example, an ∞-norm based Lyapunov function for a random three-dimensional stable system was found in 58 s using the internal global solver for bilinear problems of the optimization interface YALMIP [19] together with the BMI solver PENBMI [20] by PENOP as the local node solver. The here proposed approach found a Lyapunov function in $1.3 \cdot 10^{-3}$ sec for the very same system.

X. Software Implementation

The presented algorithm is implemented in the Multi-Parametric Toolbox (MPT) [21] for MATLAB. The toolbox can be downloaded free of charge at http://control.ee.ethz.ch/~mpt/.

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Global Stability Analysis of a Nonlinear Model of Internet Congestion Control With Delay

Matthew Peet and Sanjay Lall

***Abstract*—We address global stability of a model for the TCP/AQM congestion control protocol. This model represents the dynamics of a single link and single source, and consists of a nonlinear differential equation with a time-delay. We make use of absolute stability theory and integral-quadratic constraints to give conditions under which the dynamics are globally asymptotically stable.**

***Index Terms*—Delay, integral quadratic constraints, Internet congestion control, nonlinear, stability.**

I. Introduction

The TCP/AQM congestion control protocol used in the Internet controls the rate at which packets are sent from sources across the network. This rate evolves according to dynamics determined by the round-trip-time of the network and the control algorithms at the source and the router. We address the question of global stability of these dynamics.

There are several different protocols, some proposed and some in use, and we look at a specific protocol developed by [14], in which the following model is given:

$$\dot{p}(t) = \begin{cases} \frac{1}{c}\left(x(t) - c\right), & \text{if } x(t) > c \text{ or } p(t) > 0 \\ 0, & \text{otherwise} \end{cases}$$
$$x(t) = e^{-\alpha p(t-\tau)/\tau}. \quad (1)$$

This is a model for a network with a single source and single bottleneck link. Here, the parameter $\tau \geq 0$ is the round-trip-time, and $0 < c < 1$ is the capacity of the link. The scalar $p(t)$ is called the *price*. It is computed at the router, and is fed back to the source. The source then sets $x(t)$, its *rate*. For a system with multiple sources, the model is the same in the special case that all sources have identical roundtrip time.

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An equilibrium solution of (1) is $p(t) = p_{\mathrm{eqm}}$ for all t, where

$$p_{\mathrm{eqm}} = -\frac{\tau}{\alpha}\log c.$$

This corresponds to a rate of $x(t) = c$. The scalar $\alpha > 0$ is a parameter of the algorithm which may be chosen by the designer, called the *gain*. The tradeoff is that a large gain increases the speed of local convergence and reduces the required buffer sizes, but the system becomes unstable if the gain is too large. Hence, we would like to determine the largest α for which the system is stable. Roughly speaking, the main result of this note is that if

$$\alpha < \frac{\pi c \log c}{2(c-1)}$$

then the system is asymptotically stable, that is for all initial conditions we have

$$\lim_{t\to\infty} p(t) = p_{\mathrm{eqm}}$$

This result is stated in Theorem 9. The local stability of this system was analyzed in [14], where it was shown that linearizing (1) and applying the Nyquist theorem implies that, for all $0 < c < 1$ and all $\tau > 0$, if

$$\alpha < \frac{\pi}{2}$$

then the system is locally stable. In particular, an important practical feature is that this stability condition is independent of delay τ. This fact was one of the motivations for this protocol in [14].

The approach taken in this note is to decompose this system of differential equations into a feedback loop, and prove stability of the dynamics by analyzing the input–output properties of the components of the loop. This approach is often called absolute stability theory [3], [21]. We make use of a specific approach called integral-quadratic constraints [13], which was first used for analysis of congestion control by [19]. We use a result from [7] in our proof. We also make use of loop transformations similar to those presented in [8], [17], [19].

Prior Work: There is a substantial body of literature modeling and analyzing the dynamics of congestion control protocols, originating with the work of [9]. Several models for TCP have been motivated by gradient algorithms for constrained optimization of a utility function [9], [10], and the model (1) may also be constructed in this way.

The closest work to this note is that of [16], [19], and [22]. In [16], it is shown that the system is globally stable if

$$\alpha < c.$$

In [19] it is shown that the previous system is globally stable if

$$\alpha < \frac{c\log c}{(c-1)}.$$

In this note we show global stability for a strictly larger set of α. Another related result is as follows; if the model is modified to allow negative queue lengths, then [22] shows that stability holds if

$$\alpha \le \frac{3}{2}$$

with a specific region of attraction.

Global stability of other models for TCP has been addressed in several papers. Razumhikin theory has been used in [2], [6], [19], [20], and [23], Lyapunov functions have been used to show stability in [1], [12], [15], and [16] and an input–output approach was taken in [4], [19]. Local stability results related to the model in this note may be found in [11] and [18].

Notation: We use the following standard notation. The set $C[a,b]$ is the space of functions

$$C[a,b] = \{f\colon [a,b] \to \mathbb{R} \mid f \text{ is continuous and bounded}\}$$

We denote by L_2 the Hilbert space of measurable functions $x : [0,\infty) \to \mathbb{R}^n$ with the usual inner product and norm. For $G : [0,\infty) \to \mathbb{R}^{m\times n}$, we use $\hat{G}$ to denote the Laplace transform of G, and overload G to also mean the corresponding linear map defined by convolution. We also define the space W_2 as

$$W_2 = \{x \in L_2 \mid \dot{x} \in L_2\}$$

For $T \ge 0$ define the projector P_T by $y = P_T z$ if

$$y(t) = \begin{cases} z(t), & \text{if } t \le T,\\ 0 & \text{otherwise.}\end{cases}$$

The extended space L_{2e} is the set of measurable functions $x : [0,\infty) \to \mathbb{R}^n$ such that $P_T x \in L_2$ for all $T \ge 0$. Similarly W_{2e} is the extended space of functions x whose truncation $P_T x$ lies in W_2 for all $T \ge 0$. A function $f : [0,\infty) \to \mathbb{R}$ is called absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x_1,\ldots,x_n$ with $\sum_{i=1}^{n-1}|x_{i+1} - x_i| \le \delta$ the function f satisfies $\sum_{i=1}^{n-1}|f(x_{i+1}) - f(x_i)| \le \varepsilon$. If $f \in W_2$ and f is absolutely continuous then $f(t) \to 0$ as $t \to \infty$.

II. A Feedback Model of the Dynamics

For convenience, we use the change of variables

$$z(t) = p(t) - p_{\mathrm{eqm}}$$

so that the new system has equilibrium $z(t) = 0$. We will eliminate $x(t)$ to give a nonlinear differential equation with a delay. Define the function f by

$$f(z) = \min\{e^{-\alpha z/\tau} - 1, 1/c - 1\} \tag{2}$$

and the function $h : \mathbb{R}^2 \to \mathbb{R}$ by

$$h(x,y) = \begin{cases} f(y), & \text{if } x > -p_{\mathrm{eqm}} \text{ or } f(y) > 0\\ 0, & \text{otherwise.}\end{cases} \tag{3}$$

The initial conditions of the system are specified by a function $z_0 \in C[-\tau, 0]$. Then, the dynamics and boundary conditions are

$$\dot{z}(t) = \begin{cases} h\left(z(t), z(t-\tau)\right), & \text{if } t \ge \tau\\ h\left(z(t), z_0(t-\tau)\right), & \text{otherwise}\end{cases}$$
$$z(0) = z_0(0). \tag{4}$$

The conditional construction in (3) forces the price p to be nonnegative. Here, (2) includes a saturation, which does not change the dynamics, as we require $z(t) \ge -p_{\mathrm{eqm}}$ for all t, and if this constraint is satisfied by the initial conditions then z never leaves this set. The saturation will allow us to give the useful sector bound

$$-\beta x^2 \le x f(x) \le 0 \ \text{ for all } x \in \mathbb{R} \tag{5}$$

where $\beta > 0$ is given by

$$\beta = \frac{\alpha(c-1)}{\tau c \log c}$$

This bound is the main feature of the dynamics that ensures stability.

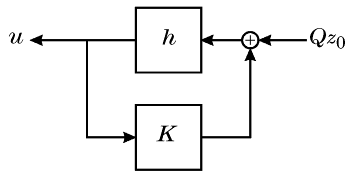

Fig. 1. System $u = h(Ku + Qz_0)$, which is equivalent to (4) when $z = Eu + Q_1 z_0$.

III. Loop Transformations

Define the integrator $E : L_{2e} \to W_{2e}$ by

$$(Eu)(t) = \int_0^t u(s)ds$$

and the delay $Z : L_{2e} \to L_{2e}$ by

$$(Zu)(t) = \begin{cases} u(t-\tau), & \text{if } t \geq \tau \\ 0, & \text{otherwise} \end{cases}$$

Define the map $K : L_{2e} \to W_{2e} \times W_{2e}$ by

$$K = \begin{bmatrix} E \\ EZ \end{bmatrix}$$

Also define the map $Q : C[-\tau, 0] \to W_{2e} \times W_{2e}$ by

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$$

$$(Q_1 z)(t) = z(0), \text{ for all } t \geq 0$$

$$(Q_2 z)(t) = \begin{cases} z(t-\tau), & \text{if } t \in [0, \tau) \\ z(0), & \text{otherwise.} \end{cases}$$

We also interpret for signals u and y the statement $u = h(y)$ to mean $u(t) = h(y(t))$ for all t. It is now easy to see that if $u \in L_{2e}$ and $z_0 \in C[-\tau, 0]$ and we define

$$z = Eu + Q_1 z_0$$

then

$$u = h(Ku + Qz_0)$$

if and only if z satisfies the differential equation and boundary conditions (4). This feedback system is illustrated in Fig. 1.

The feedback loop here is not scalar. However, we can simplify the analysis by closing one of the loops first. To do this, we will assume temporarily that $Q_1 z_0 = 0$.

Suppose $u \in L_{2e}$ and $y \in W_{2e}$, and $z = Eu$. In particular, this implies that $z(0) = 0$. Then

$$u = h(Eu, y)$$

if and only if

$$\dot{z}(t) = h\left(z(t), y(t)\right), \text{ for all } t \geq 0 \tag{6}$$

This is simply a scalar differential equation. Since h is a discontinuous function of its first argument, we interpret solutions of this equation in the sense of Filippov [5]. Specifically, the important property is that there is an associated differential inclusion for which the corresponding set-valued operator is upper semicontinuous. The function $y \in W_{2e}$, and this implies y is absolutely continuous. For any such y there exists a unique z with $z(0) = 0$ satisfying (6), and this z is both absolutely continuous and $z \in W_{2e}$. Hence, we define a map $\Gamma : W_{2e} \to L_{2e}$ by

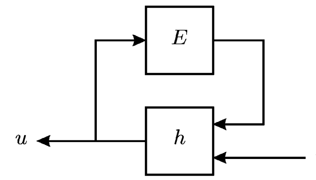

Fig. 2. System $u = \Gamma y$, which is one of the two feedback loops in Fig. 1, and is equivalent to (6).

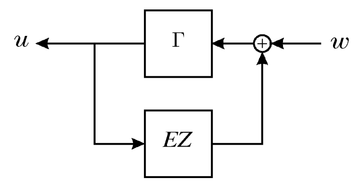

Fig. 3. System $u = \Gamma(EZu + w)$. This is equivalent to the dynamics (8).

$$u = \Gamma y$$

if and only if u is the unique solution of

$$u = h(Eu, y)$$

This corresponds to the feedback diagram shown in Fig. 2.

We now introduce an artificial input w to the system, to express the dynamics (4) in terms of Γ. For any $u \in L_{2e}$ and $w \in W_{2e}$ let $z = Eu$. Then

$$u = \Gamma(EZu + w) \tag{7}$$

if and only if z and w satisfy

$$\dot{z}(t) = \begin{cases} h\left(z(t), z(t-\tau) + w(t)\right), & \text{if } t \geq \tau \\ h\left(z(t), w(t)\right), & \text{otherwise} \end{cases} \tag{8}$$

This is illustrated in Fig. 3. Hence, in particular, if $z_0 \in C[-\tau, 0]$ with $z_0(0) = 0$, then by choosing $w = Q_2 z_0$ we have that u satisfies (7) if and only if $z = Eu$ satisfies (4).

Again, existence and uniqueness of solutions to (8) may be shown using the results of [5]. In particular, on each interval $[t, t+\tau]$ (8) is simply an ordinary differential equation of the form (6), and hence solutions exist and are unique; existence for all positive time is then shown by induction.

One further loop transformation is required, as illustrated in Fig. 4. Here, we simply add and subtract Eu to one of the signals. We will define the two inner loops to be Δ and G so that this feedback diagram is equivalent to that in Fig. 5. First, define $G = EZ - E$.

Now, define Δ as follows. For any $u \in W_{2e}$ and $v \in W_{2e}$, let $z = Eu$. Then

$$u = \Gamma(Eu + v)$$

if and only if

$$\dot{z}(t) = h\left(z(t), z(t) + v(t)\right) \text{ for all } t \geq 0 \tag{9}$$

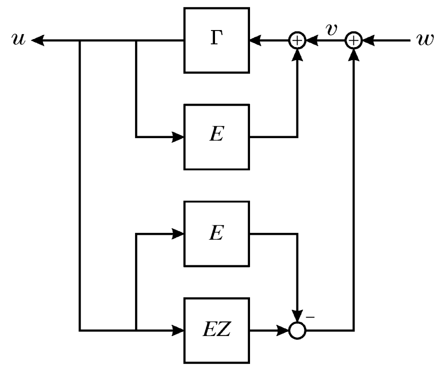

Fig. 4. System $u = \Gamma(EZu - w)$, constructed from Fig. 3 by a loop transformation.

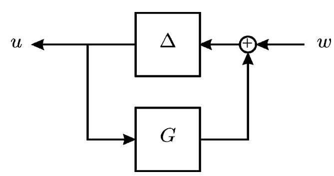

Fig. 5. System $u = \Delta(Gu + w)$, equivalent to Figs. 3 and 4, and to (8).

Similarly to the previous cases, this system has a unique solution $z \in L_{2e}$ for all $v \in W_{2e}$ and, hence, it defines a map $\Delta : W_{2e} \to W_{2e}$ such that

$$u = \Delta v$$

if and only if u is the unique solution to $u = \Gamma(Eu + v)$.

The point of this loop transformation is that the map from w to u shown in Fig. 3 is equal to that in Fig. 5, and this in turn is equivalent to the system of differential equations (8). We summarize this in the lemma that follows.

Lemma 1: For any $u \in L_{2e}$ and $w \in W_{2e}$, the equation

$$u = \Delta(Gu + w) \tag{10}$$

holds if and only if

$$u = \Gamma(EZu + w)$$

and this holds if and only if $z = Eu$ and w satisfy (8).

Proof: The proof follows immediately from the above system definitions and the loop transformation result of [3, Th. III.3]. ■

Thus, we can define the map $\Phi : W_{2e} \to L_{2e}$ by $u = \Phi w$ if u and w satisfy (7).

IV. Main Results

In this section, we prove the main results, that the system of differential equations (4) is asymptotically stable. The approach is as follows. First, we show input-output stability of (8), by constructing integral-quadratic constraints satisfied by G and Δ in Fig. 5. We then show that this implies asymptotic stability. We first prove some preliminary technical results. The sector bound implies the following.

Lemma 2: For all $v \in W_2$, we have

i) $\Delta v \in L_2$ and $\|\Delta v\| \le \beta\|v\|$;

ii) $\langle \Delta v, \beta v + \Delta v\rangle \le 0$.

Proof: Let $u = \Delta v$ and $z = Eu$. Then, z satisfies $z(0) = 0$ and the dynamics of (9). Also, since f has the above sector bound, we have

$$h(x,y)^2 \le -\beta y h(x,y), \quad \text{for all } x, y \in \mathbb{R}$$

and, hence

$$\dot z(t)^2 \le -\beta\left(z(t) + v(t)\right)\dot z(t).$$

Therefore, for all $T > 0$ we have

$$\begin{aligned}\|P_T \dot z\|^2 &\le -\beta \int_0^T \left(z(t)+v(t)\right)\dot z(t)dt\\ &= -\beta\int_0^t \dot z(t)v(t)dt - \frac{\beta}{2}z(T)^2\\ &\le -\beta\langle P_T\dot z, v\rangle\end{aligned}.$$

Hence

$$\begin{aligned}\|P_T u\|^2 &\le -\beta\langle P_T u, v\rangle\\ &\le \beta\|P_T u\|\|v\|\end{aligned} \tag{11}$$

and, therefore, $\|P_T u\| \le \beta\|v\|$ for all $T > 0$. This implies $u \in L_2$ and part i) of the desired result. Part ii) is then implied by (11). ■

Lemma 3: Consider the sequence $a_0, a_1, a_2, \ldots \in \mathbb{R}$, and suppose $a_k \ge 0$ for all k. Suppose that for all $\delta > 0$ there exists n such that for all $k > j > n$

$$a_k - a_j \le \delta$$

then the sequence $a_0, a_1, \ldots$ converges.

Proof: The proof is straightforward and, hence, omitted. ■

Lemma 4: Suppose $v \in W_2$ and let $z = E\Delta v$. Then

$$\lim_{t\to\infty} z(t) = 0.$$

Proof: Suppose $T_2 > T_1 > 0$, and let $H = P_{T_2} - P_{T_1}$. Then, as in the proof of Lemma 2

$$\begin{aligned}\|H\dot z\|^2 &= \int_{T_1}^{T_2} \dot z(t)^2 dt\\ &\le -\beta\int_{T_1}^{T_2}\left(z(t)+v(t)\right)\dot z(t)dt\\ &= -\beta\langle H\dot z, Hv\rangle - \frac{\beta}{2}\left(z(T_2)^2 - z(T_1)^2\right)\end{aligned}$$

and, hence,

$$z(T_2)^2 - z(T_1)^2 \le 2\|H\dot z\|\|Hv\|.$$

Now by choosing T_1, T_2 sufficiently large we can make $\|Hv\|$ as small as we like and, hence, for any increasing sequence $T_0, T_1, \ldots$ the result of Lemma 3 implies that the sequence $z(T_k)^2$ converges as $k \to \infty$. Since z is continuous this implies $z(t)$ tends to a limit as $t \to \infty$. Let z_∞ be this limit.

We now show that $z_\infty = 0$. To see this, notice that $v \in W_2$ and, hence, $v(t) \to 0$ as $t \to \infty$. Suppose for the sake of a contradiction that $z_\infty \ne 0$. Then, if $z_\infty < 0$ then for all t sufficiently large we have $z(t) + v(t) < 0$ and, hence

$$\begin{aligned}\dot z(t) &= h\left(z(t), z(t)+v(t)\right)\\ &= f\left(z(t)+v(t)\right).\end{aligned}$$

Similarly, if $z_\infty > 0$, then for all t sufficiently large we also have

$$\dot z(t) = f\left(z(t)+v(t)\right).$$

Then, since f is continuous, $\dot z(t)$ converges to a limit as $t \to \infty$, and since $\dot z \in L_2$ this limit must be zero. Hence, $f(z_\infty) = 0$ and, therefore, $z_\infty = 0$. ■

Lemma 5: For all $v \in W_2$, we have

$$\langle \Delta v, \dot v + \Delta v\rangle \le \beta v(0)^2$$

Proof: For convenience, let $u = \Delta v$, $z = Eu$ and $r = z + v$. Then, we have $z(0) = 0$ and

$$\dot{z}(t) = \begin{cases} f(r(t)), & \text{if } t \in V \\ 0, & \text{otherwise} \end{cases}$$

where the set V is

$$V = \left\{ t \geq 0 \mid z(t) > \frac{\tau}{\alpha} \log c \text{ or } r(t) < 0 \right\}.$$

Now, z and r are continuous and, hence, if $r(0) \geq 0$, then V is the countable union of disjoint open intervals

$$V = (a_1, b_1) \cup (a_2, b_2) \cup \ldots$$

If $r(0) < 0$, then similarly

$$V = [0, b_0) \cup (a_1, b_1) \cup (a_2, b_2) \cup \ldots$$

We claim now that for $i \geq 1$ we have $r(a_i) = 0$. To see this, notice that $a_i \notin V$ and hence $z(a_i) \leq (\tau \log c)/\alpha$ and $r(a_i) \geq 0$. Therefore, $\dot{z}(a_i) \leq 0$ and, hence, $z(a_i + \varepsilon) \leq (\tau \log c)/\alpha$ for all sufficiently small $\varepsilon > 0$. Now, since $a_i + \varepsilon \in V$ we must therefore have $r(a_i + \varepsilon) < 0$ for all $\varepsilon > 0$ sufficiently small and, therefore, $r(a_i) = 0$. Now, we have

$$\begin{aligned} \langle \Delta v, \dot{v} + \Delta v \rangle &= \langle \dot{z}, \dot{r} \rangle \\ &= \int_0^\infty \dot{z}(t)\dot{r}(t)dt \\ &= \int_V f(r(t))\dot{r}(t)dt. \end{aligned}$$

If $r(0) < 0$, then we have

$$\langle \Delta v, \dot{v} + \Delta v \rangle = \int_0^{b_0} f(r(t))\dot{r}(t)dt + \sum_{i \geq 1} \int_{a_i}^{b_i} f(r(t))\dot{r}(t)dt$$

and, hence

$$\langle \Delta v, \dot{v} + \Delta v \rangle = \int_{r(0)}^{r(b_0)} f(q)dq + \sum_{i \geq 1} \int_{r(a_i)}^{r(b_i)} f(q)dq.$$

Now, since $r(a_i) = 0$ we have

$$\langle \Delta v, \dot{v} + \Delta v \rangle = \int_{r(0)}^{0} f(q)dq + \sum_{i \geq 0} \int_{0}^{r(b_i)} f(q)dq.$$

Now since f is sector-bounded, we have for all $a \in \mathbb{R}$

$$-\beta a^2 \leq \int_0^a f(q)dq \leq 0.$$

The previous sum converges, since each term is negative and it is bounded by the value of the inner-product, which is finite since $\dot{z}, \dot{r} \in L_2$. Therefore

$$\begin{aligned} \langle \Delta v, \dot{v} + \Delta v \rangle &\leq \int_{r(0)}^{0} f(q)dq \\ &\leq \beta r(0)^2 \\ &= \beta v(0)^2 \end{aligned}$$

as desired. The same result holds for $r(0) \geq 0$ by a similar argument.■

A. Input–Output Stability

We will prove input–output stability by constructing integral-quadratic constraints satisfied by G and Δ. First, define the set of transfer functions A as follows. The function $\hat{G} \in A$ if and only if $Gu \in W_{2e}$ for all $u \in L_{2e}$ and there exists $J_0 \in L_1[0, \infty)$, real numbers $\alpha_1, \ldots, \alpha_N$ and $t_1, \ldots, t_N \geq 0$ such that $s\hat{G}(s) = \hat{J}(s)$, where J is

$$J(t) = J_0(t) + \sum_{i=1}^{N} \alpha_i \delta(t - t_i)$$

We use the following result from [7, p. 71].

Theorem 6: Suppose $\lambda \in \mathbb{R}$, and $\Pi : j\mathbb{R} \to \mathbb{C}^{n \times n}$ is bounded and measurable, and $\Pi(j\omega)$ is Hermitian for all $\omega \in \mathbb{R}$. Suppose the transfer function $\hat{G} \in A$ and $\Delta : L_{2e} \to L_{2e}$, and the following conditions hold.

i) For all $\kappa \in [0, 1]$ and $w \in W_2$ and $g \in L_{2e}$ there exists unique $v \in W_{2e}$ and $u \in L_{2e}$ such that

$$\begin{aligned} v &= Gu + w \\ u &= \kappa \Delta v + g \end{aligned}$$

and the resulting map $(w, g) \mapsto (v, u)$ is causal.

ii) There exists $\eta > 0$ such that for all $\omega \in \mathbb{R}$

$$\begin{bmatrix} \hat{G}(j\omega) \\ I \end{bmatrix}^* (\Pi(j\omega) + \Lambda(j\omega)) \begin{bmatrix} \hat{G}(j\omega) \\ I \end{bmatrix} \leq -\eta I$$

where Λ is

$$\Lambda(j\omega) = \begin{bmatrix} 0 & -j\omega\lambda \\ j\omega\lambda & 0 \end{bmatrix}.$$

iii) The map Δ is bounded on L_2 and causal, and there exists $\gamma > 0$ such that for all $\kappa \in [0, 1]$, $v \in W_2$ and $u \in L_2$ such that $u = \kappa \Delta v$ we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{u}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{u}(j\omega) \end{bmatrix} d\omega + 2\lambda \int_0^\infty u(t)^T \dot{v}(t)dt \geq -\gamma |v(0)|^2.$$

Then, there exist $\rho_1, \rho_2, \rho_3 > 0$ such that for all $w \in W_2$ and $g \in L_{2e}$ the unique solution $v \in W_{2e}$, $u \in L_{2e}$ to

$$\begin{aligned} v &= Gu + w \\ u &= \Delta v + g \end{aligned}$$

satisfies for all $T > 0$

$$\int_0^T \left(|v(t)|^2 + |u(t)|^2 \right) dt \leq \rho_1 \|w\|^2 + \rho_2 \|\dot{w}\|^2 + \rho_3 \int_0^T |g(t)|^2 dt.$$

The following result shows that the system under consideration is input–output stable.

Theorem 7: Suppose $\beta\tau < \pi/2$. There exists $\rho_1, \rho_2 > 0$ such that for any $w \in W_2$, the unique $u \in L_{2e}$ and $v \in W_{2e}$ such that

$$\begin{aligned} v &= Gu + w \\ u &= \Delta v \end{aligned}$$

satisfy

$$\|v\|^2 + \|u\|^2 \leq \rho_1 \|w\|^2 + \rho_2 \|\dot{w}\|^2.$$

Proof: We will apply Theorem 6 using $\lambda = -2/\pi$ and

$$\Pi(j\omega) = \begin{bmatrix} 0 & -\beta \\ -\beta & -2-4/\pi \end{bmatrix}.$$

To show condition i), define the map $\Psi : (v,u) \mapsto (w,g)$ by

$$\begin{aligned} w &= v - Gu \\ g &= -\kappa\Delta v + u \end{aligned}.$$

Then we need to show that Ψ is invertible and has a causal inverse for all $\kappa \in [0,1]$. To show invertibility, first notice that for any $w \in W_{2e}$ there exists a unique $u \in L_{2e}$ such that

$$u = \kappa\Delta(Gu + w)$$

since the corresponding set of differential equations is simply (8) scaled by κ. Hence, we can define the causal map $\Theta : W_{2e} \to L_{2e}$ such that $\Theta w = u$, the unique solution corresponding to w. Then, $\Psi^{-1} : (v,u) \mapsto (w,g)$ is given by

$$\begin{aligned} w &= (I + G\Theta)(v + Gu) \\ g &= \kappa\Delta(I + G\Theta)(v + Gu) + u. \end{aligned}$$

Since G, Θ and Δ are causal, the map Ψ^{-1} is also causal.

Condition iii) holds, since Δ is causal, and it is bounded by Lemma 2. Also, since with $v \in W_2$ and $u \in L_2$ we have

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{u}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{u}(j\omega) \end{bmatrix} d\omega + 2\lambda \int_0^{\infty} u(t)^T \dot{v}(t)dt \\ &\geq -\gamma\,|v(0)|^2 \\ &= -2\beta\langle u, v\rangle - \left(2 + \frac{4}{\pi}\right)\langle u, u\rangle - \frac{4}{\pi}\langle u, \dot{v}\rangle \\ &= -2\langle u, \beta v + u\rangle - \frac{4}{\pi}\langle u, u + \dot{v}\rangle. \end{aligned}$$

Now, with $u = \kappa\Delta v$ and $\kappa \in [0,1]$, this equals

$$\begin{aligned} &-2\langle u, \beta v + u\rangle - \frac{4}{\pi}\langle u, u + \dot{v}\rangle \\ &\quad = -2\langle \kappa\Delta v, \beta v + \kappa\Delta v\rangle - \frac{4}{\pi}\langle \kappa\Delta v, \kappa\Delta v + \dot{v}\rangle \\ &\quad \geq -2\kappa\langle \Delta v, \beta v + \Delta v\rangle - \frac{4\kappa}{\pi}\langle \Delta v, \Delta v + \dot{v}\rangle \\ &\quad \geq -\frac{4}{\pi}\kappa\beta\,|v(0)|^2 \\ &\quad \geq -\frac{4}{\pi}\beta\,|v(0)|^2 \end{aligned}$$

where we used Lemmas 5 and 2.

Condition ii) holds, because

$$\hat{G}(j\omega) = \frac{e^{-j\omega\tau} - 1}{j\omega}$$

and, hence, for all $\omega \in \mathbb{R}$, we have

$$\begin{aligned} &\begin{bmatrix} \hat{G}(j\omega) \\ I \end{bmatrix}^* (\Pi(j\omega) + \Lambda(j\omega)) \begin{bmatrix} \hat{G}(j\omega) \\ I \end{bmatrix} \\ &\quad = -2 - \frac{4\cos(\omega\tau)}{\pi} + \frac{2\beta\sin(\omega\tau)}{\omega} \\ &\quad < -\eta \end{aligned}$$

where the latter inequality holds for some $\eta > 0$ if $\beta\tau < \pi/2$. ■

B. Asymptotic Stability

We now address asymptotic stability. We first show that we can achieve a response corresponding to an initial condition applied to (4) by applying an appropriate input to (8).

Lemma 8: Suppose $q_0 \in C[-\tau, 0]$, and let $q \in W_{2e}$ be the unique solution to

$$\begin{aligned} \dot{q}(t) &= \begin{cases} h(q(t), q(t-\tau)) & \text{if } t \geq \tau \\ h(q(t), q_0(t-\tau)) & \text{otherwise} \end{cases} \\ q(0) &= q_0(0). \end{aligned}$$

Then, there exists $w \in W_{2e}$ and $T \geq 0$ such that $z = E\Phi w$ satisfies

$$q(t) = z(t+T), \quad \text{for all } t \geq 0.$$

Proof: First assume $q_0(-\tau) < 0$ and $q_0(0) > 0$, and define $y \in W_{2e}$ parametrized by ε and T by

$$y(t) = \begin{cases} -\varepsilon t, & \text{if } t < -q_0(-\tau)/\varepsilon \\ q_0(-\tau), & \text{if } -q_0(-\tau)/\varepsilon \leq t < T \\ q_0(t - T - \tau), & \text{if } T \leq t < T + \tau \\ q(t - T - \tau), & \text{otherwise} \end{cases}$$

Now, let $x = E\Gamma y$. Then, we have $x(0) = 0$ and

$$\dot{x}(t) = h(x(t), y(t)), \quad \text{for all } t \geq 0.$$

With this choice of y we can make $x(-q_0(-\tau)/\varepsilon)$ arbitrarily small by choosing ε sufficiently large. For $t \in [-q_0(-\tau)/\varepsilon, T]$ we have $y(t)$ is a negative constant, and hence $\dot{x}(t)$ is a positive constant, and so $x(t)$ is linearly increasing on this interval. Hence, we can choose $\varepsilon > 0$ sufficiently large and $T > 0$ such that

$$x(T) = q_0(0).$$

If the signs of $q_0(-\tau)$ and $q_0(0)$ are not as before, then a similar approach may be used to construct a continuous y such that $x(T) = q_0(0)$. Now for $t \in [T, T+\tau)$, we have

$$\dot{x}(t) = h(x(t), q_0(t - T - \tau)).$$

Now let $w = P_{T+\tau}(y - EZ\Gamma y)$ and $z = E\Phi w$. Then

$$\begin{aligned} P_{T+\tau} z &= P_{T+\tau} E\Phi P_{T+\tau}(y - EZ\Gamma y) \\ &= P_{T+\tau} E\Phi(y - EZ\Gamma y) \\ &= P_{T+\tau} E\Gamma y \\ &= P_{T+\tau} x \end{aligned}$$

where the second equation holds because $E\Phi$ is causal, and the third equation holds since for any $y \in W_{2e}$ we have $\Phi(y - EZ\Gamma y) = \Gamma y$. Then, $w \in W_{2e}$ and one may also verify that the function w is continuous, in particular at time $T+\tau$. Hence, for $t \in [T, T+\tau)$ the function z also satisfies

$$\dot{z}(t) = h(z(t), q_0(t - T - \tau))$$

and $z(T) = q_0(0)$, since this is satisfied by x. Also, since $z = E\Phi w$ we have for all $t \geq \tau$

$$\dot{z}(t) = h(z(t), z(t-\tau) + w(t))$$

and hence for $t \geq T + \tau$ we have

$$\dot{z}(t) = h(z(t), z(t-\tau))$$

as desired. ■

Asymptotic stability now follows from input–output stability, as stated later. This is the main result of this note.

Theorem 9: If α satisfies

$$\alpha < \frac{\pi c \log c}{2(c-1)}$$

then the system is asymptotically stable. That is, for any $q_0 \in C[-\tau, 0]$ let $q \in W_{2e}$ be the unique solution to

$$\dot{q}(t) = \begin{cases} h\left(q(t), q(t-\tau)\right), & \text{if } t \geq \tau \\ h\left(q(t), q_0(t-\tau)\right), & \text{otherwise} \end{cases}$$
$$q(0) = q_0(0).$$

Then

$$\lim_{t \to \infty} q(t) = 0.$$

Proof: Lemma 8 implies that there exists $w \in W_2$ such that $z = E\Phi w$ satisfies $q(t) = z(t+T)$ for all $t \geq 0$. Let $u \in L_{2e}$ and $v \in W_{2e}$ be the unique solution to

$$v = Gu + w$$
$$u = \Delta v$$

then $z = E\Delta v$. Using the definition of β, Theorem 7 now implies that $u \in L_2$ and $v \in W_{2e} \cap L_2$, and hence $\dot{v} = (Z - I)u + \dot{w}$ and, hence, $v \in W_2$. Then, Lemma 4 implies the desired result. ■

V. Summary

In this note, we have analyzed global stability of a single-source single-link model of congestion control, and have shown both input–output and asymptotic stability if

$$\alpha < \frac{\pi c \log c}{2(c-1)}$$

To do this, we made use of integral-quadratic constraints and the sector-bounded property of the system. The gap between the condition for linear stability that $\alpha < \pi/2$ and that for nonlinear stability tends to zero as the link capacity c approaches 1. It is currently unknown whether the dynamics is globally stable for all $\alpha < \pi/2$ for any capacity $0 < c < 1$.

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Assigning Frequencies via Determinantal Equations: New Counterexamples and Invariants

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***Abstract*—The problem of arbitrary pole-zero assignment by fixed structure compensators has been addressed so far in terms of either necessary or sufficient conditions. The strongest existing condition is based on the rank of the differential of a related map on a degenerate controller and holds true generically when the degrees of freedom of the compensator exceed the number of frequencies to be assigned ($mp > n$ for the output feedback problem). The complete (nongeneric) solvability of the problem is still open, even when complex controllers are considered. A simple necessary solvability condition is that the linear map, appearing as a factor of the main determinantal map, defining the assignment problem, is onto. Here we examine the special problem of assignment of matrix pencil zeros via diagonal perturbations and we present a new necessary and sufficient condition for complex solvability in terms of a new invariant involving the minors of this linear map. Based on this result, we demonstrate that for the important case where the degrees of freedom of the controller are equal to the number of frequencies to be assigned, the surjectivity of this linear map although it is necessary, it is not sufficient for the solvability of the problem.**

***Index Terms*—Algebraic geometry methods, feedback systems, linear systems, pole assignment.**

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