AN EXTENSION OF THE WEIERSTRASS APPROXIMATION THEOREM TO LINEAR VARIETIES: APPLICATION TO DELAY SYSTEMS

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Abstract: In this paper, we show that linear varieties of polynomials can be used to approximate linear varieties of the space of continuous functions. This property is important in applications where polynomial optimization is used as it allows one to impose affine constraints on the decision variables with no loss of accuracy. In particular, construction of Lyapunov functionals for systems with delay is discussed.

Keywords: Polynomial Optimization, Semidefinite Programming, Polynomial Approximation, Time-Delay, Stability

1. INTRODUCTION

In 1885, Weierstrass first published a result (Weierstrass, 1885) showing that real-valued polynomials can be used to approximate any continuous function on a compact interval to arbitrary accuracy with respect to the supremum norm. Various generalizations of the Weierstrass approximation theorem have focused on generalized mappings (Stone, 1948) and alternate topologies (Krein, 1945). More recently, the Weierstrass theorem has found applications in numerical computation due to the ease with which polynomial functions are parameterized and evaluated. In particular, the Weierstrass theorem has often been used to justify polynomial curve fitting and its generalizations in the field of machine learning (Recht, 2006).

In this paper, we reexamine the Weierstrass theorem from the relatively new perspective of polynomial optimization. These problems consider optimization over $\mathcal{C}[0,1]$, the Banach space of continuous functions on $[0,1]$. The structure of the problem is often a special case of

$$\begin{aligned} &\max Af \\ &b + Bf \geq 0 \\ &c + Cf = 0, \end{aligned}$$

where $A, B, C : \mathcal{C}[0,1] \rightarrow \mathbb{R}$ are bounded linear operators, and c and b are constants. We would like to determine if the existence of a continuous optimal solution to the problem implies the existence of a polynomial optimal solution. In particular, we consider a problem which arises in Lyapunov analysis of linear time-delay systems (Peet *et al.*, 2006). In this case, we are obliged to optimize over the set of continuous matrix-valued functions T such that

$$\int_0^1 T(s)ds = 0.$$

We would like to determine whether we can assume T is polynomial. If T is polynomial, then we can apply recent advances in sum-of-squares optimization techniques (Parrilo, 2000) and results in semialgebraic geometry (Stengle, 1973; Schmüdgen, 1991; Putinar, 1993) which make it

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possible to numerically solve polynomial optimization problems in an asymptotic manner using semidefinite programming. The main result of this paper is to show that a continuous solution to the proposed type of optimization problem implies the existence of an equivalent polynomial solution.

2. NOTATION AND BACKGROUND

Most notation is standard. $\mathbb{R}$ is the real numbers. $\mathbb{R}^{n\times m}$ is the real matrices of dimension n by m. $\mathcal{C}[I]$ is the Banach space of functions $f : I \to \mathbb{R}$ with norm

$$\|f\|_\infty = \sup_{s\in I} |f(s)|.$$

When F is a continuous matrix-valued function on I, then $\|F\|_\infty$ denotes

$$\|F\|_\infty = \sup_{s\in I} \bar{\sigma}(F(s)),$$

where $\bar{\sigma}(F)$ denotes the maximum singular value norm.

The following is a statement of the Weierstrass theorem.

Theorem 1. (Weierstrass Approximation Theorem). Let $f \in \mathcal{C}[0,1]$. Then there is a sequence of polynomials p_n such that $p_n(x)$ converges uniformly to $f(x)$ on $[0,1]$.

3. LINEAR VARIETIES

The following proposition is an extension of the Weierstrass theorem to linear varieties of the Banach space $\mathcal{C}[0,1]$.

Proposition 2. Let $L_i : \mathcal{C}[0,1] \to \mathbb{R}$ be bounded linear operators. Then for any $f \in \mathcal{C}[0,1]$ and $\delta > 0$, there exists polynomial r such that $\|f - r\|_\infty \le \delta$ and $L_i r = L_i f$ for $i = 1,\dots,k$.

PROOF. Proceed by induction. Suppose that the proposition is true for $k = m-1$. If $L_m = 0$, then let p be defined as for $m-1$. If $L_m d = 0$ for all $d \in \mathcal{C}[0,1]$, then let r be as given by the proposition for $m-1$. In this case $L_m r = L_m f = 0$. Otherwise, there exists some $g \in \mathcal{C}[0,1]$ such that $L_m g = c > 0$. Let β be a uniform bound for operators L_i in $i = 1,\dots,k$. Assume without loss of generality that $\|g\|_\infty = 1$. Let $\tilde{f} = f + \frac{\delta}{4} g$ and $\gamma = \min\{\frac{\delta}{8}, \frac{\delta c}{8\beta}\}$.

By assuming that the proposition is true for $k = m-1$, we assume there exists some polynomial p such that

$$L_i f = L_i p \text{ for } i = 1,\dots,m-1$$

and

$$\|\tilde{f} - p\|_\infty \le \gamma \le \delta/8.$$

Therefore

$$\begin{aligned}\|f-p\|_\infty &\le \|f + \frac{\delta}{4} g - p\|_\infty + \frac{\delta}{4}\|g\|_\infty \\ &\le \frac{\delta}{8} + \frac{\delta}{4} < \delta/2.\end{aligned}$$

Furthermore,

$$\begin{aligned}L_m p &= L_m f + \frac{\delta}{4} L_m g - L_m\left(f + \frac{\delta}{4} g - p\right) \\ &= L_m f + \frac{\delta}{4} c - L_m\left(f + \frac{\delta}{4} g - p\right) \\ &\ge L_m f + \frac{\delta}{4} c - \beta\gamma \ge L_m f + \frac{\delta}{8} c > L_m f.\end{aligned}$$

There also exists polynomial b with $L_m b < L_m f$ and $L_i f = L_i b$ for $i = 1,\dots,m-1$ by letting $\tilde{f} = f - \frac{\delta}{4} g$.

Now since $L_m p > L_m f$ and $L_m b < L_m f$, there exists some $\lambda \in [0,1]$ such that $\lambda L_m p + (1-\lambda) L_m b = Lf$. Now let $r = \lambda p + (1-\lambda) b$. Then r is polynomial,

$$L_m r = \lambda L_m p + (1-\lambda) L_m b = L_m f,$$

and

$$\begin{aligned}L_i r &= \lambda L_i p + (1-\lambda) L_i b \\ &= \lambda L_i f + (1-\lambda) L_i f \\ &= L_i f \qquad \text{for } i = 1,\dots,m-1,\end{aligned}$$

and

$$\begin{aligned}\|f - r\|_\infty &= \|\lambda(f-p) + (1-\lambda)(f-b)\|_\infty \\ &\le \lambda\|f-p\|_\infty + (1-\lambda)\|f-b\|_\infty \\ &\le \delta/2 + \delta/2 = \delta.\end{aligned}$$

Therefore, if the proposition is true for $k = m-1$, it is also true for $k = m$. Assume without loss of generality that $L_1 = 0$. Then the proposition is true for $k = 1$ by the Weierstrass Approximation Theorem.

4. MATRIX-VALUED VARIABLES AND CONSTRAINTS

The Proposition also works with vector and matrix-valued functions and constraints.

Corollary 3. Let $T_{i,j} : \mathcal{C}[0,1] \to \mathbb{R}$ be bounded linear operators. Then for any $f_j \in \mathcal{C}[0,1]$ and $\delta > 0$, there exist polynomials r_j such that $\|f_j - r_j\|_\infty \le \delta$ for $j = 1,\dots,n$ and $\sum_{j=1}^n T_{i,j} r_j = \sum_{j=1}^n T_{i,j} f_j$ for $i = 1,\dots,q$.

PROOF. Proceed by induction. First assume the Corollary holds for $n = t-1$. By this assumption, there exist polynomials r_j such that

$\|f_j - r_j\|_\infty \le \frac{\delta}{2}$ for $j = 1,\dots,t-1$ and $\sum_{j=1}^{t-1} T_{i,j} r_j = \sum_{j=1}^{t-1} T_{i,j} f_j$ for $i = 1,\dots,q$.

Let β be a uniform bound on the $T_{i,j}$. Let $L_i = T_{i,j}$ for $i = 1,\dots,q$. Now, by Proposition 2, there exists a polynomial r_t such that $\|f_t - r_t\|_\infty \le \frac{\delta}{2}$ and $T_{i,t} r_t = T_{i,t} f_t$ for $i = 1,\dots,q$. Therefore $\sum_{j=1}^{t} T_{i,j} r_j = \sum_{j=1}^{t} T_{i,j} f_j$ for $i = 1,\dots,q$ and the Corollary holds for $n = t$. The Corollary holds for $n = 1$ by Proposition 2.

Note: This Corollary and the equivalence of norms implies that the variables and constraints in Proposition 2 can be matrix-valued.

5. OPTIMIZATION

Consider the following problem with bounded linear operators $L_i, K_i : \mathcal{C}[0,1] \to \mathbb{R}$.

$$\begin{aligned} &\max L_0 f \\ &c_i + L_i f = 0, \qquad \text{for } i = 1,\dots,m \\ &d_i + K_i f \ge 0, \qquad \text{for } i = 1,\dots,n \end{aligned} \tag{1}$$

Proposition 4. Consider optimization problem (1). Suppose $f \in \mathcal{C}[0,1]$ is feasible with objective value h. Then there exists a feasible polynomial solution with objective value h.

PROOF. Suppose f is feasible with objective value h. By Corollary 3, there exists a polynomial p such that

$$\begin{aligned} &-h + L_0 p = 0 \\ &c_i + L_i p = c_i + L_i f = 0, \qquad \text{for } i = 1,\dots,m \\ &d_i + K_i p = d_i + K_i f \ge 0, \qquad \text{for } i = 1,\dots,n \end{aligned}$$

Therefore, p is feasible with objective value h

6. AN EXAMPLE FROM TIME-DELAY

The motivating problem for this paper arises from analysis of linear systems with discrete delays. Specifically, we are interested in systems of the form

$$\dot{x}(t) = \sum_{i=0}^{k} A_i x(t - h_i),$$

where $x(t) \in \mathbb{R}^n$. In the simplest case we are given information about the the delays $h_0,\dots,h_k$, the matrices $A_0,\dots,A_k$, and a matrix of polynomials, B, and we would like to determine whether the system is stable.

For such systems it is known that if the system is stable, then this property can be proven using a Lyapunov functional of the form

$$\begin{aligned} V(\phi) = &\int_{-h}^{0} \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix}^T M(s) \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix} ds \\ &+ \int_{-h}^{0}\int_{-h}^{0} \phi(s)^T N(s,t)\phi(t)\, ds\, dt. \end{aligned} \tag{2}$$

For simplicity, we consider a single delay and therefore M and N are continuous matrix-valued functions. Here ϕ is an element of the state space, which is the space of continuous functions mapping $[-h,0]$ to $\mathbb{R}^n$. The derivative of V has a similar structure to V and is also defined by matrix functions which are linear transformations of M and N.

We wish to prove stability by constructing functionals of the form of Equation (2) using polynomial optimization. In (Peet *et al.*, 2006), a necessary and sufficient condition was given for positivity of the first part of the functional. A version of this is as follows.

Theorem 5. Suppose $M : [-h,0] \to \mathbb{S}^{2n}$ is continuous. Then the following are equivalent.

(i) There exists an $\epsilon > 0$ so that for all continuous $y : [-h,0] \to \mathbb{R}^n$

$$\int_{-h}^{0} \begin{bmatrix}y(0)\\ y(t)\end{bmatrix}^T M(t) \begin{bmatrix}y(0)\\ y(t)\end{bmatrix} dt \ge \epsilon \|y\|_{L_2} \tag{3}$$

(ii) There exist an $\eta > 0$ and a continuous function $T : [-h,0] \to \mathbb{S}^m$ satisfies

$$M(t) + \begin{bmatrix} T(t) & 0 \\ 0 & -\eta I \end{bmatrix} \ge 0 \quad \text{for all } t \in [-h,0]$$

$$\int_{-h}^{0} T(t)\, dt = 0$$

This theorem converts positivity of an integral to *pointwise* positivity of a function with a linear constraint. If we assume M and T are polynomial, pointwise positivity is equivalent to a sum-of-squares constraint. The constraint that T integrates to zero is a bounded linear constraint. The condition that the derivative of the Lyapunov function be negative has a similar structure. For details on formulating the semidefinite program, we refer to (Peet *et al.*, 2006).

The important question in this case is whether one can assume that M and T are polynomials. The following Proposition is ***not*** a complete proof that one can assume that M and T are polynomial. This is because not all of the constraints implied by one function being the derivative of another can be represented using a bounded operator L. However, the proposition is otherwise complete and offers progress in this direction.

Proposition 6. Suppose that $L : \mathcal{C}^{m\times m}[0,1] \times \mathcal{C}^{n\times n}[0,1] \to \mathbb{R}^{p\times p}$ is a bounded linear operator. Suppose for $\epsilon > 0$, there exist continuous matrix-valued functions M and T such that for some $\epsilon > 0$

$$\begin{aligned} &M(s) + T(s) \geq \epsilon I, \\ &-D(s) + U(s) \geq \epsilon I, \\ &L(M, D) = 0, \\ &\int_0^1 T(s)ds = 0, \qquad \int_0^1 U(s)ds = 0. \end{aligned}$$

Then there exist matrix-valued polynomials N, Q, R, E such that for some $\eta > 0$

$$\begin{aligned} &N(s) + Q(s) \geq \eta I, \\ &-E(s) + R(s) \geq \eta I, \\ &L(N, E) = 0, \\ &\int_0^1 Q(s)ds = 0, \qquad \int_0^1 R(s)ds = 0. \end{aligned}$$

PROOF. By Proposition 2, there exist polynomial matrices N, Q, R, E such that

$$\begin{aligned} &\int_0^1 Q(s)ds = \int_0^1 T(s)ds = 0, \\ &\int_0^1 R(s)ds = \int_0^1 U(s)ds = 0, \\ &L(N, E) = L(M, D) = 0, \\ &\|M - N\|\infty \leq \epsilon/3, \quad \|D - E\|\infty \leq \epsilon/3, \\ &\|Q - T\|\infty \leq \epsilon/3, \quad \|R - U\|\infty \leq \epsilon/3. \end{aligned}$$

The we have

$$\begin{aligned} N + Q &= M + T + (N - M) + (Q - T) \geq \epsilon/3I \\ -E + R &= -D + U + (D - E) + (R - U) \geq \epsilon/3I, \end{aligned}$$

as desired.

7. CONCLUSION

Numerous extensions of the Weierstrass approximation theorem have been proposed in the literature. Typically, these results either alter the algebra used to approximate the continuous functions or consider continuous functions on spaces other than the reals. The contribution of this paper is to instead consider approximations on subsets of the continuous functions, and in particular those defined by affine constraints. The application we consider is stability of linear systems with delay. Additionally, Proposition 2 can be used for general problems in polynomial optimization and in particular sum-of-squares programming problems (Parrilo, 2004). Ongoing work involves investigation of unbounded operators such as differentiation and functions of multiple variables.

REFERENCES

Krein, M. G. (1945). On a problem of extrapolation of A.N. Kolmogorov. *Dokl. Akad. Nauk. SSSR* **46**, 306–309.

Parrilo, P. A. (2000). Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization. PhD thesis. California Institute of Technology.

Parrilo, P. A. (2004). Web site for SOSTOOLS. `http://www.cds.caltech.edu/sostools/`.

Peet, M., A. Papachristodoulou and S. Lall (2006). On positive forms and the stability of linear time-delay systems. In: *Proceedings of the IEEE Conference on Decision and Control*.

Putinar, M. (1993). Positive polynomials on compact semi-algebraic sets. *Indiana Univ. Math. J.* **42**(3), 969–984.

Recht, B. (2006). Convex Modeling with Priors. PhD thesis. Massachusetts Institute of Technology.

Schmüdgen, C. (1991). The K-moment problem for compact semi-algebraic sets. *Mathematische Annalen* **289**(2), 203–206.

Stengle, G. (1973). A nullstellensatz and a positivstellensatz in semialgebraic geometry. *Mathematische Annalen* **207**, 87–97.

Stone, M. H. (1948). The generalized Weierstrass approximation theorem. *Mathematics Magazine* pp. 167–183 and 237–254.

Weierstrass, K. (1885). Über die analytische darstellbarkeit sogenannter willkürlicher functionen einer reellen veränderlichen. *Sitzungsberichte der Akademie zu Berlin* pp. 633–639 and 789–805.