CODE–2007 : Conférence de la SMAI sur l'Optimisation et la Décision
Institut Henri Poincaré, Paris
18–20 avril 2007

ON POSITIVE QUADRATIC FORMS AND STABILITY OF LINEAR SYSTEMS

Matthew M. Peet[1]

[1]INRIA - Rocquencourt

The stability of many types of linear systems can be expressed using the convex cone of positive quadratic forms. In this talk, we consider linear dynamical systems with discrete delays. In this case, it is known that if the system is stable, then this property may be proven using a Lyapunov functional of the form

$$V(\phi) = \int_{-h}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds + \int_{-h}^{0} \int_{-h}^{0} \phi(s)^T N(s,t) \phi(t) \, ds \, dt.$$

In this talk, we show how to use optimization-based methods to construct the functions M and N such that the functional V is positive. Roughly speaking, in the first part we show that

$$V_1(\phi) = \int_{-h}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds$$

is positive for all ϕ if and only if there exists a piecewise continuous matrix-valued function T such that

$$M(t) + \begin{bmatrix} T(t) & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \quad \text{for all } t, \qquad \int_{-h}^{0} T(t) \, dt = 0.$$

That is, we convert positivity of the integral to *pointwise* positivity. By assuming that M is polynomial, pointwise positivity may then be easily enforced as it is equivalent to a sum-of-squares constraint. The connection between sum-of-squares programming and semidefinite programming was first made by Parillo[1]. The constraint that T integrate to 0 is linear and also easily enforced. Next, we show that if N is piecewise-polynomial and is defined by polynomials N_{ij} on the subintervals $I_i, I_j \subset [-h, 0]$, then

$$V_2(\phi) = \int_{-h}^{0} \int_{-h}^{0} \phi(s)^T N(s,t) \phi(t) \, ds \, dt$$

is positive if and only is there exists a $Q \geq 0$ such that

$$N_{ij}(s,t) = Z(s)^T Q_{ij} Z(t).$$

Here Z is a monomial basis. This result allows us to express positivity of the second part of the functional as a semidefinite programming constraint. We follow with conditions on joint positivity of the functional which combines both of these results. In the talk, numerical examples are given which show increasing accuracy as the degree of the monomials increases. These results compare favorably with existing methods of analysis.

Next we consider a number of results from semialgebraic geometry. One example of such a result is the Positivstellensatz result of Stengle [2]. This is given as follows.

Theorem 1 (Stengle) *The following are equivalent*

1. $$\left\{ x : \begin{array}{ll} p_i(x) \geq 0 & i = 1, \ldots, k \\ q_j(x) = 0 & j = 1, \ldots, m \end{array} \right\} = \emptyset$$
2. *There exist* $t_i \in \mathbb{R}[x]$, $s_i, r_{ij}, \ldots \in \Sigma_s$ *such that*

$$-1 = \sum_i q_i t_i + s_0 + \sum_i s_i p_i + \sum_{i \neq j} r_{ij} p_i p_j + \cdots$$

Here Σ_s is the set of sums of squares of polynomials. The Positivstellensatz allows us to construct parameter-dependent Lyapunov functionals in parameter α by constructing functions $M(s,\alpha)$ and $N(s,t,\alpha)$ such that

$$M(s,\alpha) + T(\alpha) \geq 0 \qquad \text{for all } \alpha \in \Delta$$

and

$$N(s,t,\alpha) = Z(s)^T Q(\alpha) Z(t), \qquad \text{where } Q(\alpha) \geq 0 \qquad \text{for all } \alpha \in \Delta.$$

In addition, we show that one can prove stability of nonlinear systems by considering non-quadratic Lyapunov functionals of the form

$$\int_{-h}^{0} Z(\phi(0),\phi(s))^T M(s) Z(\phi(0),\phi(s))\, ds + \int_{-h}^{0}\int_{-h}^{0} Z(\phi(s))^T N(s,t) Z(\phi(t))\, ds\, dt,$$

where here again $Z(x)$ is the vector of monomials in variables x. The same conditions as before now also hold in the case of non-quadratic functionals, providing a sufficient conditions for stability.

We conclude with examples from networking and epidemiology.

Bibliographie

[1] P. A. Parrilo, Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization, *Ph.D. dissertation*, California Institute of Technology, (2000).

[2] G. Stengle, A nullstellensatz and a positivstellensatzin semialgebraic geometry, *Mathematische Annalen*, **207**, (1974), 87–97.