# USING THE POSITIVSTELLENSATZ FOR STABILITY ANALYSIS OF NEUTRAL DELAY SYSTEMS IN THE FREQUENCY DOMAIN

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Abstract: This paper describes a method for determining the stability of linear time-delay systems of the neutral class. The contributions presented here are twofold. First, we show how to use sum-of-squares programming and real algebraic geometry to determine pole locations in the complex plane. Second, we summarize a number of cases where conditions on the root locations of a system have been shown to imply certain stability properties. These topics are then combined to give polynomial-time algorithms for proving stability in a certain number of cases. Numerical examples are used for illustration.

Keywords: Sum-of-Squares, delay systems, neutral systems, stability

## 1. INTRODUCTION

In this paper we are interested in the location of poles of systems with transfer function of the type

$$G(s) = \frac{r(s)}{q_0(s) + \sum_{k=1}^{n} q_k(s) e^{-\tau_k s}},$$

where $r$ and $q_i$ are polynomials in complex variable $s$ with

$$q_k(s) = \sum_{j=0}^{m} q_k^j s^j$$

for $q_k^j \in \mathbb{R}$ and $\tau_k \geq 0$. We restrict ourselves to proper systems and therefore impose the condition $\deg q_0 \geq \deg r$. We do not consider delay systems of the advanced type and therefore $\deg q_0 \geq \deg q_k$ for $k = 1, \cdots n$. We are, however, considering the neutral case and therefore we will typically have $\deg q_0 = \deg q_k$ for at least one $k$.

Both $H_\infty$ and exponential stability have previously been associated with pole locations in various circumstances and under various conditions. It is worth noting that for neutral time-delay systems, these concepts of stability are ***not*** equivalent. For a recent result relating $H_\infty$ stability to pole locations, see (Partington and Bonnet, 2004). Other sources include (Pontyagin, 1955)

The aim of this paper is to establish easy to check conditions which guarantee stability in various senses. More precisely, we are interested in answering the following question :

1. **Delay-Independent Stability:** Is the system stable for all $\tau_k \geq 0$?
2. **Robust Delay-Dependent Stability:** For given $T_k$, is the system is stable for all $\tau_k \in [0, T_k]$?

In this paper, we give results which allow us to answer questions 1) and 2) using semidefinite programming. Our approach is to develop results which allow us to formulate the problem in terms of semialgebraic geometry. We then use Positivstellensatz results which allow us to express the problem as convex optimization over sum-of-squares polynomials. We then use semidefinite programming to solve the problem numerically.

Note that a number of results exist in the literature for determining stability of time-delay systems. Some of these results also extend to systems of the neutral type. The benefits of the approach proposed in this paper is the ability to give a verifiable proof of stability using a polynomial time algorithm in a way which approaches a necessary and sufficient condition.

The paper is organized as follows. We begin by a background on Sum-of-Squares and polynomial optimization in section 2. In section 3.1 we consider the commensurate delay case and answer question 1 in the sense of exponential stability. Section 3.2 restricts to the single delay case where it is possible to answer question 1 relative to $H_\infty$-stability. Finally, numerical examples are given in section 4.

## 2. SOME BACKGROUND ON POLYNOMIAL OPTIMIZATION

*Definition 1.* A polynomial is *sum-of-squares*, denoted $p \in \Sigma_s$ if it can be written as the finite sum of squares of polynomials. i.e.

$$p(x) = \sum_{i=1}^{n} g_i(s)^2$$

**Note:** Definition 1 is used to refer to both scalar and matrix-valued polynomials.

The sum-of-squares constraint is a sufficient condition for global non-negativity. It is also necessary in a number of important cases including scalars in 2 variables and matrices in one variable. The sum-of-squares condition can be checked using semidefinite programming, since $p \in \Sigma_s$ if an only if there exists a matrix $Q \geq 0$ such that

$$p(x) = Z(x)^T Q Z(z),$$

where $Z$ is a column of independent basis functions for the polynomials of degree $d \leq \frac{\deg p}{2}$. Typically, $Z$ is a vector of monomials, for example

$$z(y)^T = \begin{bmatrix} 1 & y_1 & y_1 y_2^2 & y_3^4 \end{bmatrix}.$$

In the matrix case, we would define $Z(y) : \mathbb{R}^n \to \mathbb{R}^{Nn}$ by

$$Z(y) = I \otimes z(y).$$

Given a matrix polynomial $G(y)$, we can test whether it is a sum-of-squares by finding a matrix $U$ such that

$$\begin{aligned} G(y) &= Z^T(y) U Z(y) \\ U &\succeq 0. \end{aligned} \tag{1}$$

Equation (1) is interpreted as equating the two polynomials $G$ and $K^T U K$. Equating their coefficients gives a family of linear constraints on the matrix $U$. Therefore to find such a $U$ we need to find a positive semidefinite matrix subject to linear constraints, and this is therefore a semidefinite program.

Most questions of interest are not naturally stated as sum-of-squares programming problems. Critical in the formulation of such problems are Positivstellensatz results. See (Stengle, 1973; Schmüdgen, 1991; Putinar, 1993) for examples. The version we use is given by the following theorem.

*Theorem 2.* (Stengle). The following are equivalent

(1)
$$\left\{ x : \begin{matrix} p_i(x) \geq 0 & i = 1, \dots, k \\ q_j(x) = 0 & j = 1, \dots, m \end{matrix} \right\} = \emptyset$$

(2) There exist $t_i \in \mathbb{R}[x]$, $s_i, r_{ij}, \ldots \in \Sigma_s$ such that
$$-1 = \sum_i q_i t_i + s_0 + \sum_i s_i p_i + \sum_{i \neq j} r_{ij} p_i p_j + \cdots$$

Here $\mathbb{R}[x]$ denotes the set of real-valued polynomials in variables $x$ and $\Sigma_s$ denotes the subset of $\mathbb{R}[s]$ which admit a sum-of-squares representation. For a given degree bound, the conditions associated with Stengle's positivstellensatz can be represented by a semidefinite program. Note that in general no upper bound on the degree bound exists. This is hardly surprising, however, since many NP-hard problems can be expressed as polynomial optimization using the Positivstellensatz.

## 3. THE FREQUENCY DOMAIN AND STABILITY

In this section we give several results which relate stability of neutral time-delay systems to feasibility of certain semialgebraic subsets of the complex plane.

The next proposition gives a sufficient condition ensuring exponential stability when the delays are not necessarily commensurate.

*Lemma 3.* Let $\epsilon > 0$.

If $\{s : q_0(s) + \sum_{k=1}^{n} q_k(s) z_k = 0,$

$$\operatorname{Re} s \geq -\epsilon, |z_k| \leq e^{\epsilon T_k}\} = \emptyset$$

Then $G$ has no poles in $\{\operatorname{Re} s \geq -\epsilon\}$ for all $\tau_k \leq T_k$ and $G$ is exponentially stable for any $\tau_k \leq T_k$.

**Proof** Proof by contradiction. Suppose that there exist $(\tau_1, \cdots, \tau_n) \in \mathbb{R}^n$ with $\tau_k \leq T_k$ for

$k = 1, \cdots, n$ and $s_0 \in \{\operatorname{Re} s \geq -\epsilon\}$ such that $q_0(s) + \sum_{k=1}^{n} q_k(s) e^{-\tau_k s} = 0$. This means that we can find $s_0 \in \{\operatorname{Re} s \geq -\epsilon\}$ and $z_i \in \mathbb{C}$, $|z_i| \leq e^{\epsilon T_i}$ such that $q_0(s) + \sum_{k=1}^{n} q_k(s) z_k = 0$ and this would contradict the hypothesis.

■

**Proposition** *3.1.* Let $\epsilon > 0$.

$$\text{If } K := \{s, z_k \in \mathbb{C} : q_0(s) + \sum_{k=1}^{n} q_k(s) z_k = 0, \\ \operatorname{Re} s \geq -\epsilon, |z_k| \leq 1 + \epsilon\} = \emptyset$$

Then $G$ is exponentially stable for any finite $\tau_k \geq 0$.

**Proof** For any finite set of $T_i \geq 0$, let $\gamma \leq \min\{\frac{\log(1+\epsilon)}{T_i}, \epsilon\}$ for $i = 1, \ldots, k$. Now let

$$F := \{s : q_0(s) + \sum_{k=1}^{n} q_k(s) z_k = 0, \\ \operatorname{Re} s \geq -\gamma, |z_k| \leq e^{\gamma T_k}\}$$

Assume by contradiction that $F \neq \emptyset$. Then there exists some $(s, z_k) \in F$. Then $\operatorname{Re} s \geq -\gamma \geq \epsilon$. Also, $|z_k| \leq e^{\gamma T_k} \leq e^{\frac{\log(1+\epsilon)}{T_k} T_k} = 1 + \epsilon$. Therefore $(s, z_k) \in K$, which contradicts the assumptions. Therefore, $F = \emptyset$ and by Lemma 3, the system is exponentially stable for all $\tau_k \leq T_k$. Since the $T_i$ are arbitrary, exponential stability holds for all finite $\tau_i \geq 0$.

■

We can prove in the same way the following result which gives a sufficient condition ensuring no poles of the transfer function in the closed right half-plane independant of the delay.

**Proposition** *3.2.* If $\{s : q_0(s) + \sum_{k=1}^{n} q_k(s) z_k = 0,$

$$\operatorname{Re} s \geq 0, |z_k| \leq 1\} = \emptyset$$

Then $G$ has no poles in $\{\operatorname{Re} s \geq 0\}$ for all $\tau_k$.

### *3.1 The case of commensurate delays*

In this section, we suppose that $(\tau_1, \cdots, \tau_n) = \tau(1, \cdots, n)$ where $\tau \in \mathbb{R}$, that is that the delays are commensurate. In this case, the poles become asymptotically determinable which simplifies the analysis. The following gives necessary and sufficient conditions to characterize exponential stability independent of delay. Similar ideas are discussed in a related contex in (Niculescu and Rasvan, 2006).

**Proposition** *3.3.* The following conditions are equivalent

1) The set $\{s \in \imath\mathbb{R}^*, q_0(s) + \sum_{k=1}^{n} q_k(s) e^{-sk\tau} = 0\}$ is empty for all $\tau$.

2) The set $\{s \in \imath\mathbb{R}^*, z \in \mathbb{C}, |z| = 1, q_0(s) + \sum_{k=1}^{n} q_k(s) z^k = 0\}$ is empty.

**Proof** The fact that 2) $\Longrightarrow$ 1) is obvious.

We prove that 1) $\Longrightarrow$ 2) by contrapositive.

Suppose that there exists $\omega_0 \in \mathbb{R}^*$ and $z \in \mathbb{C}$, $|z| = 1$ such that $\sum_{k=1}^{n} q_k(s) z^k = 0$. We can write $z = e^{-\imath a}$ with $a \in \mathbb{R}$. Let $\tau_0 = \frac{a}{\omega}$, we get $q_0(\imath\omega_0) + \sum_{k=1}^{n} q_k(\imath\omega_0) e^{-\imath\omega_0 k \tau_0} = 0$ which means that there exists $s_0 = \imath\omega_0 \in \imath\mathbb{R}^*$, $\tau_0 \in \mathbb{R}$ such that $q_0(s_0) + \sum_{k=1}^{n} q_k(s_0) e^{-s_0 k \tau_0} = 0$ which contracdicts 1).

■

Now recall that

$$q_k(s) = \sum_{j=0}^{m} q_k^j s^j.$$

**Proposition** *3.4.* Suppose that

$$q_0^m + \sum_{k=1}^{n} q_k^m e^{-k\tau s}$$

has no zeroes in $\{\operatorname{Re} s \geq 0\}$ and that $G$ is stable when $\tau = 0$. Then, the following conditions are equivalent:

1) $G$ is exponentially stable for all $\tau$.

2) The set $\{s \in \imath\mathbb{R}^*, z \in \mathbb{C}, |z| = 1, q_0(s) + \sum_{k=1}^{n} q_k(s) z^k = 0\}$ is empty.

**Proof** 1) $\Longrightarrow$ 2) is obvious using Proposition 3.3

2) $\Longrightarrow$ 1). As $G$ is stable when $\tau = 0$ we only need to check if there are any destabilizing roots when $\tau$ increases. As

$$q_0^m + \sum_{k=1}^{n} q_k^m e^{-k\tau s}$$

has no zeros in

$$\{\operatorname{Re} s \geq 0\},$$

we know from (Pontyagin, 1955) that for each $\tau$, there exists $\epsilon > 0$ such that $G$ only has a finite number of poles in $\{\mathrm{Re}\, s \geq -\epsilon\}$. Indeed the infinite number of roots which appear as $\tau$ is sufficiently small are located left to an axis which is in the left half-plane and we only need to care about the behaviour of the poles of small modulus as $\tau$ increases. When 2) holds we get by Proposition 3.3 and the delay free stability hypothesis that $G$ has no poles on the imaginary axis for all $\tau$ so that there cannot be any crossing as $\tau$ increases and $G$ is $H_\infty$-stable and even exponentially stable for all $\tau$. ■

### *3.2 The single delay case*

In this section $G$ is taken to be

$$G(s) = \frac{r(s)}{q_0(s) + q_1(s)e^{-\tau s}},$$

The $H_\infty$-stability of such systems was fully analysed in (Partington and Bonnet, 2004). It was proved that if $|\alpha| = |\lim_{|s|\to\infty} q_0(s)/q_1(s)| > 1$, then the poles of large modulus are asymptotic to a vertical line strictly in the left half-plane. If $|\alpha| = 1$, the poles are asymptotic to the imaginary axis and this more delicate case is recalled below.

**Theorem** *3.1.* (Partington and Bonnet, 2004) Let $G(s) = \dfrac{r(s)}{q_0(s) + q_1(s)e^{-s\tau}}$ and suppose that

$$\frac{q_0(s)}{q_1(s)} = \alpha + \frac{\beta}{s} + \frac{\gamma}{s^2} + O\left(\frac{1}{s^3}\right) \qquad \text{as} \quad |s| \to \infty,$$

for constants $\alpha$, $\beta$ and $\gamma$ with $\alpha = \pm 1$. For sufficiently large integers $n$ let $\lambda_n = 2ni\pi$ if $\alpha = -1$ and let $\lambda_n = (2n+1)i\pi$ if $\alpha = 1$. Then the poles $s_n$ of $G$ satisfy

$$s_n = \frac{\lambda_n}{h} - \frac{\beta}{\alpha\lambda_n} + \frac{h}{\lambda_n^2}\left(\frac{\beta^2}{2} - \frac{\gamma}{\alpha}\right) + o\left(\frac{1}{n^2}\right).$$

The system has infinitely many unstable poles if $\gamma/\alpha > \beta^2/2$, and infinitely many stable poles if $\gamma/\alpha < \beta^2/2$. In the latter case there can be at most finitely many unstable poles, and, if there are none, then the transfer function $G$ lies in $H_\infty$ if and only if $\deg p \geq \deg r + 2$. If $\gamma/\alpha = \beta^2/2$, then the condition $\deg p \geq \deg r + 2$ is still necessary for stability.

It is now possible to characterize the $H_\infty$-stability independant of delays.

**Proposition** *3.5.* Suppose that $G$ is stable at $\tau = 0$ and that $|\alpha| > 1$, then the following are equivalent:

1) The set $\{s \in \imath\mathbb{R}^*, z \in \mathbb{C}, |z| = 1, q_0(s) + \sum_{k=1}^{n} q_k(s)z^k = 0\}$ is empty.

2) $G$ is $H_\infty$-stable for all $\tau \geq 0$.

From Theorem 3.1 together with Proposition 3.3, we can easily get the following proposition.

**Proposition** *3.6.* Suppose that $G$ is stable at $\tau = 0$ and that $\alpha$, $\beta$, $\gamma$ defined as in Theorem 3.1 satisfy the condition $\gamma/\alpha < \beta^2/2$.

Then the following are equivalent:

1) The set $\{s \in \imath\mathbb{R}^*, z \in \mathbb{C}, |z| = 1, q_0(s) + \sum_{k=1}^{n} q_k(s)z^k = 0\}$ is empty and $\deg p \leq \deg r + 2$.

2) $G$ is $H_\infty$-stable for all $\tau \geq 0$.

## 4. NUMERICAL EXAMPLES

We now illustrate the proposed methodology with several numerical examples.

**Example 1:** Consider a transfer function with denominator

$$g(s) = 2s + 1 + se^{-\tau s} + se^{-2\tau s} \tag{2}$$

This is a neutral system with multiple commensurate delays. When $\tau = 0$, the system is stable. By Proposition 3.4, delay independent stability of 2 is equivalent to infeasibility of the following semialgebraic set.

$$\left\{s \in \imath\mathbb{R}^*, z \in \mathbb{C} \,:\, |z| = 1, 2s + 1 + sz + sz^2 = 0\right\}$$

We convert this set to one expressed in real numbers. This can be done automatically in Matlab.

$$\begin{aligned} &\mathrm{Im}(2s + 1 + sz + sz^2) \\ &= \mathrm{Im}\left(2\imath\omega + 1 + \imath\omega(z_R + \imath z_I) + \imath\omega(z_R + \imath z_I)^2\right) \\ &= 2\omega + 1 + \omega z_R + \omega z_R^2 - \omega z_I^2 \end{aligned}$$

Similarly

$$\mathrm{Re}(2s + 1 + sz + sz^2) = -\omega z_I - 2\omega z_R z_I$$

Combining, we now consider infeasibility of

$$\Big\{(\omega, z_R, z_I) \in \mathbb{R}^3 \,:\, z_R^2 + z_I^2 - 1 = 0,\\ -\omega z_I - 2\omega z_R z_I = 0,\\ 2\omega + 1 + \omega z_R + \omega z_R^2 - \omega z_I^2 = 0\Big\}$$

By Theorem 2, infeasibility of this semialgebraic set is equivalent to existence of $t_1, t_2, t_3 \in \mathbb{R}[x, z_R, z_I]$ and $s \in \Sigma_s$ such that

$$\begin{aligned} -1 = &(z_R^2 + z_I^2 - 1)\, t_1 - (\omega z_I + 2\,\omega z_R z_I)\, t_2 \\ &+ (2\,\omega + 1 + \omega z_R + \omega z_R^2 - \omega z_I^2)\, t_3 + s. \end{aligned} \tag{3}$$

By constraining $t_i$ to be of degree 2 or less(the degree of $s$ must then be 4) and using SOSTools (Parrilo, 2004), we find condition 3 to be feasible with the following. We have scaled down the solution by a factor of 10.

$$
\begin{aligned}
t_1 =& -1.2\omega - .37z_I - .22z_R z_I \\
t_2 =& -1.06 + .16z_R - .11z_R^2 + .11z_I^2 \\
t_3 =& .37 + .08z_R + 1.9\omega^2 + .54z_R^2 - .07\omega z_I + .56z_I^2 \\
s =& -1.17\omega z_R z_I - .24z_R - .06z_R^2 - .26\omega z_I - .3z_I^2 \\
& + .46\omega^2 + .08z_R^3 + .54z_R^4 + .56z_I^4 + 1.19z_R\omega^2 \\
& + .08z_I^2 z_R + 3.11z_R^2\omega^2 + .73z_I^2\omega^2 - .33z_I^3\omega \\
& + 1.1z_I^2 z_R^2 + .71z_I z_R^2\omega + .6
\end{aligned}
$$

We therefore conclude that the system is exponentially stable independent of delay. Naturally, in practice the process is automated and takes relatively little computational time (˜.5 seconds) in the case of commensurate delays.

**Example 2:**

In order to gauge performance of the proposed method, we consider a number of parameterized transfer functions and determine the parameter range for which the system is exponentially stable independant of delay.

Table 1. Systems and stability regions.

| G(s,a) | $a_{\min}$ | $a_{\max}$ |
|---|---|---|
| $2s + 1 + ase^{-\tau s}$ | -2.00 | 2.00 |
| $s^2 + s + 1 + as^2 e^{-\tau s}$ | -.866 | .866 |
| $s^2 - .5as^2 z + 2s - .5sz - .02a^2 s^2 z^2$ $-.5asz + .04asz^2 + 1 - .5z - .02z^2$ | -1.86 | 1.86 |

Note that in the final result we let $z = e^{-\tau s}$. This example was taken from a 2-dimensional state-space representation. In addition to these results, we were also able to replicate the results published in various sources such as (Marshall *et al.*, 1992).

## 5. CONCLUSION

This paper describes ways to use recent advances in polynomial optimization to approach problems in analyzing delay-independent stability of delayed systems of the neutral type. The advantages of the approach described in this paper is that the algorithm runs in polynomial time and gives a verifiable certificate of stability in the form of a polynomial refutation. We believe these methods can also be applied to delay-dependent stability of delayed systems of the retarded type by adopting a "robust dichotomy" approach using the results of (Zhang *et al.*, 2002). This is a topic of ongoing work.

## REFERENCES


Marshall, J.E., H. Górecki, K. Walton and A. Korytowski (1992). *Time-Delay Systems: Stability and performance criteria with applications.* Simon and Schuster.

Niculescu, S.I. and V. Rasvan (2006). Delay-independant stability in lossless propagation models with applications (I): A complex domain approach. In: *Proc. 14th International Symposium on Mathematical Theory of Networks and systems, Perpignan, France.*

Parrilo, P. A. (2004). Web site for SOSTOOLS. `http://www.cds.caltech.edu/sostools/`.

Partington, J. R. and C. Bonnet (2004). $H_\infty$ and BIBO stabilization of delay systems of neutral type. *Systems Control Lett.* **52**, 283–288.

Pontyagin, L.S. (1955). On the zeros of some elementary transcendental functions. *Amer. Math. Soc. Transl.* **2**, 95–110.

Putinar, M. (1993). Positive polynomials on compact semi-algebraic sets. *Indiana Univ. Math. J.* **42**(3), 969–984.

Schmüdgen, C. (1991). The k-moment problem for compact semi-algebraic sets. *Mathematische Annalen* **289**(2), 203–206.

Stengle, G. (1973). A nullstellensatz and a positivstellensatz in semialgebraic geometry. *Mathematische Annalen* **207**, 87–97.

Zhang, J., C. Knospe and P. Tsiotras (2002). Stability of linear time-delay systems: A delay-dependant criterion with a tight conservatism bound.