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Positivity of Kernel Functions for Systems with Communication Delay

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Abstract— **The purpose of this paper is to provide further results on a method of constructing Lyapunov functionals for infinite-dimensional systems using semidefinite programming. Specifically, we give a necessary and sufficient condition for positivity of a positive integral operator described by a polynomial kernel. We then show how to combine this result with multiplier operators in order to obtain positive composite Lyapunov functionals. These types of functionals are used to prove stability of linear time-delay systems.**

I. Introduction

Consider a linear time-delay system of the form

$$\dot{x}(t) = \sum_{i=1}^{k} A_i x(t - h_i) \tag{1}$$

where $x(t) \in \mathbb{R}^n$. In the simplest case we are given information about the delays, $h_0, \ldots, h_k$, and the matrices $A_0, \ldots, A_k$ and we would like to determine whether the system is stable.

The system of equations given by Equation (1) is a special case of a class of linear systems for which there exist converse Lyapunov results wherein the structure of the Lyapunov functional is known. Examples of these results are given by [11], [1], [10]. Let $h_0 = 0$ and $h_k = h$. For this class of linear systems, a necessary and sufficient condition for stability is the existence of a Lyapunov functional which has the following form:

$$V(\phi) = \int_{-h}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds + \int_{-h}^{0}\int_{-h}^{0} \phi(s)^T N(s,t) \phi(t)\, ds\, dt$$

The derivative of this Lyapunov functional along trajectories of (1) is also a functional defined by a linear transformation of the matrix functions M and N. Thus the question of stability is the following convex feasibility problem.

$$\begin{aligned} &\text{find} && M, N \\ &\text{s.t.} && V(\phi) - \epsilon\|\phi(0)\| \geq 0 \quad \text{for all } \phi \in \mathcal{C} \\ &&& -\dot{V}(\phi) \geq 0 \quad \text{for all } \phi \in \mathcal{C} \end{aligned}$$

The goal of this paper is to use semidefinite programming to construct polynomial solutions to this feasibility problem. The question of whether polynomial solutions are equivalent to continuous solutions is not completely understood, however some affirmative results on this topic can be found in [8]. In the earlier paper [9], we gave a necessary and sufficient condition for positivity of the first part of the functional,

$$V_1(\phi) = \int_{-h}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds.$$

Roughly speaking, this result stated that the first part of the functional is positive for all ϕ if and only if there exists a matrix-valued function T such that

$$M(t) + \begin{bmatrix} T(t) & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \quad \text{for all } t$$

$$\int_{-h}^{0} T(t)\, dt = 0.$$

For polynomials this new condition is easy to implement as the pointwise positivity condition is equivalent to a sum-of-squares constraint and the integral condition is linear in the coefficients.

In this paper, we give a necessary and sufficient condition for positivity of the second part of the functional when it is defined by a polynomial, N, as

$$V_2(\phi) = \int_{-h}^{0}\int_{-h}^{0} \phi(s)^T N(s,t) \phi(t)\, ds\, dt.$$

In the simplest case, this result shows that if N is a polynomial, then V_2 is positive if and only if there exists a $Q \geq 0$ such that

$$N(s,t) = Z(s)^T Q Z(t).$$

Here Z is a specific monomial basis. This result allows us to express positivity conditions directly in terms of positive matrices. Note that simple positivity of N in this case is not sufficient for positivity of V_2 as illustrated by the counterexample $N(s,t) = (s-t)^2$, $h = 2$, and $\phi(s) = s+1$.

The second significant contribution of this paper is to show how to combine both the result on kernel functions and the result obtained in [9]. We show that positivity of the combined functional is guaranteed by the existence of a T and $Q \geq 0$, such that

$$\begin{aligned} M(s) &= Z(s)^T Q_{11} Z(s) + T(s) \\ N(s,t) &= Z(s)^T Q_{12} Z(s,t) + Z(t,s)^T Q_{21} Z(s) \\ &\quad + \int_{-h}^{0} Z(\omega,s)^T Q_{22} Z(\omega,t) d\omega \end{aligned}$$

$$\int_{-h}^{0} T(s)\, ds = 0.$$

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This condition improves on previous results by relaxing the constraint that each part of the functional be individually positive. Note that if we set $Q_{12} = Q_{21} = 0$, we recover the previous conditions.

A. Background

The study of stability of time-delay systems has long been an active area of research. Krasovskii [4] was the first to provide a converse Lyapunov result for these systems. The work by Repin [10] first provided the quadratic structure of the Lyapunov functional for linear systems. Further references may be found there as well as in [2], [3], [1] and [5].

Many algorithms for the stability analysis of linear time-delay systems have been proposed in the literature. Lyapunov functionals and eigenvalue analysis are tools frequently used in these algorithms. However, certain questions of stability of linear time-delay systems are NP-hard, such as delay-independent stability or stability with interval delay. Therefore there is unlikely to exist an algorithm which will always answer these stability questions in polynomial time. The complexity of determining whether all roots of a quasipolynomial lie in the closed left half-plane is unknown. The question of interest, therefore, is how best to construct polynomial-time algorithms for the analysis of linear time-delay systems. The algorithm proposed in this paper has certain positive traits which include little or no conservativity for relatively little computational effort and a structure based on a condition known to be necessary and sufficient.

II. Notation

Let $\mathbb{N}$ denote the set of nonnegative integers. Let $\mathbb{S}^n$ be the set of $n \times n$ real symmetric matrices, and for $X \in \mathbb{S}^n$ we write $X \succeq 0$ to mean that X is positive semidefinite. For two matrices A, B, we denote the Kronecker product by $A \otimes B$. For X any Banach space and $I \subset \mathbb{R}$ any interval, let $\Omega(I, X)$ be the space of all functions

$$\Omega(I, X) = \{\, f : I \to X \,\}$$

and let $\mathcal{C}(I, X)$ be the Banach space of bounded continuous functions

$$\mathcal{C}(I, X) = \{\, f : I \to X \mid f \text{ is continuous and bounded} \,\}$$

equipped with the norm

$$\|f\| = \sup_{t \in I} \|f(t)\|_X.$$

We will omit the range space when it is clear from the context; for example we write $\mathcal{C}[a, b]$ to mean $\mathcal{C}([a, b], X)$. A function is called $\mathcal{C}^n(I, X)$ if the i^{th} derivative exists and is a continuous function for $i = 0, \dots, n$. A function $f \in \mathcal{C}[a, b]$ is called piecewise continuous if there exists a finite number of points $a < h_1 < \cdots < h_k < b$ such that f is continuous at all $x \in [a, b] \backslash \{h_1, \dots, h_k\}$ and its right and left-hand limits exist at $\{h_1, \dots, h_k\}$.

Define also the projection $H_t : \Omega[-h, \infty) \to \Omega[-h, 0]$ for $t \ge 0$ and $h > 0$ by

$$(H_t x)(s) = x(t + s) \quad \text{for all } s \in [-h, 0].$$

We follow the usual convention and denote $H_t x$ by x_t.

A. Linear Time-Delay Systems

We consider linear differential equations with discrete delays, of the form

$$\dot{x}(t) = \sum_{i=0}^{k} A_i x(t - h_i) \quad \text{for all } t \ge 0, \tag{2}$$

where $A_0, \dots, A_k \in \mathbb{R}^{n \times n}$ and the trajectory $x : [-h, \infty) \to \mathbb{R}^n$. The boundary conditions are specified by a given function $\phi : [-h, 0] \to \mathbb{R}^n$ and the constraint

$$x(t) = \phi(t) \quad \text{for all } t \in [-h, 0]. \tag{3}$$

Let $\phi \in \mathcal{C}[-h, 0]$. Then there exists a unique continuous, differentiable function x satisfying (2) and (3). We write the solution as an explicit function of the initial conditions using the map $G : \mathcal{C}[-h, 0] \to \Omega[-h, \infty)$, defined by

$$(G\phi)(t) = x(t) \quad \text{for all } t \ge -h$$

where x is the unique solution of (2) and (3) corresponding to initial condition ϕ. Also for $s \ge 0$ define the *flow map* $\Gamma_s : \mathcal{C}[-h, 0] \to \mathcal{C}[-h, 0]$ by

$$\Gamma_s \phi = H_s G \phi$$

which maps the state of the system x_t to the state at a later time $x_{t+s} = \Gamma_s x_t$. The system is called ***exponentially stable*** if there exists $\sigma > 0$ and $a \in \mathbb{R}$ such that for every $\phi \in \mathcal{C}[-h, 0]$,

$$\|(G\phi)(t)\| \le a e^{-\sigma t} \|\phi\| \quad \text{for all } t \ge 0.$$

B. Quadratic Lyapunov Functionals

Suppose $V : \mathcal{C}[-h, 0] \to \mathbb{R}$. Define the ***Lie derivative*** of V with respect to Γ by

$$\dot{V}(\phi) = \limsup_{r \to 0+} \frac{1}{r} \big(V(\Gamma_r \phi) - V(\phi) \big).$$

$\dot{V}$ denotes both the Lie derivative and the usual derivative, The difference will be clear from context. Consider the set X of quadratic functionals, where $V \in X$ if there exist bounded piecewise $\mathcal{C}^1$ functions $M : [-h, 0) \to \mathbb{S}^{2n}$ and $N : [-h, 0) \times [-h, 0) \to \mathbb{R}^{n \times n}$ such that

$$V(\phi) = \int_{-h}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds + \int_{-h}^{0} \int_{-h}^{0} \phi(s)^T N(s, t) \phi(t) \, ds \, dt. \tag{4}$$

The following result states that for linear systems with delay, the system is exponentially stable if and only if this can be shown using a quadratic Lyapunov functional $V \in X$.

Theorem 1: The linear system defined by equations (2) and (3) is exponentially stable if and only if there exists a Lie-differentiable function $V \in X$ and $\varepsilon > 0$ such that for all $\phi \in \mathcal{C}[-h, 0]$

$$\begin{aligned} V(\phi) &\ge \varepsilon \|\phi(0)\|^2, \\ \dot{V}(\phi) &\le -\varepsilon \|\phi(0)\|^2. \end{aligned} \tag{5}$$

Further $V \in X$ may be chosen such that the corresponding functions M and N of equation (4) have the following smoothness property: $M(s)$ and $N(s,t)$ are continuous at all $s, t \neq h_i$.

Proof: See [11] for a proof. ■

III. Positivity

Suppose $0 = h_0 < h_1 < \cdots < h_k = h$. Define the sets $H = \{-h_0, \ldots, -h_k\}$ and $H^c = [-h, 0]\backslash H$. Define the intervals

$$H_i = \begin{cases} [-h_1, 0] & \text{if } i = 1 \\ [-h_i, -h_{i-1}) & \text{if } i = 2, \ldots, k. \end{cases}$$

Recall the form of the quadratic Lyapunov functional.

$$V(\phi) = \int_{-h}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds + \int_{-h}^{0} \int_{-h}^{0} \phi(s)^T N(s,t) \phi(t)\, ds\, dt$$

A necessary and sufficient condition for positivity of the first part of the functional was given in [9] by the following theorem.

Theorem 2: Suppose the piecewise continuous $M : [-h, 0] \to \mathbb{S}^{m+n}$ is continuous on H^c, then the following are equivalent.

(i) There exists an $\epsilon > 0$ so that for all $x \in \mathbb{R}^m$ and continuous $y : [-h, 0] \to \mathbb{R}^n$

$$\int_{-h}^{0} \begin{bmatrix} x \\ y(t) \end{bmatrix}^T M(t) \begin{bmatrix} x \\ y(t) \end{bmatrix} dt \geq \epsilon \|y\|^2 \qquad (6)$$

(ii) There exists an $\epsilon > 0$ and a piecewise continuous $T : [-h, 0] \to \mathbb{S}^m$ which is continuous on H^c and satisfies

$$M(t) + \begin{bmatrix} T(t) & 0 \\ 0 & -\epsilon I \end{bmatrix} \geq 0 \quad \text{for all } t \in [-h, 0]$$

$$\int_{-h}^{0} T(t)\, dt = 0.$$

We now consider the second part of the functional. Define the vector of indicator functions $g : [-h, 0] \to \mathbb{R}^k$ by

$$g_i(t) = \begin{cases} 1 & \text{if } t \in H_i \\ 0 & \text{otherwise} \end{cases}$$

for all $i = 1, \ldots, k$ and all $t \in [-h, 0]$. Let $z_d(t)$ be the vector of monomials

$$z_d(t) = \begin{bmatrix} 1 & t & t^2 & \ldots & t^d \end{bmatrix}^T$$

and for convenience also define the function $Z_{n,d} : [-h, 0] \to \mathbb{R}^{nk(d+1) \times n}$ by

$$Z_{n,d}(t) = g(t) \otimes I_n \otimes z_d(t).$$

Thus Z_d^n is a vector of block diagonal matrices with copies of $z_d g_i(t)$ on the diagonal of matrix i. A polynomial in two variables is referred to as a *binary* polynomial. A function $N : [-h, 0] \times [-h, 0] \to \mathbb{S}^n$ is called a *binary piecewise polynomial matrix* if for each $i, j \in \{1, \ldots, k\}$ the function N restricted to the set $H_i \times H_j$ is a binary polynomial matrix. It is straightforward to show that N is a symmetric binary piecewise polynomial matrix if and only if there exists a matrix $Q \in \mathbb{S}^{nk(d+1)}$ such that

$$N(s,t) = Z_{n,d}^T(s) Q Z_{n,d}(t).$$

Here d is the degree of N.

Now consider the quadratic form

$$\int_{-h}^{0} \int_{-h}^{0} \phi(s)^T N(s,t)\, \phi(t)\, ds\, dt. \qquad (7)$$

We would like to characterize the binary piecewise polynomial matrices N for which the quadratic form (7) is nonnegative for all $\phi \in \mathcal{C}([-h, 0], \mathbb{R}^n)$. We first state the following Lemma.

Lemma 3: Suppose z is the vector of monomials

$$z(t) = \begin{bmatrix} 1 & t & t^2 & \ldots & t^d \end{bmatrix}^T$$

and the linear map $A : \mathcal{C}[0,1] \to \mathbb{R}^{d+1}$ is given by

$$A\phi = \int_0^1 z(t)\phi(t)\, dt$$

Then $\operatorname{rank} A = d + 1$.

Proof: Suppose for the sake of a contradiction that $\operatorname{rank} A < d + 1$. Then $\operatorname{range} A$ is a strict subset of $\mathbb{R}^{d+1}$ and hence there exists a nonzero vector $q \in \mathbb{R}^{d+1}$ such that $q \perp \operatorname{range} A$. This means

$$\int_0^1 q^T z(t)\phi(t)\, dt = 0$$

for all $\phi \in \mathcal{C}[0,1]$. Since $q^T z$ and ϕ are continuous functions, define the function $v : [0,1] \to \mathbb{R}$ by

$$v(t) = \int_0^t q^T z(s)\, ds \qquad \text{for all } t \in [0,1].$$

Since v is absolutely continuous, we have for every $\phi \in \mathcal{C}[0,1]$ that

$$\int_0^1 \phi(t)\, dv(t) = \int_0^1 q^T z(t)\phi(t)\, dt = 0$$

where the integral on the left-hand-side of the above equation is the Stieltjes integral. The function v is also of bounded variation, since its derivative is bounded. The Riesz representation theorem implies that if v is of bounded variation and

$$\int_0^1 \phi(t)\, dv(t) = 0$$

for all $\phi \in \mathcal{C}[0,1]$, then v is constant on an everywhere dense subset of $(0,1)$. Since v is continuous, we have v is constant, and therefore $q^T z(t) = 0$ for all t. Since $q^T z$ is a polynomial, this contradicts the statement that $q \neq 0$. ■

We now state the positivity result.

Theorem 4: Suppose N is a symmetric binary piecewise polynomial matrix of degree d. Then

$$\int_{-h}^{0} \int_{-h}^{0} \phi(s)^T N(s,t) \phi(t)\, ds\, dt \geq 0 \qquad (8)$$

for all $\phi \in \mathcal{C}([-h,0],\mathbb{R}^n)$ if and only if there exists $Q \in \mathbb{S}^{nk(d+1)}$ such that

$$N(s,t) = Z_{n,d}^T(s) Q Z_{n,d}(t)$$
$$Q \succeq 0.$$

Proof: We only need to show the *only if* direction. Suppose N is a symmetric binary piecewise polynomial matrix. Let d be the degree of N. Then there exists a symmetric matrix Q such that

$$N(s,t) = Z_{n,d}^T(s) Q Z_{n,d}(t).$$

Now suppose that the inequality (8) is satisfied for all continuous functions ϕ. We will show that every such Q is positive semidefinite. To see this, define the linear map $J : \mathcal{C}([-h,0],\mathbb{R}^n) \to \mathbb{R}^{nk(d+1)}$ by

$$J\phi = \int_{-h}^{0} (g(t) \otimes I_n \otimes z(t))\phi(t)\, dt.$$

Then

$$\int_{-h}^{0}\int_{-h}^{0} \phi(s)^T N(s,t)\phi(t)\, ds\, dt = (J\phi)^T Q (J\phi).$$

The result we desire holds if $\operatorname{rank} J = nk(d+1)$, since in this case $\operatorname{range} J = \mathbb{R}^{nk(d+1)}$. Then if Q has a negative eigenvalue with corresponding eigenvector q, there exists ϕ such that $q = J\phi$ so that the quadratic form will be negative, contradicting the hypothesis.

To see that $\operatorname{rank} J = nk(d+1)$, define for each $i = 1,\ldots,k$ the linear map $L_i : \mathcal{C}[H_i] \to \mathbb{R}^n$ by

$$L_i\phi = \int_{H_i} z(t)\phi(t)\, dt$$

Then if we choose coordinates for ϕ such that

$$\phi = \begin{bmatrix} \phi|_{H_1} \\ \phi|_{H_2} \\ \vdots \\ \phi|_{H_k} \end{bmatrix}$$

where $\phi|_{H_j}$ is the restriction of ϕ to the interval H_j, then we have in these coordinates that J is

$$J = \operatorname{diag}(L_1,\ldots,L_k) \otimes I_n.$$

Further, by Lemma 3 the maps L_i each satisfy $\operatorname{rank} L_i = d+1$. Therefore $\operatorname{rank} J = nk(d+1)$ as desired. ■

The following corollary gives a tighter degree bound on the representation of N.

Corrolary 5: Let N be a binary piecewise polynomial matrix of degree $2d$ which is positive in the sense of Equation (8), then there exists a $Q \in \mathbb{S}^{nk(d+1)}$ such that

$$N(s,t) = Z_{n,d}^T(s) Q Z_{n,d}(t)$$
$$Q \succeq 0.$$

The proof is omitted due to length constraints.

For convenience, we define the set of symmetric binary piecewise polynomial matrices which define positive quadratic forms by

$$\Gamma_{n,d} = \left\{ Z_{n,d}^T(s) Q Z_{n,d}(t) \mid Q \in \mathbb{S}^{nk(d+1)}, Q \succeq 0 \right\}.$$

If we are given a binary piecewise polynomial matrix $N : [-h,0] \times [-h,0] \to \mathbb{S}^n$ of degree $2d$ and want to know whether it defines a positive quadratic form, then this is computationally checkable using semidefinite programming. The number of variables involved in this task scales as $(nk)^2(d+1)^2$.

IV. Joint Positivity

The previous section showed that one can enforce positivity on the individual parts of the quadratic functional V. We now show how one can enforce joint positivity of the functional in a way that is not less conservative than individual positivity.

Theorem 6: Suppose there exists a $Q \succeq 0$ and $T : I \to \mathbb{S}^m$ such that

$$V(c,x) = \int_I \begin{bmatrix} c \\ x(s) \end{bmatrix}^T M(s) \begin{bmatrix} c \\ x(s) \end{bmatrix} ds + \int_I \int_I x(s)^T N(s,t)\, x(t)\, ds\, dt,$$

where

$$M(s) = Z^{m+n}(s)^T Q_1 Z^{m+n}(s) + \begin{bmatrix} T(s) & 0 \\ 0 & 0 \end{bmatrix}$$
$$N(s,t) = Z^n(s)^T Q_{23}\, Z^n(s,t) + Z^n(t,s)^T Q_{32} Z^n(t) + \int_I Z^n(\omega,s)^T Q_{33}\, Z^n(\omega,t)\, d\omega,$$
$$\int_I T(s) ds = 0$$

and

$$Q_1 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \qquad Q = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & Q_{23} \\ 0 & Q_{32} & Q_{33} \end{bmatrix}.$$

Then $V(c,x) \geq 0$ for $c \in \mathbb{R}^m$ and $x \in \mathcal{C}(I)$.

Proof: Since $Q \succeq 0$, $Q = D^T D$ for $D \in \mathbb{R}^{f+g+h \times f+g+h}$ where $Q_1 \in \mathbb{S}^{f+g}$, $Q_{33} \in \mathbb{S}^h$. Partition D as

$$D = \begin{bmatrix} D_1 & D_2 & D_3 \end{bmatrix},$$

where $D_1 \in \mathbb{R}^{f+g+h \times f}$, $D_2 \in \mathbb{R}^{f+g+h \times g}$. Then

$$\begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & Q_{23} \\ 0 & Q_{32} & Q_{33} \end{bmatrix} = \begin{bmatrix} D_1^T D_1 & D_1^T D_2 & D_1^T D_3 \\ D_2^T D_1 & D_2^T D_2 & D_2^T D_3 \\ D_3^T D_1 & D_3^T D_2 & D_3^T D_3 \end{bmatrix}$$

Let

$$P(s) := \begin{bmatrix} D_1 & D_2 \end{bmatrix} Z^{m+n}(s)$$
$$R(s,t) := \begin{bmatrix} 0 & D_3 \end{bmatrix} Z^{m+n}(s,t),$$

and $A : \mathcal{C}(I) \to \mathcal{C}(I)$ be given by

$$A\phi(s) = P(s)\phi(s) + \int_I R(s,t)\phi(t) dt.$$

Let

$$\phi(s) = \begin{bmatrix} c \\ x(s) \end{bmatrix}.$$

Then

$$\begin{aligned}\langle A\phi, A\phi\rangle &= \int_I \begin{bmatrix} c \\ x(s) \end{bmatrix}^T P(s)^T P(s) \begin{bmatrix} c \\ x(s) \end{bmatrix} \\ &\quad + \int_I \int_I \begin{bmatrix} c \\ x(s) \end{bmatrix}^T \Big(P(s)^T R(s,t) + R(t,s)^T P(t) \\ &\qquad + \int_I R(\omega,s)^T R(\omega,t)\, d\omega \Big) \begin{bmatrix} c \\ x(t) \end{bmatrix} dt\, ds \\ &= \int_I \begin{bmatrix} c \\ x(s) \end{bmatrix}^T M(s) \begin{bmatrix} c \\ x(s) \end{bmatrix} \\ &\quad + \int_I \int_I \begin{bmatrix} c \\ x(s) \end{bmatrix}^T G(s,t) \begin{bmatrix} c \\ x(s) \end{bmatrix} dt\, ds.\end{aligned}$$

where

$$\begin{aligned}G(s,t) &= P(s)^T R(s,t) + R(t,s)^T P(t) \\ &\qquad + \int_I R(\omega,s)^T R(\omega,t) d\omega \\ &= Z^{m+n}(s)^T \begin{bmatrix} D_1^T \\ D_2^T \end{bmatrix} \begin{bmatrix} 0 & D_3 \end{bmatrix} Z^{m+n}(s,t) \\ &\quad + Z^{m+n}(t,s)^T \begin{bmatrix} 0 \\ D_3^T \end{bmatrix} \begin{bmatrix} D_1 & D_2 \end{bmatrix} Z^{m+n}(t) \\ &\quad + \int_I Z^{m+n}(t,s)^T \begin{bmatrix} 0 \\ D_3^T \end{bmatrix} \begin{bmatrix} 0 & D_3^T \end{bmatrix} Z^{m+n}(s,t)\, ds\, dt \\ &= \begin{bmatrix} 0 & 0 \\ 0 & N(s,t) \end{bmatrix}\end{aligned}$$

Therefore $V(c,x) = \langle A\phi, A\phi\rangle \geq 0$ for all $x \in \mathcal{C}(I)$ and $c \in \mathbb{R}^m$. ∎

Theorem 6 can be used for both the functional and its derivative by, for example in the single-delay case, letting $c = x(0)$ or

$$c = \begin{bmatrix} x(0) \\ x(-h) \end{bmatrix}.$$

We use the following notation.

$$\begin{aligned}\Gamma^k_{n,d} = \Big\{ &(M,N) : \\ &M(s) = Z_d^{(k+2)n}(s)^T Q_1 Z_d^{(k+2)n}(s); \\ &N(s,t) = Z_d^n(s)^T Q_{23}\, Z_d^n(s,t) + Z_d^n(t,s)^T Q_{32} Z_d^n(t) \\ &\qquad + \int_I Z_d^n(\omega,s)^T Q_{33}\, Z_d^n(\omega,t)\, d\omega; \\ &Q \geq 0; \\ &Q_{13} = Q_{31}^T = 0 \Big\}\end{aligned}$$

Also

$$P_{n,d} = \left\{ Z_{n,d}^T(t) U Z_{n,d}(t) \mid U \in \mathbb{S}^{(d+1)nk} \right\}$$

The assumption that M and N define individually positive forms implies $(M,N) \in \Gamma$. However, the converse may not be true.

V. The Lie Derivatives

This section is concerned with presentation of the Lie derivative of the Lyapunov functional as a linear transformation of the matrix functions M and N. Because this material has already been presented in detail in [9], we will only consider the single-delay case. Suppose that $V \in X$ is given by (4), where $M : [-h,0] \to \mathbb{S}^{2n}$ and $N : [-h,0] \times [-h,0] \to \mathbb{R}^{n\times n}$ are continuous matrix functions. Then the Lie derivative of V is

$$\begin{aligned}\dot V(\phi) &= \int_{-h}^0 \begin{bmatrix} \phi(0) \\ \phi(-h) \\ \phi(s) \end{bmatrix}^T D(s) \begin{bmatrix} \phi(0) \\ \phi(-h) \\ \phi(s) \end{bmatrix} ds \\ &\quad + \int_{-h}^0 \int_{-h}^0 \phi(s)^T E(s,t) \phi(t)\, ds\, dt. \qquad (9)\end{aligned}$$

Partition D and M as

$$M(t) = \begin{bmatrix} M_{11}(t) & M_{12}(t) \\ M_{21}(t) & M_{22}(t) \end{bmatrix} \quad D(t) = \begin{bmatrix} D_{11}(t) & D_{12}(t) \\ D_{21}(t) & D_{22}(t) \end{bmatrix}$$

so that $M_{11} : [-h,0] \to \mathbb{S}^n$ and $D_{11} : [-h,0] \to \mathbb{S}^{2n}$. The functions D and E are linearly related to M and N by the following.

Definition 7: Define the map $L : \mathcal{C}_1 \times \mathcal{C}_1 \to \mathcal{C}_1 \times \mathcal{C}_1$ by $(D,E) = L(M,N)$ if for all $t, s \in [-h,0]$ we have

$$\begin{aligned}K &= \int_{-h}^0 M_{11}(t) dt \\ D_{11} &= \begin{bmatrix} A_0^T K + K A_0 & K A_1 \\ A_1^T K & 0 \end{bmatrix} \\ &\quad + \frac{1}{h} \begin{bmatrix} M_{12}(0) + M_{21}(0) & -M_{12}(-h) \\ -M_{21}(-h) & 0 \end{bmatrix} \\ &\quad + \frac{1}{h} \begin{bmatrix} M_{22}(0) & 0 \\ 0 & -M_{22}(-h) \end{bmatrix} \\ D_{12}(t) &= \begin{bmatrix} A_0^T M_{12}(t) - \dot M_{12}(t) + N(0,t) \\ A_1^T M_{12}(t) - N(-h,t) \end{bmatrix} \\ D_{22}(t) &= -\dot M_{22}(t)\end{aligned}$$

$$E(s,t) = -\frac{\partial N(s,t)}{\partial s} - \frac{\partial N(s,t)}{\partial t}.$$

Lemma 8: Suppose $M, N \in \mathcal{C}_1$ and V is given by (4). Let $(D,E) = L(M,N)$. Then the Lie derivative of V is given by (9).

Proof: The proof is straightforward by differentiation and integration by parts of (4). ∎

VI. Stability Conditions

In the following theorem we use the results of the paper to construct semidefinite programming problems.

Theorem 9: Suppose there exist $\epsilon > 0$, $d \in \mathbb{N}$ and piecewise matrix polynomials M, T, N, D, U, E such that

$$\left(M + \begin{bmatrix} T + \epsilon I & 0 \\ 0 & 0 \end{bmatrix}, N\right) \in \Gamma^0_{n,d},$$
$$\left(-D + \begin{bmatrix} U - \epsilon I & 0 \\ 0 & 0 \end{bmatrix}, E\right) \in \Gamma^k_{n,d},$$
$$T \in P_{n,d}, \qquad U \in P_{(k+1)n,d},$$
$$(D, E) = L(M, N),$$
$$\int_{-h}^{0} T(s)\, ds = 0, \qquad \int_{-h}^{0} U(s)\, ds = 0.$$

Then the system defined by equations (2) and (3) is exponentially stable.

Proof: Assume M, T, N, D, U, E satisfy the above conditions, and define the function V by (4). Then Lemma 8 implies that $\dot{V}$ is given by (9). The function V is nonnegative by Theorem 6. The same is true for nonpositivity of $\dot{V}$. ■

The feasibility conditions of Theorem 9 are semidefinite-representable. In particular the condition that a piecewise polynomial matrix lie in Γ is a set of linear and positive semidefiniteness constraints on its coefficients. Standard semidefinite programming codes may therefore be used to efficiently find such piecewise polynomial matrices. Most such codes will also return a dual certificate of infeasibility if no such polynomials exist.

As in the Lyapunov analysis of nonlinear systems using sum-of-squares polynomials, the set of candidate Lyapunov functions is parameterized by the degree d. This allows one to search first over polynomials of low degree, and increase the degree if that search fails.

There are various natural extensions of this result. The first is to the case of uncertain systems, where we would like to prove stability for all matrices A_i in some given semialgebraic set. This is possible by extending Theorem 9 to allow Lyapunov functions which depend polynomially on unknown parameters. A similar approach may be used to check stability for systems with uncertain delays. It is also straightforward to extend the class of Lyapunov functions, since it is sometimes not necessary that functions be nonnegative on the whole real line. To do this, one can use techniques for parameterizing polynomials nonnegative on an interval; for example, every polynomial $p(x) = f(x) - (x-1)(x-2)g(x)$ where f and g are sums of squares is nonnegative on the interval $[1, 2]$.

VII. Numerical Example

In this section, we illustrate the convergence and accuracy of our method with a simple numerical example. Consider the following differential equation with a single delay.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t - \tau)$$

For the purpose of evaluating accuracy, the goal is to determine the minimum and maximum stable values of the delay, τ. This is done by bisection. The results are illustrated in Table I.

Joint Positivity Approach		
d	$\tau_{\min}$	$\tau_{\max}$
1	.10017	1.6249
2	.10017	1.7175
3	.10017	1.71785
Analytic	.10017	1.71785

TABLE I
τ_{max} AND τ_{min} FOR DEGREE $2d$ USING THE JOINT POSITIVITY APPROACH

As one can see, there is not significant conservatism for this example. A MATLAB implementation of the algorithm can be found online [7].

VIII. Conclusion

In this paper, we have supplied a necessary and sufficient condition for positivity of quadratic functionals defined by polynomial kernels. We have also given a condition for joint positivity of a quadratic functional defined by both kernel and multiplier functions. The results included in this paper help to improve the understanding of the structure of the quadratic Lyapunov functional necessary for stability of time-delay systems. The end goal of this research is to be able to give a condition using LMIs which is equivalent to stability of linear time-delay systems. It is anticipated that eventually these results will be used to prove stability of other infinite-dimensional systems. The conditions presented here have natural extensions to the cases of nonlinearity and uncertainty. See [6] for work in these areas.

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