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POSITIVE FORMS AND STABILITY OF LINEAR TIME-DELAY SYSTEMS*

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Abstract. We consider the problem of constructing Lyapunov functions for linear differential equations with delays. For such systems it is known that exponential stability implies the existence of a positive Lyapunov function which is quadratic on the space of continuous functions. We give an explicit parameterization of a sequence of finite-dimensional subsets of the cone of positive Lyapunov functions using positive semidefinite matrices. This allows stability analysis of linear time-delay systems to be formulated as a semidefinite program.

Key words. Lyapunov stability, delay systems, semidefinite programming

AMS subject classifications. 34D20, 90C22, 37F10

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1. Summary of the paper. In this paper we present an approach to the parameterization of Lyapunov functions for infinite-dimensional systems. In particular, we consider linear time-delay systems. These are systems which can be represented in the form

$$\dot{x}(t) = \sum_{i=0}^{k} A_i x(t - h_i),$$

where $x(t) \in \mathbb{R}^n$. In the simplest case we are given the delays $h_0, \dots, h_k$ and the matrices $A_0, \dots, A_k$ and we would like to determine whether the system is stable. For such systems it is known that if the system is stable, then there exists a Lyapunov function of the form

$$V(\phi) = \int_{-h}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds + \int_{-h}^{0}\int_{-h}^{0} \phi(s)^T N(s,t)\phi(t)\, ds\, dt,$$

where $h = \max\{h_0, \dots, h_k\}$ and M and N are piecewise continuous matrix-valued functions. Here $\phi : [-h, 0] \to \mathbb{R}^n$ is an element of the state space, which in this case is the space of continuous functions mapping $[-h, 0]$ to $\mathbb{R}^n$. The function V is thus a quadratic form on the state space. The derivative is also such a quadratic form, and the matrix-valued functions which define it depend linearly on M and N.

In this paper we develop an approach which uses semidefinite programming to construct piecewise continuous functions M and N such that the function V is positive and its derivative is negative. Roughly speaking, our contributions are as follows.

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Spacing functions. In Theorem 5, we show that for any piecewise continuous function M

$$V_1(\phi) = \int_{-h}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds$$

is positive for all ϕ if and only if there exists a piecewise continuous matrix-valued function T such that

$$M(t) + \begin{bmatrix} T(t) & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \quad \text{for all } t,$$

$$\int_{-h}^{0} T(t)\, dt = 0.$$

That is, we convert positivity of the integral to *pointwise* positivity of M. If we assume that M is polynomial, then pointwise positivity may be easily enforced, and in the case of positivity on the real line this is equivalent to a sum-of-squares constraint. The assumption that M is polynomial has recently been shown to be nonconservative. The constraint that T integrates to zero is a simple linear constraint on the coefficients of T. Notice that the sufficient condition that $M(s)$ be pointwise nonnegative is conservative, and as the equivalence above shows it is easy to generate examples where V_1 is nonnegative even though $M(s)$ is not pointwise nonnegative.

Positive polynomial kernels. We give a sum-of-squares characterization of positive polynomial kernels. We consider the quadratic form

$$\int_{-h}^{0} \int_{-h}^{0} \phi(s)^T \ N(s,t) \ \phi(t)\, ds\, dt.$$

In Theorem 7, we show that the quadratic form is positive if and only if there exists a positive semidefinite matrix $Q \succeq 0$ such that $N(s,t) = Z(s)^T Q Z(t)$, where Z is a vector of monomials. This condition allows us to test positivity of a polynomial kernel using semidefinite programming and implies the existence of a sum–of–squares-type representation. Note that pointwise positivity of N is not sufficient for positivity of the functional. The condition that the derivative of the Lyapunov function be negative is similarly enforced.

This paper is organized as follows. We begin with a discussion of the relevant history and prior work. We then give a definition of the system for which we will prove stability and a presentation of the class of Lyapunov functions we will construct. Following this section, we present our first result giving pointwise condition for positivity of the functional. Next, we give a parameterization of piecewise polynomial matrices. We use sum-of-squares techniques to parameterize matrices which are pointwise nonnegative, and we present a new result on the parameterization of kernel matrices which define positive quadratic forms. We then return to the problem of linear time-delay systems to document the derivative of the Lyapunov function along trajectories of the system. Finally, we use this derivative to define a semidefinite program for proving stability of linear time-delay systems.

2. Background and prior work. The use of Lyapunov functions on an infinite-dimensional space to analyze differential equations with delay originates with the work of [10]. For linear systems, quadratic Lyapunov functions were first considered by [19].

The book of [4] presents many useful results in this area, and further references may be found there as well as in [5, 9] and [13].

The idea of using semidefinite programming and sum of squares to solve polynomial optimization problems has many sources. Well-known examples include [11, 12] and [16]. Work on this problem continues actively, with recent results to be found in, e.g., [6] and [1].

An early example of using sum-of-squares polynomials together with semidefinite programming to construct polynomial Lyapunov functions for nonlinear ordinary differential equations can be found in [15]. Substantial work has been done in this area, with contributions from, e.g., [23] and [2].

The idea of constructing Lyapunov functions for linear time-delay systems using semidefinite programming is not original or new. A typical approach has been to examine subclasses of functions M and N, e.g., constant matrices, piecewise linear functions, etc. In practice, some of these approaches have been shown to be highly accurate.

The motivation for this paper is not exclusively the stability of linear time-delay systems. It is our hope, by investigating properties of a positive form known to be necessary and sufficient for stability of infinite-dimensional systems, and by using polynomial methods, that our results will also be useful in analysis of nonlinear and partial differential systems.

Building on the results of this paper, a treatment of nonquadratic Lyapunov functions for nonlinear time-delay systems appears in [14].

2.1. Notation. Let $\mathbb{N}$ denote the set of nonnegative integers. Let $\mathbb{S}^n$ be the set of $n \times n$ real symmetric matrices, and for $X \in \mathbb{S}^n$ we write $X \succeq 0$ to mean that X is positive semidefinite. For two matrices A, B, we denote the Kronecker product by $A \otimes B$. For X any Banach space and $I \subset \mathbb{R}$ any interval, let $\Omega(I, X)$ be the space of all functions

$$\Omega(I, X) = \{\, f \colon I \to X \,\}$$

and let $C(I, X)$ be the Banach space of bounded continuous functions

$$C(I, X) = \{\, f : I \to X \mid f \text{ is continuous and bounded} \,\}$$

equipped with the norm

$$\|f\| = \sup_{t \in I} \|f(t)\|_X .$$

We will omit the range space when it is clear from the context; for example we write $C[a, b]$ to mean $C([a, b], X)$. A function is called $C^n(I, X)$ if the ith derivative exists and is a continuous function for $i = 0, \ldots, n$. A function $f \in C[a, b]$ is called piecewise continuous if there exists a finite number of points $a < h_1 < \cdots < h_k < b$ such that f is continuous at all $x \in [a, b] \backslash \{h_1, \ldots, h_k\}$ and its right- and left-hand limits exist at $\{h_1, \ldots, h_k\}$.

Define also the projection $F_t : \Omega[-h, \infty) \to \Omega[-h, 0]$ for $t \geq 0$ and $h > 0$ by

$$(F_t x)(s) = x(t + s) \quad \text{for all } s \in [-h, 0].$$

We follow the usual convention and denote $F_t x$ by x_t.

3. Linear time-delay systems. Suppose $0 = h_0 < h_1 < \cdots < h_k = h$ and $A_0, \ldots, A_k \in \mathbb{R}^{n\times n}$. We consider linear differential equations with delay, of the form

$$\dot{x}(t) = \sum_{i=0}^{k} A_i x(t - h_i) \quad \text{for all } t \geq 0, \tag{1}$$

where the trajectory $x : [-h, \infty) \to \mathbb{R}^n$. The boundary conditions are specified by a given function $\phi : [-h, 0] \to \mathbb{R}^n$ and the constraint

$$x(t) = \phi(t) \quad \text{for all } t \in [-h, 0]. \tag{2}$$

If $\phi \in C[-h, 0]$, then there exists a unique function x satisfying (1) and (2). The system is called *exponentially stable* if there exist $\sigma > 0$ and $a \in \mathbb{R}$ such that for every initial condition $\phi \in C[-h, 0]$ the corresponding solution x satisfies

$$\|x(t)\| \leq a e^{-\sigma t} \|\phi\| \quad \text{for all } t \geq 0.$$

We write the solution as an explicit function of the initial conditions using the map $G : C[-h, 0] \to \Omega[-h, \infty)$, defined by

$$(G\phi)(t) = x(t) \quad \text{for all } t \geq -h,$$

where x is the unique solution of (1) and (2) corresponding to initial condition ϕ. Also for $s \geq 0$ define the *flow map* $\Gamma_s : C[-h, 0] \to C[-h, 0]$ by

$$\Gamma_s \phi = F_s G \phi,$$

which maps the state of the system x_t to the state at a later time $x_{t+s} = \Gamma_s x_t$.

3.1. Lyapunov functions. Lyapunov theory for infinite-dimensional systems closely parallels that for finite-dimensional systems. The difference is that the state space is now a function space, and therefore Lyapunov functions are actually functions of functions. For linear time-delay systems, the Lyapunov functions we define here are functions of segments of the trajectory and, in particular, of the state x_t. For a given $V : C[-h, 0] \to \mathbb{R}$, we use the standard notion of the *Lie derivative* or derivative of a function on a vector field. The Lie derivative of V is defined by the flow map, Γ, as

$$\dot{V}(\phi) = \limsup_{r \to 0^+} \frac{1}{r}\big(V(\Gamma_r \phi) - V(\phi)\big).$$

In keeping with tradition, we will use the notation $\dot{V}$ to denote the Lie derivative. We will consider the set X of quadratic functions, where $V \in X$ if there exist piecewise continuous functions $M : [-h, 0) \to \mathbb{S}^{2n}$ and $N : [-h, 0) \times [-h, 0) \to \mathbb{R}^{n\times n}$ such that

$$V(\phi) = \int_{-h}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds + \int_{-h}^{0}\int_{-h}^{0} \phi(s)^T N(s,t) \phi(t)\, ds\, dt. \tag{3}$$

The following result shows that for linear systems with delay the system is exponentially stable if and only if there exists a quadratic Lyapunov function. Define the sets $H = \{-h_0, \ldots, -h_k\}$ and $H^c = [-h, 0] \backslash H$.

THEOREM 1. *The linear system defined by* (1) *and* (2) *is exponentially stable if and only if there exists a Lie-differentiable function* $V \in X$ *and* $\varepsilon > 0$ *such that for all* $\phi \in C[-h, 0]$

$$V(\phi) \geq \varepsilon \|\phi(0)\|^2, \qquad \dot{V}(\phi) \leq -\varepsilon \|\phi(0)\|^2. \tag{4}$$

Further $V \in X$ *may be chosen such that the corresponding functions* M *and* N *of* (3) *have the following smoothness property:* $M(s)$ *and* $N(s,t)$ *are continuous on* $s, t \in [-h, 0] \backslash \{-h_0, \ldots, -h_k\}$.

Proof. See [4] or [7] for a recent proof. ∎

The consequence of this theorem is that stability of a linear time-delay system is equivalent to the existence of a Lyapunov function which decreases along segments of the trajectory. In the next few sections, we will give results which will allow us to better understand the positivity of the function.

4. Positivity of integral forms. The goal of this section is to present results which enable us to characterize functions $V \in X$ which satisfy the positivity conditions in (4) and have the form

$$V(y) = \int_{-h}^{0} \begin{bmatrix} y(0) \\ y(t) \end{bmatrix}^T M(t) \begin{bmatrix} y(0) \\ y(t) \end{bmatrix} dt.$$

Before stating the main result in Theorem 5, we give a few necessary lemmas.

LEMMA 2. *Suppose* $f \colon [-h, 0] \to \mathbb{R}$ *is piecewise continuous. Then the following are equivalent:*

(i) $\int_{-h}^{0} f(t)\, dt \geq 0$.

(ii) *There exists a function* $g \colon [-h, 0] \to \mathbb{R}$ *which is piecewise continuous and satisfies*

$$f(t) + g(t) \geq 0 \quad \textit{for all } t,$$

$$\int_{-h}^{0} g(t)\, dt = 0.$$

Proof. The direction (ii) $\Longrightarrow$ (i) is immediate. To show the other direction, suppose (i) holds, and let g be

$$g(t) = -f(t) + \frac{1}{h} \int_{-h}^{0} f(s)\, ds \quad \text{for all } t.$$

Then g satisfies (ii). ∎

The next lemma shows that minimizing over continuous functions is as good as minimizing over piecewise continuous functions.

LEMMA 3. *Suppose* $H = \{-h_0, \ldots, -h_k\}$, *and let* $H^c = [-h, 0] \backslash H$. *Let* $f \colon [-h, 0] \times \mathbb{R}^n \to \mathbb{R}$ *be continuous on* $H^c \times \mathbb{R}^n$, *and suppose there exists a bounded function* $z \colon [-h, 0] \to \mathbb{R}$, *continuous on* H^c, *such that for all* $t \in [-h, 0]$

$$f\big(t, z(t)\big) = \inf_x f(t, x).$$

Further suppose for each bounded set $X \subset \mathbb{R}^n$ *the set*

$$\{\, f(t, x) \mid x \in X, t \in [-h, 0] \,\}$$

is bounded. Then

$$\inf_{y \in C[-h,0]} \int_{-h}^{0} f\big(t, y(t)\big)\, dt = \int_{-h}^{0} \inf_x f(t,x)\, dt. \tag{5}$$

Proof. Let

$$K = \int_{-h}^{0} \inf_x f(t,x)\, dt.$$

It is easy to see that

$$\inf_{y \in C[-h,0]} \int_{-h}^{0} f\big(t, y(t)\big)\, dt \geq K$$

since, if not, then there would exist some continuous function y and some interval on which

$$f\big(t, y(t)\big) < \inf_x f(t,x),$$

which is clearly impossible.

We now show that the left-hand side of (5) is also less than or equal to K and hence equals K. We need to show that for any $\varepsilon > 0$ there exists $y \in C[-h,0]$ such that

$$\int_{-h}^{0} f\big(t, y(t)\big)\, dt < K + \varepsilon.$$

To do this, for each $n \in \mathbb{N}$ define the set $H_n \subset \mathbb{R}$ by

$$H_n = \bigcup_{i=1}^{k-1} (h_i - \alpha/n, h_i + \alpha/n)$$

and choose $\alpha > 0$ sufficiently small so that $H_1 \subset (-h, 0)$. Let z be as in the hypothesis of the lemma, and pick M and R so that

$$M > \sup_{t \in [-h,0]} \|z(t)\|,$$

$$R = \sup\big\{ \, |f(t,x)| \;\big|\; t \in [-h,0], \|x\| \leq M \,\big\}.$$

For each n choose a continuous function $x_n : [-h, 0] \to \mathbb{R}^n$ such that $x_n(t) = z(t)$ for all $t \notin H_n$ and

$$\sup_{t \in [-h,0]} \|x_n(t)\| < M.$$

This is possible, for example, by linear interpolation. Now we have for the continuous function x_n

$$\begin{aligned} \int_{-h}^{0} f\big(t, x_n(t)\big)\, dt &= K + \int_{-h}^{0} \Big(f\big(t, x_n(t)\big) - f\big(t, z(t)\big) \Big)\, dt \\ &= K + \int_{H_n} \Big(f\big(t, x_n(t)\big) - f\big(t, z(t)\big) \Big)\, dt \\ &\leq K + 4R\alpha(k-1)/n. \end{aligned}$$

This proves the desired result. $\square$

The following lemma states that when the $\arg\min_z f(t,z)$ is piecewise continuous in t we have the desired result.

LEMMA 4. *Suppose* $f\colon [-h,0]\times\mathbb{R}^n \to \mathbb{R}$ *and the hypotheses of Lemma* 3 *hold. Then the following are equivalent:*

(i) *For all* $y \in C[-h,0]$

$$\int_{-h}^{0} f\big(t, y(t)\big)\, dt \ge 0.$$

(ii) *There exists* $g : [-h,0] \to \mathbb{R}$ *which is piecewise continuous and satisfies*

$$f(t,z) + g(t) \ge 0 \quad \text{for all } t, z,$$

$$\int_{-h}^{0} g(t)\, dt = 0.$$

Proof. Again we need only show that (i) implies (ii). Suppose (i) holds; then

$$\inf_{y\in C[-h,0]} \int_{-h}^{0} f\big(t, y(t)\big)\, dt \ge 0,$$

and hence by Lemma 3 we have

$$\int_{-h}^{0} r(t)\, dt \ge 0,$$

where $r : [-h,0] \to \mathbb{R}^n$ is given by

$$r(t) = \inf_x f(t,x) \quad \text{for all } t.$$

The function r is continuous on H^c since f is continuous on $H^c \times \mathbb{R}^n$. Hence, by Lemma 2, there exists g such that condition (ii) holds, as desired. ∎

We now specialize the result of Lemma 4 to the case of quadratic functions. It is shown that in this case, under certain conditions, the $\arg\min_z f(t,z)$ is piecewise continuous.

THEOREM 5. *Suppose* $M : [-h,0] \to \mathbb{S}^{m+n}$ *is piecewise continuous, and there exists* $\varepsilon > 0$ *such that for all* $t \in [-h,0]$ *we have*

$$M_{22}(t) \ge \varepsilon I,$$

where M *is partitioned as*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

with $M_{22} : [-h,0] \to \mathbb{S}^n$. *Then the following are equivalent:*

(i) *For all* $x \in \mathbb{R}^m$ *and continuous* $y : [-h,0] \to \mathbb{R}^n$

$$\int_{-h}^{0} \begin{bmatrix} x \\ y(t) \end{bmatrix}^T M(t) \begin{bmatrix} x \\ y(t) \end{bmatrix} dt \ge 0. \tag{6}$$

(ii) *There exists a function* $T : [-h, 0] \to \mathbb{S}^m$ *which is piecewise continuous and satisfies*

$$M(t) + \begin{bmatrix} T(t) & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \quad \textit{for all } t \in [-h, 0],$$

$$\int_{-h}^{0} T(t)\, dt = 0.$$

Proof. Again we need only show that (i) implies (ii). Suppose $x \in \mathbb{R}^n$, and define

$$f(t, z) = \begin{bmatrix} x \\ z \end{bmatrix}^T M(t) \begin{bmatrix} x \\ z \end{bmatrix} \quad \text{for all } t, z.$$

Since by the hypothesis M_{22} has a lower bound, it is invertible for all t and its inverse is piecewise continuous. Therefore $z(t) = -M_{22}(t)^{-1} M_{21}(t)x$ is the unique minimizer of $f(t, z)$ with respect to z. By the hypothesis (i), we have that for all $y \in C[-h, 0]$

$$\int_{-h}^{0} f\big(t, y(t)\big)\, dt \geq 0.$$

Hence by Lemma 4 there exists a function g such that

$$(7) \qquad \begin{aligned} g(t) + f(t, z) \geq 0 \quad & \text{for all } t, z, \\ \int_{-h}^{0} g(t)\, dt = 0. & \end{aligned}$$

The proof of Lemma 2 gives one such function as

$$g(t) = -f\big(t, z(t)\big) + \frac{1}{h} \int_{-h}^{0} f(s, z(s))\, dt.$$

We have

$$f\big(t, z(t)\big) = x^T \big(M_{11}(t) - M_{12}(t) M_{22}^{-1}(t) M_{21}(t)\big) x,$$

and therefore $g(t)$ is a quadratic function of x, say $g(t) = x^T T(t) x$, and $T : [-h, 0] \to \mathbb{S}^m$ is continuous on H^c. Then (7) implies

$$x^T T(t) x + \begin{bmatrix} x \\ z \end{bmatrix}^T M(t) \begin{bmatrix} x \\ z \end{bmatrix} \geq 0 \quad \text{for all } t, z, x,$$

as required. ∎

Notice that the strict positivity assumption on M_{22} in Theorem 5 is implied by the existence of an $\epsilon > 0$ such that

$$V(x) \geq \epsilon \|x\|_2^2,$$

where $\|\cdot\|_2$ denotes the L_2-norm.

We have now shown that the convex cone of functions M such that the first term of (3) is nonnegative is exactly equal to the sum of the cone of pointwise nonnegative functions and the linear space of functions whose integral is zero. The key benefit of this is that it is easy to parameterize the latter class of functions, and in particular when M is a polynomial these constraints are semidefinite representable constraints on the coefficients of M. Note that in (6) the vectors x and y are allowed to vary independently, whereas (3) requires that $x = y(0)$. It is, however, straightforward to show using the technique in the proof of Lemma 3 that this additional constraint does not change the result.

5. Polynomial matrices and kernels. In this paper we use piecewise polynomial matrices as a conveniently parameterized class of functions to represent the functions M and N defining the Lyapunov function (3) and its derivative. Theorem 5 has reduced nonnegativity of the first term of (3) to pointwise nonnegativity of a piecewise polynomial matrix in one variable.

We first make some definitions which we will use in this paper; some details on polynomial matrices may be found in [21] and [8]. We consider polynomials in n variables. As is standard, for $\alpha \in \mathbb{N}^n$ define the monomial in n variables x^α by $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We say M is a real polynomial matrix in n variables if for some finite set $W \subset \mathbb{N}^n$ we have

$$M(x) = \sum_{\alpha \in W} A_\alpha x^\alpha,$$

where A_α is a real matrix for each $\alpha \in W$. A convenient representation of polynomial matrices is as a quadratic function of monomials. Suppose z is a vector of monomials in the variables x, such as

$$z(x) = \begin{bmatrix} 1 \\ x_1 \\ x_1 x_2^2 \\ x_3^4 \end{bmatrix}.$$

For convenience, assume the length of z is $d+1$. Let $Q \in \mathbb{S}^{n(d+1)}$ be a symmetric matrix. Then the function M defined by

$$M(x) = (I_n \otimes z(x))^T Q (I_n \otimes z(x)) \tag{8}$$

is an $n \times n$ symmetric polynomial matrix, and every real symmetric polynomial matrix may be represented in this way for some monomial vector z. If we partition Q as

$$Q = \begin{bmatrix} Q_{11} & \dots & Q_{1n} \\ \vdots & & \vdots \\ Q_{n1} & \dots & Q_{nn} \end{bmatrix},$$

where each $Q_{ij} \in \mathbb{R}^{(d+1)\times(d+1)}$, then the i, j entry of M is

$$M_{ij}(x) = z(x)^T Q_{ij} z(x).$$

Given a polynomial matrix M, it is called a *sum of squares* if there exist a vector of monomials z and a positive semidefinite matrix Q such that (8) holds. In this case,

$$M(x) \succeq 0 \quad \text{for all } x,$$

and therefore the existence of such a Q is a sufficient condition for the polynomial M to be globally pointwise positive semidefinite. A matrix polynomial in one variable is pointwise nonnegative semidefinite on the real line if and only if it is a sum of squares; see [3]. Given a matrix polynomial $M(x)$, we can test whether it is a sum of squares by testing whether there is a matrix Q such that

$$\begin{aligned} M(x) &= (I_n \otimes z(x))^T Q (I_n \otimes z(x)), \\ Q &\succeq 0, \end{aligned} \tag{9}$$

where z is the vector of all monomials with degree half the degree of M. Equation (9) is interpreted as equality of polynomials, and equating their coefficients gives a finite set of linear constraints on the matrix Q. Therefore to find such a Q we need to find a positive semidefinite matrix subject to linear constraints, and this is therefore testable via semidefinite programming. See [25] for background on semidefinite programming.

5.1. Piecewise polynomial matrices. Define the intervals

$$H_i = \begin{cases} [-h_1, 0] & \text{if } i = 1, \\ [-h_i, -h_{i-1}) & \text{if } i = 2, \dots, k. \end{cases}$$

A matrix-valued function $M : [-h, 0] \to \mathbb{S}^n$ is called a *piecewise polynomial matrix* if for each $i = 1, \dots, k$ the function M restricted to the interval H_i is a polynomial matrix. We represent such piecewise polynomial matrices as follows. Define the vector of indicator functions $g : [-h, 0] \to \mathbb{R}^k$ by

$$g_i(t) = \begin{cases} 1 & \text{if } t \in H_i, \\ 0 & \text{otherwise} \end{cases}$$

for all $i = 1, \dots, k$ and all $t \in [-h, 0]$. Let $z(t)$ be the vector of monomials

$$z(t) = \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^d \end{bmatrix}$$

and for convenience also define the function $Z_{n,d} : [-h, 0] \to \mathbb{R}^{nk(d+1) \times n}$ by

$$Z_{n,d}(t) = g(t) \otimes I_n \otimes z(t).$$

Then it is straightforward to show that M is a piecewise matrix polynomial if and only if there exist matrices $Q_i \in \mathbb{S}^{n(d+1)}$ for $i = 1, \dots, k$ such that

$$M(t) = Z_{n,d}(t)^T \operatorname{diag}(Q_1, \dots, Q_k) Z_{n,d}(t). \tag{10}$$

The function M is pointwise positive semidefinite, i.e.,

$$M(t) \succeq 0 \quad \text{for all } t \in [-h, 0],$$

if there exist positive semidefinite matrices Q_i satisfying (10). We refer to such functions as *piecewise sum-of-squares matrices*, and we define the set of such functions

$$\Sigma_{n,d} = \Big\{ Z_{n,d}^T(t) Q Z_{n,d}(t) \;\Big|\; Q = \operatorname{diag}(Q_1, \dots, Q_k),\ Q_i \in \mathbb{S}^{n(d+1)}, Q_i \succeq 0 \Big\}.$$

If we are given a function $M : [-h, 0] \to \mathbb{S}^n$ which is piecewise polynomial and want to know whether it is piecewise sum of squares, then this is computationally checkable using semidefinite programming. Naturally, the number of variables involved in this task scales as $kn^2(d+1)^2$ when the degree of M is $2d$.

5.2. Piecewise polynomial kernels. We consider functions N of two variables s, t which we will use as a kernel in the quadratic form

$$\int_{-h}^{0} \int_{-h}^{0} \phi(s)^T N(s,t)\, \phi(t)\, ds\, dt \tag{11}$$

which appears in the Lyapunov function (3). A polynomial in two variables is referred to as a *binary* polynomial. A function $N : [-h, 0] \times [-h, 0] \to \mathbb{S}^n$ is called a *binary piecewise polynomial matrix* if for each $i, j \in \{1, \dots, k\}$ the function N restricted to the set $H_i \times H_j$ is a binary polynomial matrix. It is straightforward to show that N is a symmetric binary piecewise polynomial matrix if and only if there exists a matrix $Q \in \mathbb{S}^{nk(d+1)}$ such that

$$N(s,t) = Z_{n,d}^T(s) Q Z_{n,d}(t).$$

Here d is the degree of N, and recall that

$$Z_{n,d}(t) = g(t) \otimes I_n \otimes z(t).$$

We now proceed to characterize the binary piecewise polynomial matrices N for which the quadratic form (11) is nonnegative for all $\phi \in C([-h, 0], \mathbb{R}^n)$. We first state the following lemma.

LEMMA 6. *Suppose z is the vector of monomials*

$$z(t) = \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^d \end{bmatrix}$$

and the linear map $A : C[0,1] \to \mathbb{R}^{d+1}$ is given by

$$A\phi = \int_0^1 z(t)\phi(t)\, dt.$$

Then $\operatorname{rank} A = d + 1$.

Proof. Suppose for the sake of a contradiction that $\operatorname{rank} A < d+1$. Then $\operatorname{range} A$ is a strict subset of $\mathbb{R}^{d+1}$, and hence there exists a nonzero vector $q \in \mathbb{R}^{d+1}$ such that $q \perp \operatorname{range} A$. This means

$$\int_0^1 q^T z(t)\phi(t)\, dt = 0$$

for all $\phi \in C[0,1]$. Since $q^T z$ and ϕ are continuous functions, define the function $v : [0,1] \to \mathbb{R}$ by

$$v(t) = \int_0^t q^T z(s)\,ds \quad \text{for all } t \in [0,1].$$

Since v is absolutely continuous, we have for every $\phi \in C[0,1]$ that

$$\begin{aligned}\int_0^1 \phi(t)\,dv(t) &= \int_0^1 q^T z(t)\phi(t)\,dt \\ &= 0,\end{aligned}$$

where the integral on the left-hand side of the above equation is the Stieltjes integral. The function v is also of bounded variation, since its derivative is bounded. The Riesz representation theorem [20] implies that if v is of bounded variation and

$$\int_0^1 \phi(t)\,dv(t) = 0$$

for all $\phi \in C[0,1]$, then v is constant on an everywhere dense subset of $(0,1)$. Since v is continuous, we have that v is constant, and therefore $q^T z(t) = 0$ for all t. Since $q^T z$ is a polynomial, this contradicts the statement that $q \neq 0$. ∎

We now state the positivity result.

THEOREM 7. *Suppose N is a symmetric binary piecewise polynomial matrix of degree d. Then*

$$\int_{-h}^0 \int_{-h}^0 \phi(s)^T N(s,t)\phi(t)\,ds\,dt \geq 0 \tag{12}$$

for all $\phi \in C([-h,0],\mathbb{R}^n)$ if and only if there exists $Q \in \mathbb{S}^{nk(d+1)}$ such that

$$\begin{aligned}N(s,t) &= Z_{n,d}^T(s) Q Z_{n,d}(t), \\ Q &\succeq 0.\end{aligned}$$

Proof. We need only show the *only if* direction. Suppose N is a symmetric binary piecewise polynomial matrix. Let d be the degree of N. Then there exists a symmetric matrix Q such that

$$N(s,t) = Z_{n,d}^T(s) Q Z_{n,d}(t).$$

Now suppose that the inequality (12) is satisfied for all continuous functions ϕ. We will show that every such Q is positive semidefinite. To see this, define the linear map $J : C([-h,0],\mathbb{R}^n) \to \mathbb{R}^{nk(d+1)}$ by

$$J\phi = \int_{-h}^0 (g(t) \otimes I_n \otimes z(t))\phi(t)\,dt.$$

Then

$$\int_{-h}^0 \int_{-h}^0 \phi(s)^T N(s,t)\phi(t)\,ds\,dt = (J\phi)^T Q (J\phi).$$

The result we desire holds if $\operatorname{rank} J = nk(d+1)$, since in this case $\operatorname{range} J = \mathbb{R}^{nk(d+1)}$. If Q has a negative eigenvalue with corresponding eigenvector q, then there exists ϕ such that $q = J\phi$ so that the quadratic form will be negative, contradicting the hypothesis.

To see that $\operatorname{rank} J = nk(d+1)$, define for each $i = 1, \ldots, k$ the linear map $L_i : C[H_i] \to \mathbb{R}^n$ by

$$L_i \phi = \int_{H_i} z(t)\phi(t)\, dt.$$

If we choose coordinates for ϕ such that

$$\phi = \begin{bmatrix} \phi|_{H_1} \\ \phi|_{H_2} \\ \vdots \\ \phi|_{H_k} \end{bmatrix},$$

where $\phi|_{H_j}$ is the restriction of ϕ to the interval H_j, then we have in these coordinates that J is

$$J = \operatorname{diag}(L_1, \ldots, L_k) \otimes I_n.$$

Further, by Lemma 6 the maps L_i each satisfy $\operatorname{rank} L_i = d+1$. Therefore $\operatorname{rank} J = nk(d+1)$, as desired. □

The following corollary gives a tighter degree bound on the representation of N.

COROLLARY 8. *Let N be a binary piecewise polynomial matrix of degree $2d$ which is positive in the sense of (12); then there exists $Q \in \mathbb{S}^{nk(d+1)}$ such that*

$$N(s,t) = Z_{n,d}^T(s) Q Z_{n,d}(t),$$

$$Q \succeq 0.$$

Proof. The binary representation used in Theorem 7 had the form

$$N(s,t) = Z_{n,2d}^T(s) P Z_{n,2d}(t),$$

$$P \succeq 0,$$

where $P \in \mathbb{S}^{nk(2d+1)}$. However, in any such representation, it is clear that $P_{ij,ij} = 0$ for $i = d+2, \ldots, 2d+1$ and $j = 1, \ldots, kn$. Therefore, since $P \succeq 0$, these rows and columns are 0 and can be removed. Define Q to be the reduction of P. $Z_{n,d}$ is the corresponding reduction of $Z_{n,2d}$. Then $Q \in \mathbb{S}^{nk(d+1)}$, $Q \succeq 0$, and

$$N(s,t) = Z_{n,2d}^T(s) P Z_{n,2d}(t) = Z_{n,d}^T(s) Q Z_{n,d}(t). \qquad \square$$

For convenience, we define the set of symmetric binary piecewise polynomial matrices which define positive quadratic forms by

$$\Gamma_{n,d} = \left\{\, Z_{n,d}^T(s) Q Z_{n,d}(t) \mid Q \in \mathbb{S}^{nk(d+1)}, Q \succeq 0 \,\right\}.$$

As for $\Sigma_{n,d}$, if we are given a binary piecewise polynomial matrix $N : [-h,0] \times [-h,0] \to \mathbb{S}^n$ of degree $2d$ and want to know whether it defines a positive quadratic form, then this is computationally checkable using semidefinite programming. The number of variables involved in this task scales as $(nk)^2(d+1)^2$.

6. Derivatives of the Lyapunov function. In this section we will take the opportunity to define the relationship between the functions M and N, which define the Lyapunov function V, and the functions D and E, which define the derivative of the Lyapunov function along trajectories of the system. As the results are well known, we will not give detailed derivations for these derivatives. More expository explanations can be found in, e.g., [4] or [13].

6.1. Single delay case. We first present the single delay case, as it will illustrate the formulation in the more complicated case of several delays. Suppose that $V \in X$ is given by (3), where $M : [-h, 0] \to \mathbb{S}^{2n}$ and $N : [-h, 0] \times [-h, 0] \to \mathbb{R}^{n\times n}$. Since there is only one delay, if the system is exponentially stable, then there always exists a Lyapunov function of this form with continuous functions M and N. Then the Lie derivative of V is

$$\dot{V}(\phi) = \int_{-h}^{0} \begin{bmatrix} \phi(0) \\ \phi(-h) \\ \phi(s) \end{bmatrix}^T D(s) \begin{bmatrix} \phi(0) \\ \phi(-h) \\ \phi(s) \end{bmatrix} ds + \int_{-h}^{0}\int_{-h}^{0} \phi(s)^T E(s,t)\phi(t)\, ds\, dt.$$

Partition D and M as

$$M(t) = \begin{bmatrix} M_{11} & M_{12}(t) \\ M_{21}(t) & M_{22}(t) \end{bmatrix}, \quad D(t) = \begin{bmatrix} D_{11} & D_{12}(t) \\ D_{21}(t) & D_{22}(t) \end{bmatrix}$$

so that $M_{11} \in \mathbb{S}^n$ and $D_{11} \in \mathbb{S}^{2n}$. Without loss of generality we have assumed that M_{11} and D_{11} are constant. The functions D and E are linearly related to M and N by

$$\begin{aligned}
D_{11} &= \begin{bmatrix} A_0^T M_{11} + M_{11}A_0 & M_{11}A_1 \\ A_1^T M_{11} & 0 \end{bmatrix} \\
&\quad + \frac{1}{h}\begin{bmatrix} M_{12}(0) + M_{21}(0) & -M_{12}(-h) \\ -M_{21}(-h) & 0 \end{bmatrix} \\
&\quad + \frac{1}{h}\begin{bmatrix} M_{22}(0) & 0 \\ 0 & -M_{22}(-h) \end{bmatrix}, \\
D_{12}(t) &= \begin{bmatrix} A_0^T M_{12}(t) - \dot{M}_{12}(t) + N(0,t) \\ A_1^T M_{12}(t) - N(-h,t) \end{bmatrix}, \\
D_{22}(t) &= -\dot{M}_{22}(t), \\
E(s,t) &= \frac{\partial N(s,t)}{\partial s} + \frac{\partial N(s,t)}{\partial t}.
\end{aligned}$$

6.2. Multiple-delay case. Recall that we define the intervals

$$H_i = \begin{cases} [-h_1, 0] & \text{if } i = 1, \\ [-h_i, -h_{i-1}) & \text{if } i = 2, \dots, k. \end{cases}$$

We first give the complete class of functions which define the Lyapunov function, V, and its derivative:

$$\begin{aligned} Y_1 = \Big\{ M : [-h, 0] \to \mathbb{S}^{2n} \mid \\ & M_{11}(t) \text{ is constant} && \text{for all } t \in [-h, 0], \\ & M \text{ is } C^1 \text{ on } H_i && \text{for all } i = 1, \dots, k \Big\}, \end{aligned}$$

$$\begin{aligned} Y_2 = \Big\{ N : [-h, 0] \times [-h, 0] \to \mathbb{S}^{n} \mid \\ & N(s,t) = N(t,s)^T && \text{for all } s, t \in [-h, 0], \\ & N \text{ is } C^1 \text{ on } H_i \times H_j && \text{for all } i, j = 1, \dots, k \Big\}, \end{aligned}$$

and, for its derivative, define

$$\begin{aligned} Z_1 = \Big\{ D : [-h, 0] \to \mathbb{S}^{(k+2)n} \mid \\ & D_{ij}(t) \text{ is constant} && \text{for all } t \in [-h, 0] \\ & && \text{for } i, j = 1, \dots, 3, \\ & D \text{ is } C^0 \text{ on } H_i && \text{for all } i = 1, \dots, k \Big\}, \end{aligned}$$

$$\begin{aligned} Z_2 = \Big\{ E : [-h, 0] \times [-h, 0] \to \mathbb{S}^{n} \mid \\ & E(s,t) = E(t,s)^T && \text{for all } s, t \in [-h, 0], \\ & E \text{ is } C^0 \text{ on } H_i \times H_j && \text{for all } i, j = 1, \dots, k \Big\}. \end{aligned}$$

Here $M \in Y_1$ is partitioned according to

$$M(t) = \begin{bmatrix} M_{11} & M_{12}(t) \\ M_{21}(t) & M_{22}(t) \end{bmatrix}, \tag{13}$$

where $M_{11} \in \mathbb{S}^n$ and $D \in Z_1$ are partitioned according to

$$D(t) = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14}(t) \\ D_{21} & D_{22} & D_{23} & D_{24}(t) \\ D_{31} & D_{32} & D_{33} & D_{34}(t) \\ D_{41}(t) & D_{42}(t) & D_{43}(t) & D_{44}(t) \end{bmatrix}, \tag{14}$$

where D_{11}, D_{33}, $D_{44} \in \mathbb{S}^n$ and $D_{22} \in \mathbb{S}^{(k-1)n}$. Let $Y = Y_1 \times Y_2$ and $Z = Z_1 \times Z_2$. Notice that if $M \in Y_1$, then M need not be continuous at h_i for $1 \le i \le k-1$; however, we require it to be right continuous at these points. We also define the derivative $\dot{M}(t)$ at these points to be the right-hand derivative of M. We define the continuity and derivatives of functions in Y_2, Z_1, and Z_2 similarly.

We define the jump values of M and N at the discontinuities as follows:

$$\Delta M(h_i) = \lim_{t \to (-h_i)_+} M(t) - \lim_{t \to (-h_i)_-} M(t)$$

for each $i = 1, \dots, k-1$, and similarly we define

$$\Delta N(h_i, t) = \lim_{s \to (-h_i)_+} N(s,t) - \lim_{s \to (-h_i)_-} N(s,t).$$

The derivative of a Lyapunov function can be defined as a linear map $Y \mapsto Z$. This is made explicit in the following definition.

DEFINITION 9. *Define the map $L : Y \to Z$ by $(D, E) = L(M, N)$ if for all $t, s \in [-h, 0]$ we have*

$$\begin{aligned}
D_{11} &= A_0^T M_{11} + M_{11} A_0 \\
&\qquad + \frac{1}{h}\big(M_{12}(0) + M_{21}(0) + M_{22}(0)\big), \\
D_{12} &= \begin{bmatrix} M_{11}A_1 & \cdots & M_{11}A_{k-1} \end{bmatrix} \\
&\qquad - \begin{bmatrix} \Delta M_{12}(h_1) & \cdots & \Delta M_{12}(h_{k-1}) \end{bmatrix}, \\
D_{13} &= \frac{1}{h}\big(M_{11}A_k - M_{12}(-h)\big), \\
D_{22} &= \frac{1}{h}\operatorname{diag}\big(-\Delta M_{22}(h_1), \dots, -\Delta M_{22}(h_{k-1})\big), \\
D_{23} &= 0, \\
D_{33} &= -\frac{1}{h} M_{22}(-h), \\
D_{14}(t) &= N(0,t) + A_0^T M_{12}(t) - \dot{M}_{12}(t), \\
D_{24}(t) &= \begin{bmatrix} \Delta N(-h_1, t) + A_1^T M_{12}(t) \\ \vdots \\ \Delta N(-h_{k-1}, t) + A_{k-1}^T M_{12}(t) \end{bmatrix}, \\
D_{34}(t) &= A_k^T M_{12}(t) - N(-h,t), \\
D_{44}(t) &= -\dot{M}_{22}(t)
\end{aligned}$$

and

$$E(s,t) = \frac{\partial N(s,t)}{\partial s} + \frac{\partial N(s,t)}{\partial t}.$$

Here M is partitioned as in (13)*, D is partitioned as in* (14)*, and the remaining entries are defined by symmetry.*

The map L is the Lie derivative operator applied to the set of functions specified by (3); this is stated precisely below. Notice that this implies that L is a linear map.

LEMMA 10. *Suppose $M \in Y_1$ and $N \in Y_2$ and V is given by* (3). *Let $(D, E) = L(M, N)$. Then the Lie derivative of V on the vector field of* (1) *is given by*

$$\dot{V}(\phi) = \int_{-h}^{0} \begin{bmatrix} \phi(-h_0) \\ \vdots \\ \phi(-h_k) \\ \phi(s) \end{bmatrix}^T D(s) \begin{bmatrix} \phi(-h_0) \\ \vdots \\ \phi(-h_k) \\ \phi(s) \end{bmatrix} ds + \int_{-h}^{0}\int_{-h}^{0} \phi(s)^T E(s,t)\phi(t)\, ds\, dt. \tag{15}$$

7. Stability conditions. We can now use the results of the paper and the linear map from Definition 9 to give stability conditions.

THEOREM 11. *Suppose there exist $d \in \mathbb{N}$ and piecewise matrix polynomials M, T, N, D, U, E such that*

$$M + \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} \in \Sigma_{2n,d},$$

$$-D + \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} \in \Sigma_{(k+2)n,d},$$

$$N \in \Gamma_{n,d},$$

$$-E \in \Gamma_{n,d},$$

$$(D, E) = L(M, N),$$

$$\int_{-h}^{0} T(s)\, ds = 0,$$

$$\int_{-h}^{0} U(s)\, ds = 0,$$

$$M_{11} \succ 0,$$

$$D_{11} \prec 0.$$

Then the system defined by (1) *and* (2) *is exponentially stable.*

Proof. Assume M, T, N, D, U, E satisfy the above conditions, and define the function V by (3). Then Lemma 10 implies that $\dot{V}$ is given by (15). The function V is the sum of two terms, each of which is nonnegative. The first is nonnegative by Theorem 5, and the second is nonnegative since $N \in \Gamma_{n,d}$. The same is true for $\dot{V}$. The strict positivity conditions of equations (4) hold since $M_{11} \succ 0$ and $-D_{11} \succ 0$, and Theorem 1 then implies stability. ∎

The feasibility conditions specified in Theorem 11 are semidefinite representable. In particular the condition that a piecewise polynomial matrix lie in Σ is a set of linear and positive semidefinite constraints on its coefficients. Similarly, the condition that T and U integrate to zero is simply a linear equality constraint on its coefficients. Standard semidefinite programming codes may therefore be used to efficiently find such piecewise polynomial matrices. Most such codes will also return a dual certificate of infeasibility if no such polynomials exist.

As in the Lyapunov analysis of nonlinear systems using sum-of-squares polynomials, the set of candidate Lyapunov functions is parameterized by the degree d. This allows one to search first over polynomials of low degree and increase the degree if that search fails.

There are various natural extensions of this result. The first is to the case of uncertain systems, where we would like to prove stability for all matrices A_i in some given semialgebraic set. This is possible by extending Theorem 11 to allow Lyapunov functions which depend polynomially on unknown parameters. A similar approach may be used to check stability for systems with uncertain delays. Additionally, stability of systems with distributed delays defined by polynomial kernels can be verified. It is also straightforward to extend the class of Lyapunov functions, since it is not necessary that each piece of the piecewise sums-of-squares functions be nonnegative on the whole real line. To do this, one can use techniques for parameterizing polynomials nonnegative on an interval; for example, every polynomial $p(x) = f(x)-(x-1)(x-2)g(x)$ where f and g are sums of squares is nonnegative on the interval $[1, 2]$.

8. Numerical examples. In this section we present the results of some example computations using the approach described above. The computations were performed using MATLAB software, together with the SOSTOOLS [18] toolbox and SeDuMi [22] code for solving semidefinite programming problems.

8.1. Illustration. Consider the process of proving stability using the results of this paper. The following system has well-known stability properties:

$$\dot{x}(t) = -x(t-1). \tag{16}$$

A MATLAB implementation of the algorithm in this paper has been developed and is available online, along with several tools for polynomial matrix manipulation [17]. This implementation returns the following Lyapunov function for system (16). For symmetric matrices, subdiagonal elements are suppressed:

$$V(x) = \int_{-1}^{0} \begin{bmatrix} x(0) \\ x(s) \end{bmatrix}^T M(s) \begin{bmatrix} x(0) \\ x(s) \end{bmatrix} ds + \int_{-1}^{0}\int_{-1}^{0} x(s)^T R(s,t) x(t)\, ds\, dt,$$

where

$$M(s) = \begin{bmatrix} 27.3 & -16.8 + 2.74s \\ & 24.3 + 8.53s \end{bmatrix}$$

and

$$R(s,t) = 9.08.$$

Positivity is proven using the function

$$t(s) = -.915 + 1.83s$$

and the sum-of-squares functions

$$Q(s) = \begin{bmatrix} 13 & -3.3 \\ & 12.2 \end{bmatrix} \geq 0$$

and

$$V(s) = Z(s)^T L\, Z(s),$$

where

$$Z(s) = \begin{bmatrix} 1 & s & 0 & 0 \\ 0 & 0 & 1 & s \end{bmatrix}^T$$

and

$$L = \begin{bmatrix} 28.215 & 5.585 & -16.8 & -1.973 \\ & 13 & 1.413 & -3.3 \\ & & 24.3 & 10.365 \\ & & & 12.2 \end{bmatrix} \geq 0.$$

This is because $-s(s+1) \geq 0$ for $s \in [-1, 0]$ and

$$M(s) + \begin{bmatrix} t(s) & 0 \\ 0 & 0 \end{bmatrix} = -s(s+1)Q(s) + V(s).$$

Furthermore,

$$R(s) = 9.08 \geq 0.$$

Therefore, by Theorems 11 and 5, the Lyapunov function is positive.

The derivative of the function is given by

$$\dot{V}(x) = \int_{-1}^{0} \begin{bmatrix} x(0) \\ x(-1) \\ x(s) \end{bmatrix}^T D(s) \begin{bmatrix} x(0) \\ x(-1) \\ x(s) \end{bmatrix} ds,$$

where

$$-D(s) = \begin{bmatrix} 9.3 & 7.76 & -6.34 \\ & 15.77 & -7.72 + 2.74s \\ & & 8.53 \end{bmatrix}.$$

Negativity of the function is proven using the function

$$U(s) = \begin{bmatrix} .0055 + .011s & -.272 - .544s \\ & -.458 - .916s \end{bmatrix},$$

where

$$\int_{-1}^{0} U(s)\, ds = 0,$$

and the sum-of-squares functions

$$X(s) = \begin{bmatrix} 8.86 & 1.90 & -3.23 \\ & 11.54 & -3.71 \\ & & 7.74 \end{bmatrix}$$

and

$$Y(s) = Z(s)^T L\, Z(s),$$

where

$$Z(s) = \begin{bmatrix} 1 & s & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & s & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & s \end{bmatrix}^T$$

and

$$L = \begin{bmatrix} 9.3 & 4.43 & 7.76 & .61896 & -6.34 & -1.0371 \\ & 8.86 & 1.281 & 1.9 & -2.1929 & -3.23 \\ & & 15.77 & 5.77 & -7.72 & .01171 \\ & & & 11.54 & -.98171 & -3.71 \\ & & & & 8.53 & 3.87 \\ & & & & & 7.74 \end{bmatrix} \geq 0.$$

Negativity follows since

$$-D(s) + \begin{bmatrix} U(s) & 0 \\ 0 & 0 \end{bmatrix} = -s(s+1)X(s) + Y(s).$$

Therefore, by Theorem 11, the derivative of the Lyapunov function is negative. Stability follows by Theorem 1.

8.2. A single delay. We consider the following instance of a system with a single delay:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-h).$$

For a given h, we use semidefinite programming to search for a Lyapunov function of degree d that proves stability. Using a bisection search over h, we determine the maximum and minimum h for which the system may be shown to be stable. These are shown below.

d	h_{min}	h_{max}
1	.10017	1.6249
2	.10017	1.7172
3	.10017	1.71785

When the degree $d = 3$, the bounds h_{min} and h_{max} are tight [4]. For comparison, we include here the bounds obtained by Gu, Kharitonov, and Chen [4] using a piecewise linear Lyapunov function with n segments.

n	h_{min}	h_{max}
1	.1006	1.4272
2	.1003	1.6921
3	.1003	1.7161

8.3. Multiple delays. Consider the system with two delays below:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & \frac{1}{10} \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x(t - \tfrac{h}{2}) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-h).$$

As above, using a bisection search over h, we prove stability for the range of h below.

d	$h_{\min}$	$h_{\max}$
1	.20247	1.354
2	.20247	1.3722

Here for degree $d = 2$, the bounds obtained are tight. Again we include here bounds obtained by Gu, Kharitonov, and Chen [4] using a piecewise linear Lyapunov function with n segments.

n	$h_{\min}$	$h_{\max}$
1	.204	1.35
2	.203	1.372

9. Summary. In this paper we developed an approach for computing Lyapunov functions for linear systems with delay. In general this is a difficult computational problem, and certain specific classes of this problem are known to be NP-hard [24]. However, the set of Lyapunov functions is convex, and this enables us to effectively test feasibility of a subset of this set. Specifically, we parameterize a convex set of positive quadratic functions using the set of polynomials as a basis, and the main results here are Theorems 5 and 7. Combining these results with the well-known approach using sum-of-squares polynomials allows one to use standard semidefinite programming software to compute Lyapunov functions. This gives a nested sequence of computable sufficient conditions for stability of linear delay systems, indexed by the degree of the polynomial. In principle this enables searching over increasing degrees to find a Lyapunov function, although further work is needed to enhance existing semidefinite programming codes to make this more efficient in practice.

It is possible that Theorems 5 and 7 are applicable more widely, specifically to stability analysis of nonlinear and partial differential systems, as well as to controller synthesis. One specific extension that is possible is analysis of delay systems with uncertain parameters, for which sufficient conditions for existence of a Lyapunov function may be given using convex relaxations. It is also possible to analyze stability of nonlinear delay systems, in the case that the dynamics are defined by polynomial delay differential equations. Preliminary work has also been done on stability analysis of certain types of partial differential equations. Further extensions to allow synthesis of stabilizing controllers are of interest and may be possible.

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