomials, then the conditions we need to test take the form:

$$M(\theta) \geq 0, \theta \in \Theta, \theta \in \mathbb{R}^m \quad (13)$$

where $M(\theta) \in \mathbb{R}^{n\times n}$ and Θ is a set described by polynomial inequalities:

$$\Theta = \{\theta \in \mathbb{R}^m \mid g_i(\theta) \leq 0, \quad i = 1,\ldots,M\}$$

where $g_i(\theta)$ are polynomial functions. In order to test condition (13) we construct a vector of variables $y = (y_1,\ldots,y_n)$ and look for sum of squares polynomials $p_i(\theta, y)$, second order with respect to the y variables, so that

$$y^T M(\theta)y + \sum_{i=1}^{M} g_i(\theta)p_i(\theta,y) \geq 0$$

$$\begin{aligned}
\dot V(t,x_t) &= x^T(t)\left(A_0^T(t)P_{11}(t)+P_{11}(t)A_0(t)+\frac{\partial P_{11}(t)}{\partial t}\right)x(t)+x^T(t-\tau(t))A_1^T(t)P_{11}(t)x(t)\\
&+x^T(t)P_{11}(t)A_1(t)x(t-\tau(t))+x^T(t)A_0^T(t)\int_{-\tau(t)}^0 P_{12}(\theta,t)x(t+\theta)d\theta+x^T(t-\tau(t))A_1^T(t)\int_{-\tau(t)}^0 P_{12}(\theta,t)x(t+\theta)d\theta\\
&+\int_{-\tau(t)}^0 x^T(t+\theta)P_{12}^T(\theta,t)A_0(t)x(t)d\theta+\int_{-\tau(t)}^0 x^T(t+\theta)P_{12}^T(\theta,t)A_1(t)x(t-\tau(t))d\theta\\
&-\int_{-\tau(t)}^0 x^T(t+\theta)\left(\frac{\partial P_{12}^T(\theta,t)}{\partial\theta}-\frac{\partial P_{12}^T(\theta,t)}{\partial t}\right)x(t)d\theta-\int_{-\tau(t)}^0 x^T(t)\left(\frac{\partial P_{12}(\theta,t)}{\partial\theta}-\frac{\partial P_{12}(\theta,t)}{\partial t}\right)x(t+\theta)d\theta\\
&+x^T(t)(P_{12}(0,t)+P_{12}^T(0,t))x(t)+x^T(t)P_{12}(-\tau(t),t)(\dot\tau(t)-1)x(t-\tau(t))+x^T(t-\tau(t))P_{12}^T(-\tau(t),t)(\dot\tau(t)-1)x(t)\\
&+x^T(t)P_{22}(0,t)x(t)+(\dot\tau(t)-1)x^T(t-\tau(t))P_{22}(-\tau(t),t)x(t-\tau(t))\\
&-\int_{-\tau(t)}^0 x^T(t+\theta)\left(\frac{\partial P_{22}(\theta,t)}{\partial\theta}-\frac{\partial P_{22}(\theta,t)}{\partial t}\right)x(t+\theta)d\theta\\
&+\int_{-\tau(t)}^0 x^T(t)R(0,\theta,t)x(t+\theta)d\theta+(\dot\tau(t)-1)\int_{-\tau(t)}^0 x^T(t-\tau(t))R(-\tau(t),\theta,t)x(t+\theta,t)d\theta\\
&+\int_{-\tau(t)}^0 x^T(t+\theta)R(\theta,0,t)x(t)d\theta+(\dot\tau(t)-1)\int_{-\tau(t)}^0 x^T(t+\theta)R(\theta,-\tau(t),t)x(t-\tau(t))d\theta\\
&-\int_{-\tau(t)}^0\int_{-\tau(t)}^0 x^T(t+\theta)\left(\frac{\partial R(\theta,\xi,t)}{\partial\theta}+\frac{\partial R(\theta,\xi,t)}{\partial\xi}-\frac{\partial R(\theta,\xi,t)}{\partial t}\right)x(t+\xi)d\theta d\xi\\
&=\frac{1}{\tau(t)}\int_{-\tau(t)}^0\begin{bmatrix}x(t)\\x(t-\tau(t))\\x(t+\theta)\end{bmatrix}^T\begin{bmatrix}Q_{11} & Q_{12} & \tau(t)Q_{13}\\ \star^T & Q_{22} & \tau(t)Q_{23}\\ \star^T & \star^T & \tau(t)\left(-\frac{\partial P_{22}(\theta,t)}{\partial\theta}+\frac{\partial P_{22}(\theta,t)}{\partial t}\right)\end{bmatrix}\begin{bmatrix}x(t)\\x(t-\tau(t))\\x(t+\theta)\end{bmatrix}d\theta\\
&-\int_{-\tau(t)}^0\int_{-\tau(t)}^0 x^T(t+\theta)\left(\frac{\partial R}{\partial\theta}+\frac{\partial R}{\partial\xi}-\frac{\partial R}{\partial t}\right)x(t+\xi)d\theta d\xi \qquad (12)
\end{aligned}$$

Indeed, the above condition guarantees that when $\theta\in\Theta$, we have $y^TM(\theta)y\ge-\sum_{i=1}^M g_i(\theta)p_i(\theta,y)\ge 0$ since $g_i\le 0$ and $p_i\ge 0$, and therefore $M(\theta)\ge 0$ for those θ.

We note that the above construction is much more general, in the sense that the vector θ could include unknown parameters etc., thus facilitating the analysis of time-varying, uncertain systems.

V. Examples with fixed and variable delays

As mentioned in the Introduction, some of the approaches in the control literature considered the problem of characterizing the *allowable* time-varying uncertainty around some nominal delay value [10], [18] (see also the references therein) by using Lyapunov-Krasovskii functionals [10] or appropriate IQCs in frequency-domain [18]. Perhaps the best way to address stability in the presence of time-varying delays is to consider the time-varying uncertainty around some nominal delay value. Then delay is a function of time given by:

$$\tau(t):=\tau_0+\eta(t) \qquad (14)$$

where η is assumed to be a bounded differentiable function. If the original system is asymptotically stable for $\tau(t)=\tau_0$, then the problem is to give constraints on the allowable time-varying uncertainty for which the stability is preserved.

The algorithm proposed in the previous section allows us to give numerical refinements to some of the existing bounds in the literature. In particular, given μ and ν we can constrain the admissible variations η to be those such that

$$|\eta(t)|\le\nu \quad\text{and}\quad |\dot\eta(t)|\le\mu$$

for all $t\ge 0$. Thus for $\mu=0$ we have a fixed delay and as μ increases, a higher variation in the time-delay value can be allowed. In an example later on in this section, we observe how changes in μ affect the maximum value of ν for which stability is guaranteed, around some mid-point τ_0.

A. Further time-varying delay problems and comments

Before giving larger examples, we will show how our technique can be used to explore the complexity of the following simple and much-studied problem:

$$\dot x=-ax(t-\tau(t)) \qquad (15)$$

It is well known that for a *constant delay*, the *optimal bound* on the delay size guaranteeing asymptotic stability is given by $\tau^*<\pi/2a$. Now, if the delay is time-varying, it is intuitively obvious that the optimal bound reduces. In fact, it has been illustrated that there exist delay functions with $|\tau(t)|\le\frac{3}{2a}$ but which produce oscillating solutions [8], [30]. In this case the corresponding delay function is piecewise continuous. In this context, a natural question arises: What happens during the transition from fixed delay to piecewise continuous?

Let us fix $a=1$ and examine the behavior of the stability window. Figure 1 shows an estimate of the upper bound

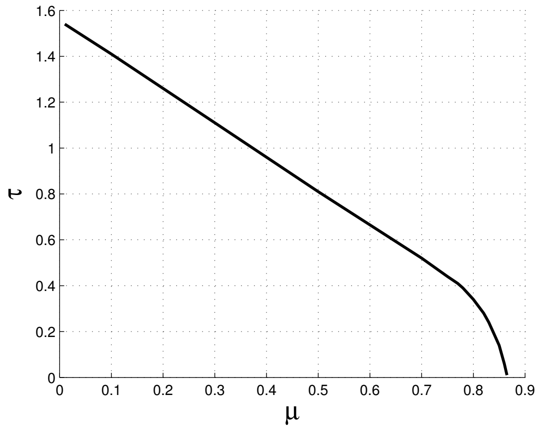

Fig. 1. Maximum stable $\overline{\tau}$ for $\underline{\tau} \simeq 0$ as a function of μ.

Order of P, S_1 and S_2 in θ	2	3	4
$\mu = 0.1, \quad \overline{\tau}$	3.92	4.20	4.35
$\mu = 0.5, \quad \overline{\tau}$	1.48	1.56	1.62
$\mu = 0.9, \quad \overline{\tau}$	0.31	0.33	0.35

TABLE I

DELAY INTERVALS FOR WHICH STABILITY IS GUARANTEED THROUGH THE CONSTRUCTION OF A LYAPUNOV FUNCTIONAL OF THE FORM (11) FOR SYSTEM (16).

on $\overline{\tau}$ for which stability can be guaranteed for $\underline{\tau} \simeq 0$. As the μ transitions from 0 we see that $\overline{\tau}$ decreases until eventually we observe the possibility of quenching - we cannot guarantee any stability for sufficiently large rates of variations in the delay. This demonstrates that allowing the delay to be piecewise continuous is critical in obtaining the $3/2$ counterexample in [7], [30]. Note that in Figure 1, the constraint that $\underline{\tau} \simeq 0$ is important. If we allow $\underline{\tau}$ to increase, then our estimate on the bound of $\overline{\tau}$ will often increase as well. This suggests that not only the size of the delay matters, but the size of the stability window, as well.

Consider now the time-delayed system

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1(t-\tau(t)) \\ x_2(t-\tau(t)) \end{bmatrix}. \qquad (16)$$

The results are summarized in Table I. There we can see how increasing the allowable time-variation in the delay leads to a compromise in the allowable overbound on the delay size. Figure 2 shows how the variation of the upper bound on the delay size changes as a function of μ, for a Lyapunov-Krasovskii functional of a particular structure.

In the next section we will talk briefly about the quenching phenomenon and we will give a more interesting example with an interval delay.

VI. SOME REMARKS ON THE QUENCHING PHENOMENON

We have already mentioned that in this paper we will discuss briefly the quenching phenomenon. Here, we will review some background material before giving an example

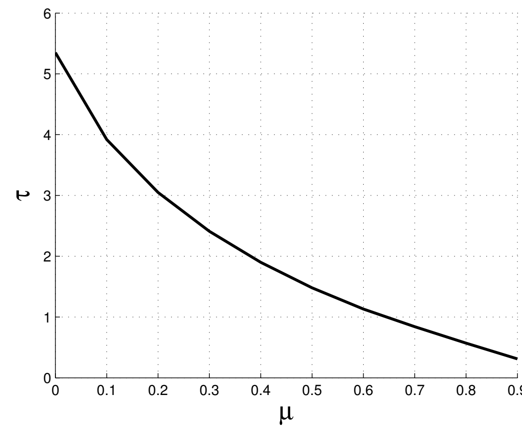

Fig. 2. Estimates of $\overline{\tau}$ versus μ for system (16). The order of P, S_1 and S_2 with respect to θ was 2, when constructing the Lyapunov Krasovskii functional using proposition 2.

– more details can be found in [14] and [15]. Consider the following linear system with a single constant delay:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau), \qquad (17)$$

where $x \in \mathbb{R}^n$, and $A_0, A_1 \in \mathbb{R}^{n\times n}$, along with appropriate initial conditions. Consider the delay τ as a free parameter. Under the notations above, the *quenching problem* can be summarized as follows: *finding conditions* such that:

(a) System (17) which is asymptotically stable for all (constant) delays $\tau \in [\underline{\tau}, \overline{\tau}]$ becomes unstable if the delay τ is assumed time-varying in the same range;

(b) reciprocally, system (17), which is unstable for all (constant) delays $\tau \in [\underline{\tau}, \overline{\tau}]$ becomes stable if the delay τ is assumed time-varying in the same range.

The quenching was proven by [15] by using a discontinuous periodic delay function. The question is to find if a differentiable delay function $\tau(t)$ can emphasize such a property. The tools we have developed in the previous section will allow us to understand, to some extent, this phenomenon, which will be better illustrated by a simple example.

Consider system (17) with

$$A_0 = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}.$$

This example comes from [15]. Using a simple frequency domain argument, it can be shown that this system is stable for constant $\tau \in [0.445, 1.502]$ and unstable otherwise, i.e., for $\tau = 0$ the system is unstable and there is an interval delay bound which stabilizes it. These intervals can be verified by constructing the matrix functions in Proposition 2 with $\dot{\tau}(t) = 0$ using the sum of squares decomposition and SOSTOOLS. For example, upper and lower bounds on this interval of stability can be obtained as shown in Table II for different order matrices P, S_1 and S_2 in θ. The values in this table converge to the true interval, thus illustrating the usefulness of the SOS methodology.

We now consider the same example, and record for different time-delay intervals, the maximum variation we can construct a Lyapunov functional as above. For this example,

Order of P, S_1 and S_2 in θ	2	3	4
Upper delay bound	1.20	1.38	1.47
Lower delay bound	0.45	0.45	0.45

TABLE II

FIXED DELAY INTERVALS FOR WHICH STABILITY IS GUARANTEED THROUGH THE CONSTRUCTION OF A LYAPUNOV FUNCTIONAL OF THE FORM (4).

$h > 0$, such that $\|\dot{\tau}(t)\| \leq \mu$	0.13	0.09	0.03
$\overline{\tau}$	0.9999	1.0264	1.0528
$\underline{\tau}$	0.9471	0.9207	0.8942

TABLE III

DELAY INTERVALS AND MAXIMUM VARIATIONS ALLOWED HIGH STABILITY IS GUARANTEED THROUGH THE CONSTRUCTION OF A LYAPUNOV FUNCTIONAL OF THE FORM (11).

we set τ_0 to be the mid-point of the delay interval in the fixed delay case, i.e.,

$$\tau_0 = \frac{0.445 + 1.502}{2} = 0.9735$$

and we look at how the size of the interval changes as variations in the allowable delay size change, i.e., we look at the maximum $|\eta(t)|$ in equation (14) (this is the variable ν introduced earlier) that can be obtained for some $\mu = |\dot{\eta}(t)|$.

The results are shown in Table III. The matrices P, S_1 and S_2 in Proposition 2 are cubic in θ. We see that as h increases, the allowable time-varying delay range decreases.

VII. CONCLUDING REMARKS

In this paper we have considered the stability analysis of a class of time-varying linear systems with time-varying delays. We developed robust stability conditions that can be tested using SDP, based on the SOS decomposition of multivariate polynomials and the Lyapunov-Krasovskii theorem commenting on the so-called *quenching phenomenon*.

One of the strengths of this work is that if the time-variation is known exactly as a function of time, this can be included in the analysis rather than having a crude bound on its variation. However, this is many times difficult as the variation is bounded, and a polynomial description of the time variation would be difficult in that respect. Future research will concentrate on this topic.

Another issue we have not explored in this paper is the appearance of stability if the variation of the time-delay is assumed large enough, while if the variation is zero the system is unstable. This case is more challenging and we are currently investigating ways to approach it.

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