Inverses of Positive Linear Operators and State Feedback Design for Time-Delay Systems ⋆

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Abstract: The problem of designing feedback controllers for dynamical systems with time-delay is addressed in this paper. Previous work has imposed significant restrictions on the structure of the candidate Control Lyapunov Functions in order to develop appropriate LMI conditions for the design. This paper addresses this issue and provides two new results. The first result is a step towards controller synthesis using the "complete quadratic" Lyapunov functional. Specifically, given such a "complete quadratic" functional, defined by polynomials, we give an algorithm for constructing the inverse of the linear operator which defines that functional. Following this, we derive semidefinite programming conditions, expressed as a Sum-of-Squares program, for state feedback synthesis of these systems using a restricted structure of the Lyapunov functional.

Keywords: Delay, Lyapunov, Polynomials, Controller Synthesis, Sum-of-Squares

1. INTRODUCTION

Models containing time-delays are very frequently used to describe systems that involve transport and propagation of data in, e.g., communication networks (Srikant [2003]), or systems that have an aftereffect (Kuang [1993]), e.g., population dynamics. The analysis and control of these systems has attracted significant attention in the past few years and several approaches have been developed for addressing these questions. There is a wealth of modelling frameworks (e.g., distributed/discrete/time-varying/multiple delays) and a series of analysis and design tools for linear model descriptions (Lyapunov Razumikhin, Lyapunov Krasovskii, frequency-domain methods) that can be used to address either delay-independent or delay-dependent analysis questions.

What complicates the analysis and design of time-delay systems is their infinite dimensional nature. For example, the stability analysis of a linear system of the form

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau)$$

by constructing a Lyapunov function is not an easy task, despite the fact that the structure of a complete Lyapunov function is known (Gu, Kharitonov, and Chen [2003]). If $A_1 = 0$, the problem becomes finite-dimensional and stability is equivalent to finding a $P > 0$ for which $A^T P + PA < 0$, i.e., by solving a semidefinite programme – the corresponding Lyapunov function is $V(x) = x^T P x$. Semidefinite programming can be also used to test the stability of the time-delayed system, but the functional $V(x_t)$ is more complicated, and it is not until recently that methods have been developed to construct it: one involving a discretization of the Lyapunov functional as in Gu [1997] and one using Sum of Squares as in Peet, Papachristodoulou, and Lall [2009].

For the more interesting problem of synthesizing state feedback controllers for a system of the form

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + Bu_t,$$

there is currently not an easy, algorithmic way to construct a $V(x_t)$ that acts as a control Lyapunov functional for which no special constraints are imposed on its structure. Recall that in the finite dimensional case,

$$\dot{x}(t) = A_0 x(t) + B_0 u(t),$$

the controller $u(t) = Kx(t)$ can be designed by finding $Q > 0$ and R such that $QA_0^T + A_0 Q + B_0 R + R^T B_0^T < 0$, in which case we have that $K = RQ^{-1}$. The control Lyapunov function in this case is $V(x_t) = x^T Q^{-1} x$, but unless the resulting condition $A_0^T Q^{-1} + Q^{-1} A_0 + Q^{-1} B_0 K + K^T B^T Q^{-1} < 0$ is manipulated by pre- and post-multiplication by Q, and renaming $KQ = R$ it is not clear how this is an LMI (Boyd, El Ghaoui, Feron, and Balakrishnan [1994]). Following the same steps for the

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time-delay case is not easy without reducing the dimension of the problem or imposing several constraints on the structure of the Lyapunov functional and/or the controller.

In this paper we consider the problem of state feedback control design for linear time-delay systems from a different point of view, which allows the construction of a control Lyapunov function. In particular, we construct the inverse to the operator that defines an appropriate Lyapunov functional. We also use a state transformation that allows for synthesis of stabilizing controllers as a Linear Matrix Inequality. Note that although we do not impose constraints on the structure of the controller, we do impose constraints on the structure of the Lyapunov functional. Thus the results here do not take full advantage of the inverse operators discussed above. This shortcoming will be addressed in future work.

The paper is organized as follows: In Section 2 we review the literature on analysis and feedback design for time-delay systems and how sum of squares methods can be used to address the analysis question. In Section 3 we show how the inverse to a linear operator relevant to time-delay systems can be constructed and in Section 4 we present our approach for state feedback synthesis. The paper in concluded in Section 5.

1.1 Notation

The notation we use is standard and can be found in Hale and Lunel [1993]. $\mathbb{R}$ denotes the reals and $\mathbb{R}^n$ the $n-$dimensional Euclidean space with norm $|\cdot|$. $\mathcal{S}^n \subset \mathbb{R}^{n\times n}$ denotes the subspace of symmetric matrices. $C_n = C([a,b],\mathbb{R}^n)$ denotes the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. For $[a,b]=[-\tau,0]$ we designate the norm of an element $\phi \in C([-\tau,0],\mathbb{R}^n)$ by $\|\phi\| = \sup_{-\tau\leq\theta\leq 0}|\phi(\theta)|$. Given $\sigma \in \mathbb{R}$ and $A \geq 0$ and $x \in C([\sigma-\tau,\sigma+A],\mathbb{R}^n)$ then for any $t \in [\sigma,\sigma+A]$ we let $x_t \in C$ be defined by $x_t(\theta)=x(t+\theta)$, $-\tau\leq\theta\leq 0$.

2. BACKGROUND AND PROBLEM STATEMENT

2.1 Problem Statement and Background Research

There have been various approaches for state feedback controller synthesis for time-delay systems, based on whether the feedback considered is instantaneous (memoryless) or contains delayed information (memory); whether the stabilization is for specific delay sizes (delay-dependent) or not (delay-independent); whether a cost function is being minimized (optimal/guaranteed control) or the pure stabilization problem is being considered; and whether the systems considered are linear or nonlinear, uncertain or not. Memory controllers are a more natural choice for feedback control, as time-delay systems are infinite dimensional. Such controllers can achieve better performance than memoryless controllers; in some cases memoryless controllers are incapable of stabilizing the system.

There is a series of recent papers concerned with the design of state feedback controllers for robust feedback stabilization of linear time delay systems, see for example, Li, Niculescu, Dugard, and Dion [1997] or Xia and Jia [2003]. Similar results were obtained for output feedback compensators (Haddad, Kapila, and Abdallah [1997]). As far as optimal control is concerned, controllers for robust optimal control of linear time delay systems have been developed, such as H_∞ (Azuma, Sagara, Fujita, and Uchida [2003], de Souza and Li [1999], Niculescu [1998]) and with guaranteed cost (Lee, Moon, and Kwon [2001], Moheimani and Petersen [1997], Nian and Feng [2003]). Some of the above methods take the size of the delay into account during the controller synthesis (delay-dependent stabilization), and some do not (delay-independent stabilization). In Gu and Han [2000], the authors consider the construction of Lyapunov Krasovskii functionals using the discretization approach proposed in Gu [1997], by solving the resulting infinite dimensional LMIs.

In this paper we consider a linear time-delay system of the form

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau) + u_t$$

for which we want to construct a controller u_t which consists of three terms,

$$u_t = B_1u_1(t) + B_2u_2(t) + \int_{-\tau}^{0} B_3(s)u_3(t,s)ds$$

The way we do that is by proposing semidefinite programming based conditions, that guarantee that a controller of the above general form is stabilizing.

2.2 Polynomial computing using SOS

Polynomial computing refers to the manipulation of polynomials by computational algorithms. Optimization of polynomials - in particular positive polynomials - has been a rapidly developing field in recent years. The most serious problem with computation using polynomials as opposed to matrices is that there is no exact way of telling when a polynomial is positive. Computationally, the question of polynomial positivity is intractable (NP-hard). However, there are several classical alternatives that we can use. The alternative that has been most popular lately has been to consider squared polynomials. This approach is extremely attractive because there is a one-to-one correspondence between positive matrices and squared polynomials. This encourages us to think that some of the methods developed for positive matrices can also be used for positive polynomials.

In this paper, we present the controller synthesis problem as a polynomial feasibility problem where the variables are matrices and polynomial matrices and the constraints are equality constraints and linear polynomial positivity constraints. These can be thought of as the polynomial equivalent of semidefinite programming. To implement the solutions presented in this paper, we recommend the SOSTOOLS package for MATLAB (Prajna, Papachristodoulou, and Parrilo [2002]) which implements the sum-of-squares approach to polynomial positivity.

2.3 Testing Lyapunov-Krasovskii functional conditions

For the autonomous system of the form

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau)$$

the analysis question has been addressed in Peet et al. [2009]. There, the so-called "complete Lyapunov-Krasovskii functional,"

$$V(x_t) = \int_{-\tau}^{0} \begin{bmatrix} x(t) \\ x(t+s) \end{bmatrix}^T \begin{bmatrix} M_{11} & \tau M_{12}(s) \\ \tau M_{12}^T(s) & \tau M_{22}(s) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t+s) \end{bmatrix} ds + \int_{-\tau}^{0}\int_{-\tau}^{0} x^T(t+s) N(s,w) x(t+w) ds dw$$

was constructed using semidefinite programming. The derivative condition is of the form

$$\dot V(x_t) = \int_{-\tau}^{0} \begin{bmatrix} x(t) \\ x(t-\tau) \\ x(t+s) \end{bmatrix}^T R \begin{bmatrix} x(t) \\ x(t-\tau) \\ x(t+s) \end{bmatrix} ds - \int_{-\tau}^{0}\int_{-\tau}^{0} x^T(t+s)\Big(\frac{\partial N(s,w)}{\partial s} + \frac{\partial N(s,w)}{\partial w} \Big) x(t+w) ds dw$$

where the matrix R is given by (1). The negativity of this can also be tested using Linear Matrix Inequalities. In particular, the following results are known: Any piecewise continuous function

$$\int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T \begin{bmatrix} M_{11} & \tau M_{12}(s) \\ \tau M_{12}^T(s) & \tau M_{22}(s) \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds$$

is positive for all ϕ if and only if there exists a piecewise continuous matrix-valued function T such that

$$\begin{bmatrix} M_{11} + T(s) & \tau M_{12}(s) \\ \tau M_{12}^T(s) & \tau M_{22}(s) \end{bmatrix} \geq 0 \text{ for all s,}$$

$$\int_{-\tau}^{0} T(s) ds = 0.$$

At the same time, the quadratic form

$$\int_{-\tau}^{0}\int_{-\tau}^{0} x^T(t+s) N(s,w) x(t+w) ds dw$$

is positive if and only if there exists a positive semidefinite matrix Q such that $N(s,w) = Z(s)^T Q Z(w)$, where Z is a vector of monomials.

3. THE INVERSE OF A LINEAR OPERATOR

In the previous section, we introduced the complete quadratic Lyapunov function. As is typically the case for any positive form, the set of quadratic Lyapunov functionals is convex. Whereas for linear finite dimensional systems the Lyapunov function was defined by a positive semidefinite matrix as $V(x) = x^T Q x$ with $Q > 0$, for time-delay systems, the complete-quadratic Lyapunov function can be defined by a positive operator,

$$A : \begin{bmatrix} x(0) \\ x(s) \end{bmatrix} \mapsto M(s) \begin{bmatrix} x(0) \\ x(s) \end{bmatrix} + \int_{-\tau}^{0} \begin{bmatrix} 0 & 0 \\ 0 & N(s,\theta) \end{bmatrix} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta,$$

with $V(x) = \langle x, Ax \rangle_2$ defined using the L_2 inner product. For linear finite-dimensional systems, the convexification of the synthesis process is achieved through the use of the composite variable $R = KQ$, with controller K realized as $K = RQ^{-1}$. For time-delay systems, a similar process of convexification can be used, but such an approach requires a method to invert the positive operator A.

In this section, we show that when the operator, A, is defined as above for a polynomial function M and polynomial N which meet the conditions of Subsection 2.3, then we can construct an inverse operator $\hat A$ which is defined by continuous functions $\hat M$ and $\hat N$.

Theorem 1. Consider the linear operator A defined by

$$Ax(s) = M(s)x(s) + \int_I \begin{bmatrix} 0 & 0 \\ 0 & N(s,\theta) \end{bmatrix} x(\theta) d\theta,$$

where $M(s) > 0$ for all $s \in I$ and N has a representation $N(s,\theta) = Z(s)^T R Z(\theta)$ where Z is a vector of basis functions and $R > 0$. Define the linear operator $\hat A$ by

$$\hat A x(s) = M(s)^{-1} x(s) + \int_I \hat N(s,\theta) x(\theta) d\theta$$

Where

$$\hat N(s,\theta) = M(s)^{-1} \begin{bmatrix} 0 & 0 \\ 0 & Z(s)^T Q Z(\theta) \end{bmatrix} M(\theta)^{-1}$$

$$Q = -R(S^{-1} + R)^{-1} S^{-1}$$

$$S = \int_I Z(s) M_{22}(s)^{-1} Z(s)^T ds.$$

Then $\hat A A x = A \hat A x = x$ for any integrable function x.

Proof. The proof is through direct substitution.

$$(\hat A A x)(s) = P(s)^{-1} P(s) x(s) + \int_I P(s)^{-1} \begin{bmatrix} 0 & 0 \\ 0 & N(s,\theta) \end{bmatrix} x(\theta) d\theta + \int_I \hat N(s,\theta) P(\theta) x(\theta) d\theta + \int_I \int_I \hat N(s,\nu) \begin{bmatrix} 0 & 0 \\ 0 & N(\nu,\theta) \end{bmatrix} x(\theta) d\nu d\theta.$$

And so,

$$(\hat A A x)(s) = x(s) + \int_I P(s)^{-1} \begin{bmatrix} 0 & 0 \\ 0 & Z(s)^T R Z(\theta) \end{bmatrix} x(\theta) d\theta + \int_I P(s)^{-1} \begin{bmatrix} 0 & 0 \\ 0 & Z(s)^T Q Z(\theta) \end{bmatrix} P(\theta)^{-1} P(\theta) x(\theta) d\theta + \int_I \int_I P(s)^{-1} \begin{bmatrix} 0 & 0 \\ 0 & Z(s)^T Q Z(\nu) \end{bmatrix} \cdot P(\nu)^{-1} \begin{bmatrix} 0 & 0 \\ 0 & Z(\nu)^T R Z(\theta) \end{bmatrix} x(\theta) d\nu d\theta$$

$$= x(s) + \int_I P(s)^{-1} \begin{bmatrix} 0 & 0 \\ 0 & D(s,\theta) \end{bmatrix} x(\theta) d\theta,$$

where

$$\begin{aligned} D(s,\theta) &= Z(s)^T (Q + R) Z(\theta) \\ &\quad + Z(s)^T \left(Q \left(\int_I Z(\nu) P_{22}(\nu)^{-1} Z(\nu)^T d\nu \right) R \right) Z(\theta) \\ &= Z(s)^T (QSR + Q + R) Z(\theta) \\ &= Z(s)^T (Q(I + SR) + R) Z(\theta) \\ &= Z(s)^T \big(-R(S^{-1} + R)^{-1} S^{-1} (I + SR) + R \big) Z(\theta) \\ &= Z(s)^T \big(-R(S^{-1} + R)^{-1} (S^{-1} + R) + R \big) Z(\theta) \\ &= Z(s)^T (-R + R) Z(\theta) = 0 \end{aligned}$$

Thus

$$(\hat A A x)(s) = x(s)$$

The proof is similar for the case $A \hat A x = x$.

Note that the proof is constructive in that, given polynomials $M(s) > 0$ and $N(s,\theta) = Z(s)^T R Z(\theta)$ with $R \geq 0$, we can easily construct the inverse functions $\hat M$ and $\hat N$.

$$R = \begin{bmatrix} A_0^T M_{11} + M_{11}A_0 + M_{12}(0) + M_{12}^T(0) + M_{22}(0) & M_{11}A_1 - M_{12}(-\tau) & \tau A_0^T M_{12}(s) - \tau \dot{M}_{12}(s) + \tau N(0,s) \\ A_1^T M_{11} - M_{12}^T(-\tau) & -M_{22}(-\tau) & \tau A_1^T M_{12}(s) - \tau N(-\tau, s) \\ \tau M_{12}^T(s)A_0 - \tau \dot{M}_{12}^T(s) + \tau N(s,0) & \tau M_{12}^T(s)A_1 - \tau N(s,-\tau) & -\tau \dot{M}_{22}(s) \end{bmatrix} \tag{1}$$

If $M(s) \not\geq 0$, one can use the equivalent operator $M(s) + \begin{bmatrix} T(s) & 0 \\ 0 & 0 \end{bmatrix} \geq 0$. Also note that we have not assumed that R is invertible. The following is an obvious corollary.

Corollary 2. Consider the linear operator A defined by

$$(Ax)(s) = M(s)x(s) + \int_I N(s,\theta)x(\theta)d\theta$$

Where $M(s) > 0$ for all $s \in I$ and N has a representation $N(s,\theta) = Z(s)^T R Z(\theta)$ where Z is a vector of basis functions and $R > 0$. Define the linear operator $\hat{A}$ by

$$(\hat{A}x)(s) = M(s)^{-1}x(s) + \int_I \hat{N}(s,\theta)x(\theta)d\theta$$

Where

$$\hat{N}(s,\theta) = M(s)^{-1}Z(s)^T Q Z(\theta)M(\theta)^{-1}$$
$$Q = -R(S^{-1} + R)^{-1}S^{-1}$$
$$S = \int_I Z(s)M(s)^{-1}Z(s)^T ds.$$

Then $\hat{A}Ax = A\hat{A}x$ for any integrable function x.

4. STATE FEEDBACK SYNTHESIS FOR LINEAR TIME-DELAY SYSTEMS

Now that we are able to construct the inverse to the Lyapunov operator for linear time-delay systems, let us turn to the problem of full state feedback synthesis. For simplicity, we consider only a single delay and we restrict the control Lyapunov function to have the structure:

$$V(x_t) = \int_{-\tau}^0 \begin{bmatrix} x(t,0) \\ x(t,s) \end{bmatrix}^T \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22}(s) \end{bmatrix} \begin{bmatrix} x(t,0) \\ x(t,s) \end{bmatrix} ds + \int_{-\tau}^0 \int_{-\tau}^0 x(t,s)^T N(s,\theta)x(t,\theta)dsd\theta.$$

The structural restriction, $M_{12} = M_{21} = 0$, is significant and conservative. However, the use of this restriction allows us to separate $x(t,0)$ and $x(t,s)$.

We start by assuming the system has the general form

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau) + B_1u_1(t) + B_2u_2(t) + \int_{-\tau}^0 B_3(s)u_3(t,s)ds, \tag{2}$$

where $A_0, A_1 \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m_1}$, $B_2 \in \mathbb{R}^{n\times m_2}$, and $B_3(s) \in \mathbb{R}^{n\times m_3}$. In this description, the input generally lies in the same space as the state. It may be instructive to think of u_1 as the instantaneous feedback, u_2 as the delayed feedback and u_3 as the distributed feedback. Of course, the structure of the feedback can be restricted by setting any $B_i = 0$. The input signal itself is assumed to be generated from the state using a controller of the form

$$u_1(t) = K_{11}x(t), \qquad u_2(t) = K_\tau x(t-\tau),$$
$$u_3(t,s) = K_{22}(s)x(t+s) + \int_{-\tau}^0 K(s,\theta)x(t+\theta)d\theta$$

Because we are dealing with the full-state feedback scenario, we can make no restrictions on the structure of the controller and we assume that we are able to directly measure $x(t+s)$ for $s \in [-\tau, 0]$. The following theorem gives an polynomial computing characterization of the full-state feedback synthesis problem for linear time-delay systems using polynomial positivity constraints. The conditions can be implemented through polynomial computing packages such as SOSTOOLS. This is done by changing constraints like $M(s) \geq 0$ to the constraint that M is sum-of-squares.

Theorem 3. Suppose there exist polynomial matrices M_{22}, T, L_{22} and L (with $M_{22}(s) \in \mathcal{S}^n$, $T(s) \in \mathcal{S}^{2n}$, $L_{22}(s) \in \mathcal{S}^n$ and $L(s,\theta) \in \mathbb{R}^{n\times m_3}$), and matrices $M_{11} \in \mathcal{S}^n$, $L_{11} \in \mathbb{R}^{n\times m_1}$, $L_\tau \in \mathbb{R}^{n\times m_2}$, $Q_1 \geq 0$ and $Q_2 \geq 0$ such that the following hold:

$$M_{11} > 0, \qquad M_{22}(s) \geq 0, \qquad N(s,\theta) = Z(s)^T Q_1 Z(\theta)$$

$$\begin{bmatrix} A_0M_{11} + M_{11}A_0^T + M_{22}(0) & A_1M_{11} & \tau N(0,s) \\ M_{11}A_1^T & -M_{22}(-\tau) & -\tau N(-\tau,s) \\ \tau N(s,0) & -\tau N(s,-\tau) & -\tau \dot{M}_{22}(s) \end{bmatrix} + \begin{bmatrix} B_1L_{11} + L_{11}B_1^T & *^T & *^T \\ L_\tau B_2^T & 0 & *^T \\ \tau L_{22}(s)B(s)^T + \int_{-\tau}^0 L(\theta,s)^T B_3(\theta)^T d\theta & 0 & 0 \end{bmatrix} + \begin{bmatrix} T_{11}(s) & T_{12}(s) & 0 \\ T_{21}(s) & T_{22}(s) & 0 \\ 0 & 0 & 0 \end{bmatrix} \leq 0$$

$$\frac{d}{ds}N(s,\theta) + \frac{d}{d\theta}N(s,\theta) = Z(s)^T Q_2 Z(\theta)$$

$$\int_{-\tau}^0 T(s)ds = 0,$$

where $Z(s)$ is the vector of monomial bases in s as described previously. Then if we let

$$u_1(t) = K_{11}x(t), \qquad u_2(t) = K_\tau x(t-\tau),$$
$$u_3(t,s) = K_{22}(s)x(t+s) + \int_{-\tau}^0 K(s,\theta)x(t+\theta)d\theta,$$

where

$$\begin{bmatrix} K_{11} & 0 & 0 \\ 0 & K_{-\tau} & 0 \\ 0 & 0 & K_{22}(s) \end{bmatrix} = \begin{bmatrix} L_{11}M_{11}^{-1} & 0 & 0 \\ 0 & L_\tau M_{11}^{-1} & 0 \\ 0 & 0 & L_{22}(s)M_{11}^{-1} \end{bmatrix}$$

$$K(s,\theta) = L(s,\theta)M_{11}^{-1},$$

the controlled system defined by Equation 2 is stable.

Proof. Consider the Lyapunov functional

$V(x) =$

$$\frac{1}{\tau}\int_{-\tau}^0 \begin{bmatrix} x(t) \\ x(t+s) \end{bmatrix}^T \begin{bmatrix} M_{11}^{-1} & 0 \\ 0 & \tau M_{11}^{-1}M_{22}(s)M_{11}^{-1} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t+s) \end{bmatrix} ds + \int_{-\tau}^0 \int_{-\tau}^0 x(t+s)^T M_{11}^{-1} N(s,\theta) M_{11}^{-1} x(t+\theta)dsd\theta$$

Positivity of the first part of the functional follows since $M_{11} > 0$ and $M_{22}(s) \geq 0$. Positivity of the second part is because $Q_1 \geq 0$ which, as discussed n Section 2.3 and Peet et al. [2009], is equivalent to positivity of

$$\int_{-\tau}^0 \int_{-\tau}^0 x(t+s)^T M_{11}^{-1} Z(s)^T Q_1 Z(\theta) M_{11}^{-1} x(t+\theta) ds d\theta.$$

Now define the following new variables

$$\begin{aligned} z_1(t) &= M_{11}^{-1} x(t), \qquad z_2(t) = M_{11}^{-1} x(t-\tau), \\ z_3(t+s) &= M_{11}^{-1} x(t+s) \end{aligned}$$

The we can substitute the following expressions for x into the functional.

$$\begin{aligned} x(t-\tau) &= M_{11} z_2(t), \\ x(t+s) &= M_{11} z_3(t+s) \end{aligned}$$

This yields

$$V(x) = x(t) M_{11}^{-1} x(t) + \int_{-\tau}^0 z_3(t+s)^T M_{22}(s) z_3(t+s) ds + \int_{-\tau}^0 \int_{-\tau}^0 z_3(t+s)^T N(s,\theta) z_3(t+\theta) ds\, d\theta.$$

We separate the functional as $V = V_{a2} + V_{a1}$ where

$$V_{a1} = \int_{-\tau}^0 z_3(t+s)^T M_{22}(s) z_3(t+s) ds + \int_{-\tau}^0 \int_{-\tau}^0 z_3(t+s)^T N(s,\theta) z_3(t+\theta) ds\, d\theta.$$

Now z_3 satisfies $z_3(t,0) = z_1(t)$, $z_3(t,-\tau) = z_2(t)$, and $\frac{d}{dt} z_3(t+s) = \frac{d}{ds} z_3(t+s)$. So, through a well-known series of manipulations,

$$\begin{aligned} \dot{V}_{a1} &= \int_{-\tau}^0 \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t+s) \end{bmatrix} \begin{bmatrix} \frac{1}{\tau} M_{22}(0) & 0 & N(0,s) \\ 0 & -\frac{1}{\tau} M_{22}(-\tau) & -N(-\tau,s) \\ N(s,0) & -N(s,-\tau) & -\dot{M}_{22}(s) \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t+s) \end{bmatrix} ds \\ &- \int_{-\tau}^0 \int_{-\tau}^0 z_3(t+s)^T \left(\frac{d}{ds} N(s,\theta) + \frac{d}{d\theta} N(s,\theta) \right) z_3(t+\theta) ds d\theta \\ &= \frac{1}{\tau} \int_{-\tau}^0 \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t+s) \end{bmatrix} \begin{bmatrix} M_{22}(0) & 0 & \tau N(0,s) \\ 0 & -M_{22}(-\tau) & -\tau N(-\tau,s) \\ \tau N(s,0) & -\tau N(s,-\tau) & -\tau \dot{M}_{22}(s) \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t+s) \end{bmatrix} ds \\ &- \int_{-\tau}^0 \int_{-\tau}^0 z_3(t+s)^T \left(\frac{d}{ds} N(s,\theta) + \frac{d}{d\theta} N(s,\theta) \right) z_3(t+\theta) ds d\theta \end{aligned}$$

As for the first term, $V_{a2}(t) = x(t)^T M_{11}^{-1} x(t)$, this has derivative

$$\dot{V}_{a2}(t) = V_{da2}(t) + V_{dc}(t),$$

where V_{da2} is the uncontrolled dynamics and V_{dc} is the controller portion.

$$V_{da2}(t) = \frac{1}{\tau} \int_{-\tau}^0 \begin{bmatrix} x(t) \\ x(t-\tau) \\ x(t+s) \end{bmatrix}^T \begin{bmatrix} M_{11}^{-1} A_0 + A_0^T M_{11}^{-1} & M_{11}^{-1} A_1 & 0 \\ A_1^T M_{11}^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \\ x(t+s) \end{bmatrix} ds,$$

and

$$\begin{aligned} V_{dc}(t) = {} & x(t)^T M_{11}^{-1} \left(B_1 u_1(t) + B_2 u_2(t) + \int_{-\tau}^0 B_3(s) u_3(t,s) ds \right) \\ &+ \left(B_1 u_1(t) + B_2 u_2(t) + \int_{-\tau}^0 B_3(s) u_3(t,s) ds \right)^T M_{11}^{-1} x(t). \end{aligned}$$

Direct substitution for z_1, z_2, and z_3 on V_{da2} yields

$$\begin{aligned} V_{da2}(t) &= \frac{1}{\tau} \int_{-\tau}^0 \begin{bmatrix} x(t) \\ x(t-\tau) \\ x(t+s) \end{bmatrix}^T \begin{bmatrix} M_{11}^{-1} A_0 + A_0^T M_{11}^{-1} & M_{11}^{-1} A_1 & 0 \\ A_1^T M_{11}^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \\ x(t+s) \end{bmatrix} ds \\ &= \int_{-\tau}^0 \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t+s) \end{bmatrix}^T \begin{bmatrix} A_0 M_{11} + M_{11} A_0^T & A_1 M_{11} & 0 \\ M_{11} A_1^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t+s) \end{bmatrix} ds. \end{aligned}$$

Now, for the controlled part, to simplify, we take only the first part. The other half follows from symmetry. Let

$$B(s) = \left[B_1 \;\; B_2 \;\; B_3(s) \right].$$

Then

$$\begin{aligned} \tfrac{1}{2} V_{dc}(t) = {} & \frac{1}{\tau} \int_{-\tau}^0 x(t)^T M_{11}^{-1} B(s) \begin{bmatrix} K_{11} & & \\ & K_{-\tau} & \\ & & \tau K_{22}(s) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \\ x(t+s) \end{bmatrix} ds \\ &+ \int_{-\tau}^0 \int_{-\tau}^0 x(t)^T M_{11}^{-1} B(s) \begin{bmatrix} 0 & & \\ & 0 & \\ & & K(s,\theta) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \\ x(t+\theta) \end{bmatrix} ds d\theta \\ &= \frac{1}{\tau} \int_{-\tau}^0 z_1(t)^T B(s) \begin{bmatrix} K_{11} M_{11} & & \\ & K_{-\tau} M_{11} & \\ & & \tau K_{22}(s) M_{11} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t+s) \end{bmatrix} ds \\ &+ \int_{-\tau}^0 \int_{-\tau}^0 z_1(t)^T B(s) \begin{bmatrix} 0 & & \\ & 0 & \\ & & K(s,\theta) M_{11} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t+\theta) \end{bmatrix} ds d\theta \\ &= \frac{1}{\tau} \int_{-\tau}^0 z_1(t)^T B(s) \begin{bmatrix} L_{11} & & \\ & L_\tau & \\ & & \tau L_{22}(s) \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t+s) \end{bmatrix} ds \\ &+ \int_{-\tau}^0 \int_{-\tau}^0 z_1(t)^T B(s) \begin{bmatrix} 0 & & \\ & 0 & \\ & & L(s,\theta) \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t+\theta) \end{bmatrix} ds d\theta \\ &= \frac{1}{\tau} \int_{-\tau}^0 z_1(t)^T \left[B_1 L_{11} \;\; B_2 L_\tau \;\; \tau B_3(s) L_{22}(s) \right] \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t+s) \end{bmatrix} ds \\ &+ \int_{-\tau}^0 z_1(t)^T \left[0 \;\; 0 \;\; \int_{-\tau}^0 B_3(s) L(s,\theta) ds \right] \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t+\theta) \end{bmatrix} d\theta. \end{aligned}$$

Summing the parts,

$$\dot{V}(t) = \dot{V}_{a1}(t) + V_{da2}(t) + V_{dc}(t).$$

Collecting terms, we have

$$\dot{V}(t) = \frac{1}{\tau}\int_{-\tau}^{0} \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t+s) \end{bmatrix}^T G(s) \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t+s) \end{bmatrix} ds - \int_{-\tau}^{0}\int_{-\tau}^{0} z_3(t+s)^T H(s,\theta) z_3(t+\theta) ds\, d\theta,$$

where

$$G(s) = \begin{bmatrix} A_0 M_{11} + M_{11} A_0^T + M_{22}(0) & A_1 M_{11} & \tau N(0,s) \\ M_{11} A_1^T & -M_{22}(-\tau) & -\tau N(-\tau, s) \\ \tau N(s,0) & -\tau N(s,-\tau) & -\tau \dot{M}_{22}(s) \end{bmatrix} + \begin{bmatrix} B_1 L_{11} + L_{11} B_1^T & *^T & *^T \\ L_\tau B_2^T & 0 & *^T \\ \tau L_{22}(s) B(s)^T + \int_{-\tau}^{0} L(\theta,s)^T B_3(\theta)^T d\theta & 0 & 0 \end{bmatrix}$$

and

$$H(s,\theta) = \frac{d}{ds} N(s,\theta) + \frac{d}{d\theta} N(s,\theta)$$

Now, we use the slack variable, T, as described in Section 2.3 and introduced in Peet et al. [2009] and the inequality

$$G(s) + \begin{bmatrix} T(s) & 0 \\ 0 & 0 \end{bmatrix} \le 0$$

to conclude that the first term in the Lyapunov derivative described above is negative. In addition, as discussed in the beginning of the proof, because $Q_2 \ge 0$ we have that $H(s,\theta) = Z(s) Q_2 Z(\theta)$ implies that

$$\int_{-\tau}^{0}\int_{-\tau}^{0} z_3(t+s)^T H(s,\theta) z_3(t+\theta) ds \ge 0.$$

This completes the proof.

5. CONCLUSIONS AND FUTURE WORK

In this paper we have made progress towards a non-conservative treatment of full-state feedback synthesis for linear time-delay systems. We have developed a method of constructing the inverse of complete quadratic Lyapunov functionals defined by polynomials. This result can be used to create LMI conditions for full-state feedback controller synthesis through the use of an appropriate Control Lyapunov Function. Note that such a functional will be defined on a different space that the primal state space. This work is left for future research. We also gave an LMI condition for controller synthesis by using a restricted functional structure. The structure of the controller is dictated by the structure of the Lyapunov functional.

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