A Converse Sum-of-Squares Lyapunov Result: An Existence Proof Based on the Picard Iteration

Matthew M. Peet and Antonis Papachristodoulou

Abstract— **In this paper, we show that local exponential stability of a polynomial vector field implies the existence of a Lyapunov function which is a sum-of-squares of polynomials. To do that, we use the Picard iteration. This result shows that local stability of polynomial vector fields can be computed in a relatively efficient manner using semidefinite programming.**

I. INTRODUCTION

Recent years have seen extensive use of computational methods to solve nonlinear and infinite-dimensional control problems. Many difficult problems in control and dynamical systems can be formulated as polynomial non-negativity conditions. For example, the ability to optimize over the set of positive polynomials using the sum-of-squares relaxation has opened up new ways for addressing nonlinear control problems, in much the same way Linear Matrix Inequalities are used to address analysis questions for linear finite-dimensional systems. Roughly speaking, we now have the ability to optimize polynomial Lyapunov functions for nonlinear systems in a convex manner. This means that determining the question of global stability for polynomial systems is relatively straightforward.

In this paper we focus on developing a sound mathematical basis for the use of sum-of-squares Lyapunov functions for analysis of nonlinear systems - we do not detail the process of actually computing sum-of-squares Lyapunov functions, as this can be found in other references. For references on early work on optimization of polynomials, see [1], [2], and [3]. For more recent work see [4] and [5]. Today, there exist a number of software packages for optimization over positive polynomials, with prominent examples being given by SOSTOOLS [6] and GloptiPoly [7]. The main subject of this paper is local stability of the zero equilibrium of

$$\dot{x}(t) = f(x(t)),$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is polynomial. For local positivity of polynomials, we apply Positivstellensatz results [8], [9], [10]. A number of authors have discussed the use of polynomial optimization algorithms for constructing polynomial Lyapunov functions to show stability of this type of system. Substantial contributions to this area include [3], [11], [12], [13], and [14]. A Survey appears in [15].

In this paper we address the question of whether an exponentially stable nonlinear system will have a sum-of-squares Lyapunov function which establishes this property.

The results of this paper are the latest in a long line of enquiry which seeks to determine whether stability of a system implies the existence of a Lyapunov function and what are the properties of that function. There are too many results in this vein to list them all. However, particularly relevant work includes the flurry of activity on continuity properties in the 1950s including the work of [16], [17] and [18] and the overview in [19]. Infinitely-differentiable functions were explored in the work [20], [21]. Other innovative results are found in [22] and [23]. The reader is also referred to the books [24] and [25] for further treatment of converse theorems of Lyapunov.

There are two technical contributions in this paper to the development of converse Lyapunov theory. Like the previous paper [26], this paper fundamentally uses approximation theory. However, unlike the work in [26], this approximation is more closely tied to systems theory in that we approximate the solution map rather than the Lyapunov function directly. This approximate solution map is used in a standard form of converse Lyapunov function. The first key insight is to note that, due to the structure of this converse functional, if the approximation to the solution map is polynomial, then the Lyapunov function will be sum-of-squares. The second key insight is to use the Picard iteration to approximate the solution map instead of standard polynomial approximations such as the Bernstein polynomials. The reason is that the Picard iteration retains several key features of the solution map. It is well-known that the Picard iteration is not ideally suited for approximation of functions in a general context, as it only converges on a short interval. However, when approximating the solution map for a stable system, we show how the Picard iteration can be extended indefinitely, while still retaining the properties of the solution map. Moreover, because the Picard iteration inductively integrates the vector field, if the vector field is polynomial, the Picard iteration will be polynomial at each iteration.

Treatment of converse Lyapunov functions from the perspective of computation is relatively new. Due to the limitations of computation, such converse functions must be defined on a finite-dimensional space. One such space is polynomials of bounded degree. In the previous work [26], we were able to show that local stability on a bounded region implies the existence of a polynomial Lyapunov function. The results of the present paper improve on this previous work in that they show that stability implies the existence of a **sum-of-squares** Lyapunov function. This improvement is important because it is possible to optimize over sum-of-squares polynomials, while it is not currently possible to optimize over positive polynomials. Finally we note that a more complete version of this proof, in addition to a degree bound, appear in the journal version of this paper, current under review.

II. Notation and Background

Denote the Euclidean unit ball centered at 0 of radius r by B_r. For a function of several arguments, $f(x_1, \cdots, x_n)$, we will sometimes use $\partial/\partial i f$ to denote the partial derivative with respect to the ith argument.

The core concept we use in this paper is the Picard iteration. We use this to construct an approximation to the solution map and then use the approximate solution map to construct the Lyapunov function. Construction of the Lyapunov function will be discussed in more depth later on. However, at this point we review the Picard iteration - a standard method for proving the existence of solutions.

Consider an ordinary differential equation of the form

$$\dot{x}(t) = f(x(t)), \qquad x(a) = x_0, \qquad f(0) = 0.$$

The solution map is a function ϕ which satisfies

$$\frac{\partial}{\partial t}\phi(t,a,x) = f(\phi(t,a,x)) \qquad \text{and} \qquad \phi(a,a,x) = x.$$

For a time-invariant system the solution map can be simplified to the form $\phi(s,t)$. However, we do not make this simplification for technical reasons which arise in the proof.

Definition 1: Let X be a metric space. We say a mapping $T : X \to X$ is **contractive** with coefficient $c \in [0,1)$ if

$$\|Tx - Ty\| \leq c\,\|x-y\| \qquad x,y \in X$$

The following is a *Fixed-Point* Theorem.

Theorem 2 (Contraction Mapping Principle): Let X be a complete metric space and let $T : X \to X$ be a contraction with coefficient c. Then there exists a unique $a \in X$ such that

$$Ta = a.$$

Furthermore

$$\left\|T^k x_0 - a\right\| \leq c^k \|x_0 - a\|.$$

To apply these results to the existence of the solution map, we use the Picard iteration.

Definition 3: For give T and r, define the metric space

$$X := \left\{ z(t,a,x) : \begin{array}{l} \sup_{t\in[a,a+T]} \|z(t,a,x)\| \leq 2r, \\ z \text{ is continuously differentiable.} \end{array} \right\} \qquad (1)$$

with norm $\|z\| = \sup_{t\in[a,a+T]} \|z(t,a,x)\|$.

Define the **Picard iteration**,

$$(Pz)(t,a,x) = x + \int_a^t f(z(s,a,x))ds.$$

In the following few sections, we will show that the Picard iteration is contractive on X for some r and T.

III. Picard Invariance Lemma

The first result is a technical Lemma showing that the Picard iteration satisfies a certain property of the solution map.

Lemma 4: Let $z(s,t,x) = 0$. For a time-invariant system, the Picard iteration satisfies

$$P^k(s,t,x) = P^k(s-a,t-a,x)$$

Proof: Proof by induction. At the first iteration,

$$Pz(s,t,x) = Pz(s-a,t-a,x) = x.$$

Suppose that

$$P^k z(s,t,x) = P^k z(s-a,t-a,x).$$

Then

$$\begin{aligned} P^{k+1}z(s,t,x) &= x + \int_t^s f(P^k z(\omega,t,x))d\omega \\ &= x + \int_{t-a}^{s-a} f(P^k z(\omega + a,t,x))d\omega \\ &= x + \int_{t-a}^{s-a} f(P^k z(\omega,t-a,x))d\omega \\ &= P^{k+1}z(s-a,t-a,x). \end{aligned}$$

Therefore, the Lemma holds by induction. ■

IV. Picard Iteration

We begin this section by showing that for any radius r, the Picard iteration is contractive on X for some T.

Lemma 5: Given $r > 0$, let $T < \min\{\frac{r}{c}, \frac{1}{L}\}$ where f has Lipschitz factor L on B_{2r} and $c = \sup_{x\in B_{2r}} f(x)$. Then for any $x \in B_r$, $P : X \to X$ and there exists some $\phi \in X$ such that for $t \in [a, a+T]$,

$$\frac{d}{dt}\phi(t,a,x) = \phi(t,a,x), \qquad \phi(0,0,x) = x$$

and for any $z \in X$,

$$\left\|\phi - P^k z\right\| \leq (TL)^k \|\phi - z\|.$$

Proof: First show $P : X \to X$. If $x \in B_r$ and $z \in X$, then $\|z(s,t,x)\| \leq 2r$ and so

$$\begin{aligned} \sup_{t\in[a,a+T]} \|Pz(t,a,x)\| &= \sup_{t\in[a,a+T]} \left\| x + \int_a^t f(z(s,a,x)) \right\| ds \\ &\leq \|x\| + \int_a^{a+T} \|f(z(s,a,x))\| ds \\ &\leq r + Tc < 2r \end{aligned}$$

Thus we conclude that $Pz \in X$. Furthermore, for $z_1, z_2 \in X$,

$$\begin{aligned} \|Pz_1 - Pz_2\| &= \sup_{t\in[a,a+T]} \left\| \int_a^t (f(z_1(s,a,x)) - f(z_2(s,a,x)))\, ds \right\| \\ &\leq \int_a^{a+T} \|f(z_1(s,a,x)) - f(z_2(s,a,x))\| ds \\ &\leq TL \sup_{s\in[a,a+T]} \|z_1(s,a,x) - z_2(s,a,x)\| \\ &= TL\|z_1 - z_2\| \end{aligned}$$

Therefore, by the contraction mapping theorem, the Picard iteration converges on $[0,T]$ with convergence rate $(TL)^k$. ■

V. Picard Extension Convergence Lemma

In this section we propose a new way of extending the Picard iteration. We use the final value of the previous Picard iteration as the initial condition for a new round of Picard iteration. This is done to achieve convergence on an arbitrary interval while maintaining the polynomial nature of the approximation.

Lemma 6: Suppose that the solution map ϕ exists on $s - t \in [0,\infty]$ and $\|\phi(s,t,x)\| \leq K\|x\|$ for any $x \in B_r$. Suppose that f is Lipschitz on B_{2Kr} with factor L and bounded with bound c. Let $T < \min\{\frac{r}{c}, \frac{1}{L}\}$. Then let $z = 0$ and define

$$G_0^k(s,t,x) = (P^k z)(s,t,x)$$

and for $i > 0$, define G_i recursively as

$$G_{i+1}^k(s,t,x) = (P^k z)(s,t,G_i^k(T,0,x)).$$

Then the G_i^k are polynomials for any i,k and the composite function

$$G^k(s,t,x) := G_i(s - iT, t, x) \quad \forall \quad s \in [t + iT, t + iT + T]$$

is continuously differentiable. For any $\delta > 0$, let k be sufficiently large. Then for any $x \in B_{r/(2K)}$, $G^k \in Y$ where

$$Y := \left\{ z(t,a,x) : \begin{array}{l} \sup_{t\in[a,a+\delta]} \|z(t,a,x)\| \le r, \\ z \text{ is continuously differentiable and} \\ G^k(s,t,x) = G^k(s-a,t-a,x). \end{array} \right\}. \tag{2}$$

Furthermore $\left\|G^k(s,0,x) - \phi(s,0,x)\right\| \le c(k)\|x\|$ where

$$\lim_{k\to\infty} c(k) = 0.$$

Proof: The first thing to note is that since the conditions of Lemma 5 is satisfied, if $x \in B_r$, P^k converges to ϕ on $[0,T]$. Let $d = TL$. Then

$$\begin{aligned}
&\sup_{s\in[0,T]} \left\|G_0^k(s,0,x) - \phi(s,0,x)\right\| \\
&= \sup_{s\in[0,T]} \left\|P^k(s,0,x) - \phi(s,0,x)\right\| \\
&\le d^k \|\phi(s,0,x)\| \le K d^k \|x\|.
\end{aligned}$$

Thus G converges to ϕ on the interval $[0,T]$. Now suppose that $\left\|G^k - \phi\right\| \le c_{i-1}(k)\|x\|$ on interval $[iT - T, iT]$ with $\lim_{k\to\infty} c_i(k) = 0$.

$$\begin{aligned}
&\sup_{s\in[iT,iT+T]} \left\|G^k(s,0,x) - \phi(s,0,x)\right\| \\
&= \sup_{s\in[iT,iT+T]} \left\|G_i^k(s-iT,0,x) - \phi(s,0,x)\right\| \\
&= \sup_{s\in[iT,iT+T]} \left\|P^k(s-iT,0,G_{i-1}^k(T,0,x)) - \phi(s,0,x)\right\| \\
&= \sup_{s\in[iT,iT+T]} \left\|P^k(s-iT,0,G_{i-1}^k(T,0,x)) - \phi(s-iT,0,\phi(iT,0,x))\right\| \\
&\le \sup_{s\in[iT,iT+T]} \left\|P^k(s-iT,0,G_{i-1}^k(T,0,x)) - \phi(s-iT,0,G_{i-1}^k(T,0,x))\right\| \\
&+ \sup_{s\in[iT,iT+T]} \left\|\phi(s-iT,0,G_{i-1}^k(T,0,x)) - \phi(s-iT,0,\phi(iT,0,x))\right\|
\end{aligned}$$

We treat these terms separately. First note that

$$\begin{aligned}
&\left\|G_{i-1}^k(T,0,x)\right\| \\
&\le \|\phi(iT,0,x)\| + \left\|\phi(iT,0,x) - G_{i-1}^k(T,0,x)\right\| \\
&\le K\|x\| + c_{i-1}(k)\|x\| \\
&\le K(1 + c_{i-1}(k))\|x\|.
\end{aligned}$$

Since $\lim_{k\to\infty} c_i(k) = 0$, for sufficiently large k, $\left\|G_{i-1}^k(T,0,x)\right\| \le K(1 + c_{i-1}(k))\|x\| \le r$. Hence

$$\begin{aligned}
&\sup_{s\in[iT,iT+T]} \left\|P^k(s-iT,0,G_{i-1}^k(T,0,x)) - \phi(s-iT,0,G_{i-1}^k(T,0,x))\right\| \\
&\le \sup_{s\in[iT,iT+T]} d^k \left\|\phi(s-iT,0,G_{i-1}^k(T,0,x))\right\| \\
&\le K d^k \left\|G_{i-1}^k(T,0,x)\right\| \\
&\le K^2 d^k (1 + c_{i-1}(k))\|x\|.
\end{aligned}$$

Now, if $x, y \in B_r$, $\|\phi(s,0,x)\|, \|\phi(s,0,y)\| \le Kr$ on $[0,T]$ and hence $\phi(s,0,x) - \phi(s,0,y) \le e^{Ls}\|x - y\|$. Now, for sufficiently large k, $\left\|G_{i-1}^k(T,0,x)\right\| \le r$ and so

$$\begin{aligned}
&\sup_{s\in[iT,iT+T]} \left\|\phi(s-iT,0,G_{i-1}^k(T,0,x)) - \phi(s-iT,0,\phi(iT,0,x))\right\| \\
&\le \sup_{s\in[iT,iT+T]} e^{L(s-iT)} \left\|G_{i-1}^k(T,0,x) - \phi(iT,0,x)\right\| \\
&\le e^{LT} c_{i-1}(k)\|x\|
\end{aligned}$$

Combining, we conclude that

$$\begin{aligned}
&\sup_{s\in[iT,iT+T]} \left\|G_i^k(s-iT,0,x) - \phi(s,0,x)\right\| \\
&\le e^{TL} c_{i-1}(k)\|x\| + K^2 d^k (1 + c_{i-1}(k))\|x\| \\
&= ((e^{TL} + K^2 d^k) c_{i-1}(k) + K^2 d^k)\|x\| \\
&= c_i(k)\|x\|.
\end{aligned}$$

where we define $c_i(k) = (e^{TL} + K^2 d^k) c_{i-1}(k) + K^2 d^k$. Since by assumption $\lim_{k\to\infty} c_{i-1}(k) = 0$ and $\lim_{k\to\infty} d^k = 0$, it is readily seen that $\lim_{k\to\infty} c_i(k) = 0$.

It is easy to see that G^k is continuously differentiable by evaluation at the points of interpolation. To show $G^k(s,t,x) = G^k(s-a,t-a,x)$, recall

$$G^k(s,t,x) := \Big\{ G_i(s - iT, t, x) \quad s \in [t + iT, t + iT + T]$$

and

$$G_{i+1}^k(s,t,x) = P^k(s,t,G_i^k(T,0,x)).$$

So $G^k(s-a,t-a,x) = G_i(s - iT - a, t - a, x)$ for $s - a \in [t - a + iT, t - a + iT + T]$, which means $G^k(s-a,t-a,x) = G_i(s - iT - a, t - a, x)$ for $s \in [t + iT, t + iT + T]$. Now $G_i(s - iT - a, t - a, x) = P^k(s - a, t - a, G_{i-1}^k(T,0,x)) = P^k(s,t,G_{i-1}^k(T,0,x)) = G_i(s - iT, t, x)$. Therefore, $G^k(s-a,t-a,x) = G_i^k(s - iT, t, x) = G^k(s,t,x)$ for $s \in [t + iT, t + iT + T]$. This implies that $G^k(s-a,t-a,x) = G^k(s,t,x)$ for all s. ■

Note that there is considerable flexibility in choosing the time interval, T, in the extended Picard iteration. The shorter the interval, the faster the convergence on the interval. However, more intervals mean more complicated approximations. This issue is discussed in detail in the journal version of this paper [under review] wherein the convergence rate is used to obtain a degree bound on the Lyapunov function.

VI. Derivative Inequality Lemma

In this critical lemma, we show that the Picard iteration approximately retains the differentiability properties of the solution map. The proof is based on induction and is inspired by an approach in [27]. This lemma is then adapted to the extended Picard iteration introduced in the previous section.

Lemma 7: Suppose that the conditions of Lemma 5 are satisfied. Then for any $x \in B_r$ and any $k \geq 0$,

$$\sup_{t\in[a,a+T]} \left\| \frac{\partial}{\partial a}(P^k z)(t,a,x) + \frac{\partial}{\partial x}(P^k z)(t,a,x)^T f(x) \right\| \leq \frac{(TL)^k}{T} \|x\|$$

Proof: Begin with the identity for $k \geq 1$

$$(P^k z)(t,a,x) = x + \int_a^t f((P^{k-1}z)(s,a,x))ds.$$

Then

$$\begin{aligned} &\frac{\partial}{\partial a}(P^k z)(t,a,x) \\ &\quad = -f((P^{k-1}z)(a,a,x)) \\ &\qquad + \int_a^t \nabla f((P^{k-1}z)(s,a,x)) \frac{\partial}{\partial a}(P^{k-1}z)(s,a,x)ds \\ &\quad = -f(x) + \int_a^t \nabla f((P^{k-1}z)(s,a,x)) \frac{\partial}{\partial a}(P^{k-1}z)(s,a,x)ds, \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial}{\partial x}(P^k z)(t,a,x) \\ &\quad = I + \int_a^t \nabla f((P^{k-1}z)(s,a,x))^T \frac{\partial}{\partial x}(P^{k-1}z)(s,a,x)ds. \end{aligned}$$

Now define for $k \geq 1$,

$$y_k(t,a,x) := \frac{\partial}{\partial a}(P^k z)(t,a,x) + \frac{\partial}{\partial x}(P^k z)(t,a,x)^T f(x).$$

For $k \geq 2$, we have

$$\begin{aligned} &y_k(t,a,x) := \frac{\partial}{\partial a}(P^k z)(t,a,x) + \frac{\partial}{\partial x}(P^k z)(t,a,x)^T f(x) \\ &= \int_a^t \nabla f((P^{k-1}z)(s,a,x)) \frac{\partial}{\partial a}(P^{k-1}z)(s,a,x)ds \\ &\qquad + \int_a^t \nabla f((P^{k-1}z)(s,a,x))^T \frac{\partial}{\partial x}(P^{k-1}z)(s,a,x)f(x)ds \\ &= \int_a^t \nabla f((P^{k-1}z)(s,a,x)) \\ &\qquad \left[\frac{\partial}{\partial a}(P^{k-1}z)(s,a,x) + \frac{\partial}{\partial x}(P^{k-1}z)(s,a,x)f(x)\right] ds \\ &= \int_a^t \nabla f((P^{k-1}z)(s,a,x)) y_{k-1}(s,a,x)ds. \end{aligned}$$

This means that since $(P^{k-1}z)(t,a,x) \in B_{2r}$, by induction

$$\begin{aligned} &\sup_{[a,a+T]} \|y_k(t)\| \\ &\leq T \sup_{t\in[a,a+T]} \left\|\nabla f((P^{k-1}z)(t,a,x))\right\| \sup_{t\in[a,a+T]} \|y_{k-1}(t,a,x)\| \\ &\leq TL \sup_{t\in[a,a+T]} \|y_{k-1}(t,a,x)\| \\ &\leq (TL)^{(k-1)} \sup_{t\in[a,a+T]} \|y_1(t,a,x)\| \end{aligned}$$

For $k = 1$, $(Pz)(t,a,x) = x$, so $y_1(t) = f(x)$, so

$$\sup_{[a,a+T]} \|y_1(t)\| \leq L\|x\|.$$

Thus

$$\sup_{[a,a+T]} \|y_k(t)\| \leq \frac{(TL)^k}{T} \|x\|.$$

■

We now adapt this lemma to the extended Picard iteration.

Lemma 8: Suppose that the conditions of Lemma 6 are satisfied. Let k be sufficiently large. Then for any $x \in B_{r/(2K)}$,

$$\begin{aligned} &\sup_{t\in[a,a+T]} \left\| \frac{\partial}{\partial a}G^k(t,a,x) + \frac{\partial}{\partial x}G^k(t,a,x)^T f(x) \right\| \\ &\qquad \leq K\frac{(TL)^k}{T}(1+c(k))\|x\| \end{aligned}$$

Proof: Recall

$$G^k(s,t,x) := \Big\{ G_i(s-iT,t,x) \quad s \in [t+iT, t+iT+T]$$

and

$$G^k_{i+1}(s,t,x) = P^k(s,t,G^k_i(T,0,x)).$$

For $t \in [a+iT, a+iT+T]$,

$$\begin{aligned} &\left\| \frac{\partial}{\partial a}G^k(t,a,x) + \frac{\partial}{\partial x}G^k(t,a,x)^T f(x) \right\| \\ &= \left\| \frac{\partial}{\partial a}G^k_i(t-iT,a,x) + \frac{\partial}{\partial x}G^k(t-iT,a,x)^T f(x) \right\| \\ &= \left\| \frac{\partial}{\partial a}P^k(t-iT,a,G^k_i(T,0,x)) + \frac{\partial}{\partial x}P^k(t-iT,a,G^k_i(T,0,x))^T f(x) \right\| \\ &\leq \frac{(TL)^k}{T} \left\|G^k_i(T,0,x)\right\| \end{aligned}$$

As we have already shown,

$$\left\|G^k_i(T,0,x)\right\| \leq (K + Kc_i(k))\|x\|.$$

Thus

$$\begin{aligned} &\left\| \frac{\partial}{\partial a}G^k(t,a,x) + \frac{\partial}{\partial x}G^k(t,a,x)^T f(x) \right\| \\ &\qquad \leq \frac{(TL)^k}{T}(K + Kc_i(k))\|x\| \end{aligned}$$

Since the c_i are increasing,

$$\begin{aligned} &\sup_{t\in[a,a+\delta]} \left\| \frac{\partial}{\partial a}G^k(t,a,x) + \frac{\partial}{\partial x}G^k(t,a,x)^T f(x) \right\| \\ &\qquad \leq \frac{(TL)^k}{T}(K + Kc(k))\|x\|. \end{aligned}$$

■

VII. Main Result - A Converse SOS Lyapunov Function

In this section, we combine the previous results in a relatively straightforward manner to obtain a converse Lyapunov function which is also a sum-of-squares polynomial. Specifically, we use a standard form of converse Lyapunov function and simply substitute the our extended Picard iteration for the solution map. Consider the system

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0. \tag{3}$$

Theorem 9: Suppose that f is polynomial and that system (3) is exponentially stable with

$$\|x(t)\| \leq K\|x(0)\| e^{-\lambda t}$$

for some $\lambda > 0$, $K \geq 1$ and for any $x(0) \in M$, where M is a bounded region. Then there exist $\alpha, \beta, \gamma > 0$ and a sum-of-squares polynomial $V(x)$ such that for any $x \in M$,

$$\alpha\|x\|^2 \leq V(x) \leq \beta\|x\|^2$$
$$\nabla V(x) f(x) \leq -\gamma\|x\|^2.$$

Proof: Let $\delta = \frac{\log 2K^2}{2\lambda}$ and r be such that $M \subset B_{r/2K}$. Since a polynomial is Lipschitz on any bounded region, Let L be the Lipschitz factor for f on B_{2Kr}. Then by Lemma 5, for some $T > 0$, the Picard iteration converges on $[0,T]$ with rate $d(k) = (TL)^k$. Choose $N > \frac{\delta}{T}$. Define G^k as in Lemma 6. By Lemma 6, there is a $c(k)$ such that $\left\|G^k - \phi\right\| \leq c(k)\|\phi\|$ on $[0,\delta]$ with rate $\lim_{k\to\infty} c(k) = 0$. Choose k sufficiently large such that

$$c(k)^2 + \delta K \frac{d(k)}{T}(1+c(k))(K+c(k)) \leq \frac{1}{4}$$

and

$$c(k)^2 < \frac{1}{4KL\delta}(1-e^{-2L\delta}).$$

We propose the following Lyapunov function, indexed by k.

$$V_k(x) := \int_0^\delta G^k(s,0,x)^T G^k(s,0,x) ds$$

The proof is divided into three parts:

a) Upper and Lower Bounded: To prove that V_k is a valid Lyapunov function, first consider upper boundedness. Suppose $x \in B$ and $s \in [0,\delta]$. Then

$$\left\|G^k(s,0,x)\right\|^2 = \left\|\phi(s,0,x) + \left[G^k(s,0,x)^T - \phi(s,0,x)\right]\right\|^2$$
$$\leq \|\phi(s,0,x)\|^2 + \left\|\left[G^k(s,0,x)^T - \phi(s,0,x)\right]\right\|^2$$

As per Lemma 6,

$$\left\|G^k(s,0,x) - \phi(s,0,x)\right\| \leq c(k)\,\|\phi(s,0,x)\| \leq Kc(k)\|x\|.$$

From stability we have $\|\phi(s,0,x)\| \leq K\|x\|$. Hence,

$$V_k(x) = \int_0^\delta \left\|G^k(s,0,x)\right\|^2 ds \leq \delta K^2\left(1+c(k)^2\right)\|x\|^2.$$

Therefore the upper boundedness condition is satisfied for any $k \geq 0$ with $\beta = \delta K^2(1+c(k)^2)$.

Next we consider the strict positivity condition. First we note

$$\|\phi(s,0,x)\|^2 = \left\|G^k(s,0,x) + \left[\phi(s,0,x) - G^k(s,0,x)\right]\right\|^2$$
$$\leq \left\|G^k(s,0,x)\right\|^2 + \left\|\phi(s,0,x) - G^k(s,0,x)\right\|^2$$

which implies

$$\left\|G^k(s,0,x)\right\|^2 \geq \|\phi(s,0,x)\|^2 - \left\|\phi(s,0,x) - G^k(s,0,x)\right\|^2$$

By Lipschitz continuity of f, as mentioned earlier,

$$\|\phi(s,0,x)\|^2 \geq e^{-2Ls}\|x\|^2,$$

and as noted earlier

$$\left\|G^k(s,0,x) - \phi(s,0,x)\right\| \leq Kc(k)\|x\|.$$

Thus

$$V_k(x) = \int_0^\delta \left\|G^k(s,0,x)\right\|^2 ds$$
$$\geq \left(\frac{1}{2L}(1-e^{-2L\delta}) - \delta Kc(k)^2\right)\|x\|^2.$$

Thus for k as defined previously, the positivity condition holds with $\alpha = \frac{1}{4L}(1-e^{-2L\delta})$.

b) Negativity of the Derivative: Finally, we prove the derivative condition. Recall

$$V_k(x) := \int_0^\delta G^k(s,0,x)^T G^k(s,0,x) ds$$
$$= \int_t^{t+\delta} G^k(s,t,x)^T G^k(s,t,x) ds$$

then since

$$\nabla V(x(t)) f(x(t)) = \frac{d}{dt} V(x(t)),$$

we have by the Leibniz rule for differentiation of integrals,

$$\frac{\partial}{\partial t} V_k(x(t)) = \left[G^k(t+\delta,t,x(t))^T G^k(t+\delta,t,x(t))\right]$$
$$- \left[G^k(t,t,x(t))^T G^k(t,t,x(t))\right]$$
$$+ \int_t^{t+\delta} 2G^k(s,t,x(t))^T \frac{\partial}{\partial 2} G^k(s,t,x(t)) ds$$
$$+ \int_t^{t+\delta} 2G^k(s,t,x(t))^T \frac{\partial}{\partial 3} G^k(s,t,x(t)) f(x(t)) ds$$
$$= \left\|G^k(\delta,0,x(t))\right\|^2 - \|x(t)\|^2 + \int_t^{t+\delta} 2G^k(s,t,x(t))^T$$
$$\left[\frac{\partial}{\partial 2} G^k(s,t,x(t)) + \frac{\partial}{\partial 3} G^k(s,t,x(t)) f(x(t))\right] ds$$

As per Lemma 8, we have

$$\left\|\frac{\partial}{\partial 2} G^k(s,t,x(t)) + \frac{\partial}{\partial 3} G^k(s,t,x(t))^T f(x(t))\right\|$$
$$\leq K\frac{d(k)}{T}(1+c(k))\|x(t)\|$$

and as previously noted,

$$\left\|G^k(s,t,x(t))\right\|^2 \leq (K^2 e^{-2\lambda(s-t)} + c(k)^2)\|x(t)\|^2.$$

Similarly, $\left\|G^k(s,t,x(t))\right\| \leq (K+c(k))\|x(t)\|$. We conclude

$$\frac{\partial}{\partial t} V_k(x(t)) \leq (K^2 e^{-2\lambda\delta} + c(k)^2)\|x(t)\|^2 - \|x(t)\|^2$$
$$+ 2\delta \frac{d(k)}{T}(1+c(k))(K+c(k))\|x(t)\|^2$$
$$\leq \left(K^2 e^{-2\lambda\delta} + c(k)^2 - 1 + 2\delta K\frac{d(k)}{T}(1+c(k))(K+c(k))\right)\|x(t)\|^2$$

Therefore, we have strict negativity of the derivative since

$$\begin{aligned}
&K^2 e^{-2\lambda\delta} + c(k)^2 + 2\delta \frac{d(k)}{T}(1+c(k))(K+c(k)) \\
&= \frac{1}{2} + c(k)^2 + 2\delta K \frac{d(k)}{T}(1+c(k))(K+c(k)) \\
&\leq \frac{3}{4} < 1
\end{aligned}$$

Thus

$$\frac{\partial}{\partial t} V_k(x(t)) \leq -\frac{1}{4}\|x(t)\|^2.$$

c) Sum of Squares: Since f is polynomial and z is trivially polynomial, $(P^k z)(s,0,x)$ is a polynomial in x and s. Therefore, $V_k(x)$ is a polynomial for any $k > 0$. To show that V is sum-of-squares, we first rewrite the function

$$V(x) = \sum_{i=1}^{N} \int_{iT-T}^{T} \left[G_i^k(s-iT,0,x)^T G_i^k(s-iT,0,x) \right] ds.$$

Since $G_i^k z$ is a polynomial in all of its arguments, $G_i^k(s-iT,0,x)^T G_i^k(s-iT,0,x)$ is sum-of-squares. It can therefore be represented as $R_i(x)^T Z_i(s)^T Z_i(s) R_i(x)$ for some polynomial vector R_i and matrix of monomial bases Z_i. Then

$$\begin{aligned}
V(x) &= \sum_{i=1}^{N} R_i(x)^T \int_{iT-T}^{iT} Z_i(s)^T Z_i(s) ds R_i(x) \\
&= \sum_{i=1}^{N} R_i(x)^T M_i R_i(x)
\end{aligned}$$

Where $M_i = \int_{iT-T}^{iT} Z_i(s)^T Z_i(s) ds \geq 0$ is a constant matrix. This proves that V is sum-of-squares since it is a sum of sums-of-squares.

■

Note: If interested in a degree bound, this can be obtained given a bound on k: if f is a polynomial of degree q, and z is a polynomial of degree d in x, then Pz will be a polynomial of degree $\max\{1, dq\}$ in x. Thus the degree of $P^k z$ will be dq^k. Thus since $z = 0$ the first term in the Lyapunov function will be degree q^k. However, the other terms iterate on $P^k z$, so the second term will be degree q^{2k} and so on. Thus the final degree will be q^{Nk} in x where $N > \frac{\delta}{T}$. Note that finding a closed-form expression for k is quite involved.

VIII. Conclusion

In this paper, we have used the Picard iteration to construct an approximation to the solution map on arbitrarily long intervals. We have used this approximation to prove that local exponential stability of a polynomial vector field implies the existence of a Lyapunov function which is a sum-of-squares of polynomials. The immediate question is whether we can obtain a degree bound for this functional. This is possible, but due to the complexity of the proof this has been omitted from this paper. Another question which arises is whether this function will have a negative derivative which is a negative of a sum-of-squares derivative. The answer cannot be found through direct analysis of the proof. In addition, a further investigation of the tradeoff between complexity and accuracy is warranted. Still unresolved is the fundamental question of whether *globally* stable vector fields will also admit sum-of-squares Lyapunov functions.

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