

# Reducing the Computational Cost of the Sum-of-Squares Stability Test for Time-Delayed Systems

Yashun Zhang, Matthew Peet and Keqin Gu

**Abstract**—This paper considers the problem of reducing the computational complexity associated with the Sum-of-Squares approach to stability analysis of time-delay systems. Specifically, this paper considers systems with a large state-space but with relatively few delays—the most common situation in practice. The paper uses the general framework of coupled differential-difference equations with delays in low-dimensional feedback channels. This framework includes both the standard delayed and neutral-type systems. The approach is based on recent results which introduced a new type of Lyapunov-Krasovskii form which was shown to be necessary and sufficient for stability of this class of systems. This paper shows how exploiting the structure of the new functional can yield dramatic improvements in computational complexity. Numerical examples are given to illustrate this improvement.

**Index Terms**—Lyapunov-Krasovskii; Time delay; Semidefinite programming; Sum-of-Squares; Complexity.

## I. INTRODUCTION

In this paper we consider stability of linear time-delay systems with fixed delay. The existence of a monotonically decreasing quadratic Lyapunov function is necessary and sufficient for stability of these systems [4], [7], [10]. As is customary, we refer to these Lyapunov functions as Lyapunov-Krasovskii functionals when the state-space is infinite dimensional. The problem of finding such a functional is considered computationally intractable. An obvious solution is to use simplified versions of the functional. Naturally, however, stability conditions derived in such a manner will be conservative [4]. A solution to this dilemma was proposed in [2] which used a “discretized” version of the Lyapunov-Krasovskii functional. The product was a series of sufficient conditions which appears to converge to necessity as the level of discretization is increased. The significance of this work is that it gives a quantifiable tradeoff between computational complexity and accuracy of the stability test. In [15], [16], the problem was approached using polynomials instead of discretized functionals. We refer to this result as the Sum-of-Squares method. The advantage of the Sum-of-Squares approach is that it is easily generalized to nonlinear and uncertain systems [12]. In both cases, the conditions are expressed using semidefinite programming (SDP) [9], [13]. A problem with both the discretized functional method and the Sum-of-Squares method is that the computational cost increases quickly for large systems with multiple delays.

In most practical systems, although the number of state variables is rather large, there are relatively few delayed elements and these delayed elements enter through low-rank coefficient matrices. However, this feature is not typically leveraged when deriving stability conditions. In this paper, we reformulate the standard model of time-delayed equations by using coupled differential-difference equations with a single delay in each feedback channel. The idea is that if the dimension of the feedback channels is substantially smaller than the number of states, then this formulation allows us to exploit this low-dimensional structure to potentially reduce the computational cost of stability analysis [5]. In addition, using coupled differential-difference equations allows us to address a larger class of systems that includes time delay systems of both retarded and neutral type.

Naturally, this paper is not the first to consider stability of coupled differential-difference equations. In particular, asymptotic stability was investigated in [18] under the assumption that the difference equations are input-to-state stable. The stability result in [18] was strengthened to uniform asymptotic stability and extended to the general coupled differential-functional equations in [6], which also considered the possibility of reducing the complexity of the discretized Lyapunov-Krasovskii functional method. A reformulation of coupled differential-difference equations with single independent delay in each channel was proposed in [5] with a discretized Lyapunov implementation in [8]. For other research on this formulation, see [1], [17], [20], and the references therein.

The purpose of this paper is to adapt the Sum-of-Squares approach to coupled differential-difference equations and a structured Lyapunov functional. The goal is to realize a complexity reduction of several orders of magnitude in systems with low-dimensional delay channels. The paper is structured as follows. We begin by introducing the coupled formulation and give regularity conditions. This is accompanied by a necessary and sufficient quadratic Lyapunov result and some basics on Sum-of-Squares. In Section III, we adapt the Sum-of-Squares approach to positivity of the Lyapunov functional introduced previously. In Section IV, we give the derivative of the Lyapunov functional and apply our results to enforce negativity. In Section V, we combine our results to give a asymptotically exact, semidefinite-programming-based approach to stability of linear time-delay systems. Finally, we discuss computational complexity and use numerical examples to illustrate the advantages of the current approach.

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## A. Notation

$\mathbb{Z}$  denotes the set of positive integers.  $\mathbb{R}^n$  denotes the  $n$ -dimensional Euclidean space, and  $\mathbb{R}^{p \times q}$  denotes the set of all  $p \times q$  real matrices.  $\mathbb{S}^n$  denotes all the  $n \times n$  symmetric real matrices. For  $X \in \mathbb{S}^n$ , the notation  $X \geq 0$  ( $X > 0$ ) means that  $X$  is positive semidefinite (definite).  $I$  denotes the identity matrix with appropriate dimension. For  $n \in \mathbb{Z}$  and a given positive real number  $r$  we use  $\mathcal{PC}(r, n)$  to denote the vector space of bounded functions  $f : [-r, 0) \rightarrow \mathbb{R}^n$  which are right continuous everywhere and continuous everywhere except possibly at a finite number of points in the interval. Unless otherwise stated,  $\|f\|$  denotes either the 2-norm if  $f \in \mathbb{R}^n$  or the  $L_2$ -norm if  $f$  is a square-integrable function. Sometimes we use the notation  $\|f\|_{L_2}$  for additional clarity. For a given function  $y$ , if  $y$  is defined on  $[t - r, t]$ , we will use  $y_{r,t}$  to denote the segment of  $y$  on this interval, but translated to the origin. Specifically,  $y_{r,t}(s) = y(t + s)$ , for  $s \in [-r, 0)$ . Through some abuse of notation, we will occasionally use  $\phi = (\phi_1, \phi_2, \dots, \phi_K) \in \mathcal{PC}$  to denote  $\phi_i \in \mathcal{PC}(r_i, m_i)$ ,  $i = 1, 2, \dots, K$ , where  $r_i$  and  $m_i$  will be clear from context.

## II. PRELIMINARIES

### A. Coupled Differential-Difference Equations

Consider a linear time-delay system described by the coupled differential-difference equation

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^K B_j y_j(t - r_j), \quad (1)$$

$$y_i(t) = C_i x(t) + \sum_{j=1}^K D_{ij} y_j(t - r_j), \quad i = 1, 2, \dots, K, \quad (2)$$

where  $x(t) \in \mathbb{R}^n$ ,  $y_i(t) \in \mathbb{R}^{m_i}$ , and  $K$  is the number of delay ‘‘channels’’. Without loss of generality, we assume the delays are in ascending order  $0 < r_1 < \dots < r_K$ . The initial conditions are given by  $x(0) = \psi \in \mathbb{R}^n$  and  $y_{i,r_i,0} = \phi_i \in \mathcal{PC}(r_i, m_i)$ . To simplify things, we use the notation  $y_t := (y_{1,r_1,t}, \dots, y_{K,r_K,t})$ , so that the state of the system at time  $t$  is  $(x(t), y_t)$ . Now consider the subsystem given by Equation (2) where we let  $x$  be the input. Input-to-state stability of this subsystem is required for stability of the coupled system. An LMI test for input-to-state stability of this subsystem, as given in [5], is quoted below.

*Proposition 1:* The difference equation (2) is input-to-state stable if there exist  $L_i \in \mathbb{S}^{m_i}$ ,  $L_i > 0$ ,  $i = 1, 2, \dots, K$  such that the following LMI is satisfied.

$$D^T U D - U < 0, \quad \text{where } U = \text{diag}(L_1, L_2, \dots, L_K)$$

and  $D = [D_{ij}]_{K \times K}$  is the matrix formed by using the matrices  $D_{ij}$  as sub-blocks.

### B. Quadratic Lyapunov-Krasovskii Functionals

The following is a necessary and sufficient condition for stability of the system defined by the coupled equations (1) and (2). This result supposes that the map  $x \rightarrow y$  given by

the uncoupled Equation (2) is input-to-state stable and that this can be proven using Proposition 1.

*Theorem 2 ([5]):* Suppose the conditions of Proposition 1 are satisfied. Then the coupled Equations (1) and (2) are uniformly asymptotically stable if and only if there exist an  $\varepsilon > 0$ , a matrix  $P \in \mathbb{S}^n$ , and functions  $Q_i(s) \in \mathbb{R}^{n \times m_i}$ ,  $R_{ij}(s, \eta) \in \mathbb{R}^{m_i \times m_j}$ , and  $S_i(s) \in \mathbb{S}^{m_i}$  such that for any  $\psi \in \mathbb{R}^n$  and  $\phi = (\phi_1, \phi_2, \dots, \phi_K) \in \mathcal{PC}$ , we have that

$$V(\psi, \phi) \geq \varepsilon \psi^T \psi \quad \text{and} \quad \dot{V}(\psi, \phi) \leq -\varepsilon \psi^T \psi,$$

where

$$\begin{aligned} V(\psi, \phi) = & \psi^T P \psi + 2\psi^T \sum_{i=1}^K \int_{-r_i}^0 Q_i(s) \phi_i(s) ds \\ & + \sum_{i=1}^K \sum_{j=1}^K \int_{-r_i}^0 \int_{-r_j}^0 \phi_i^T(s) R_{ij}(s, \eta) \phi_j(\eta) d\eta ds \\ & + \sum_{i=1}^K \int_{-r_i}^0 \phi_i^T(s) S_i(s) \phi_i(s) ds. \end{aligned} \quad (3)$$

### C. Sum-of-Squares

Sum-of-Squares is a branch of polynomial computing which considers the positivity of polynomials. Although the question of polynomial positivity is NP-hard, testing whether a polynomial can be represented as the sum of squares of polynomials is tractable. For a given  $d > 0$ , let  $Z_d$  be the vector of monomials

$$Z_d(s) := [1 \quad s \quad s^2 \quad \dots \quad s^d]^T.$$

Let  $Z_{n,d} : \mathbb{R} \rightarrow \mathbb{R}^{n(d+1) \times n}$  be defined as  $Z_{n,d}(s) = I_{n \times n} \otimes Z_d(s)$ , where  $\otimes$  denotes the Kronecker product. A  $n \times n$  symmetric polynomial matrix  $G(s)$  is sum-of-squares if and only if it can be represented as  $G(s) = Z_{n,d}^T(s) J Z_{n,d}(s)$  for some positive semidefinite matrix  $J$ . This is a convex constraint on the coefficients of  $G$ . We will denote the sum-of-squares constraint on the coefficients of a polynomial by

$$G \in \Sigma_{n,d} := \{G : \mathbb{R} \rightarrow \mathbb{S}^n : G(s) = Z_{n,d}^T(s) J Z_{n,d}(s)\}.$$

That a polynomial be sum-of-squares is sufficient for positivity. However, it is necessary only in special cases. Furthermore, positivity over compact semialgebraic subsets can be tested through a combination of sum-of-squares and Positivstellensatz results in a manner akin to the S-procedure. See [14] for details. Through considerable abuse of notation, for a semialgebraic set  $H$ , we will allow ‘‘ $G(s) \in \Sigma_{n,d}$  for  $s \in H$ ’’ to denote conditions derived using a Positivstellensatz. Because of the variety and complexity of Positivstellensatz results, we do not list the conditions explicitly. The software package SOSTOOLS may be used to set up and solve the associated sum-of-squares optimization problems [19].

## III. POSITIVITY CONDITIONS

The purpose of this paper is to use polynomial computing to construct a matrix  $P$  and continuous matrix-valued functions  $Q_i(s)$ ,  $S_i(s)$ , and  $R_{ij}(s, \eta)$  such that the conditions

of Theorem 2 hold. First, we present three previous results. The first is a restatement of a result from [16] and relates positivity of the first part of the functional to positivity of the function  $M$ .

*Lemma 3 ([16]):* Suppose that  $M : [-r, 0] \rightarrow \mathbb{S}^{n+m}$  is piecewise continuous on  $s \in [-r, 0]$  with  $\{-r_0, -r_1, \dots, -r_K\}$  as possible singular points. Then there exists an  $\epsilon > 0$  such that

$$V(\psi, \phi) = \int_{-r}^0 \begin{bmatrix} \psi \\ \phi(s) \end{bmatrix}^T M(s) \begin{bmatrix} \psi \\ \phi(s) \end{bmatrix} ds \geq \epsilon \|\phi\|_{L_2}^2$$

for all  $\psi \in \mathbb{R}^n$ , and continuous  $\phi : [-r, 0] \rightarrow \mathbb{R}^m$  if and only if there exists a  $\epsilon > 0$  and a function  $T : [-r, 0] \rightarrow \mathbb{S}^n$ , piecewise continuous on  $s \in [-r, 0]$ , such that

$$M(s) + \begin{bmatrix} T(s) & 0 \\ 0 & -\epsilon I \end{bmatrix} \geq 0 \text{ for all } s \in [-r, 0]$$

$$\text{and } \int_{-r}^0 T(s) ds = 0.$$

The second result relates positivity of the second part of the functional to positivity of the matrix representation of  $N$ .

*Lemma 4 ([16]):* For any piecewise continuous polynomial function  $N : [-r, 0] \times [-r, 0] \rightarrow \mathbb{R}^{n \times n}$ ,

$$\int_{-r}^0 \int_{-r}^0 \phi^T(s) N(s, \eta) \phi(\eta) d\eta ds \geq 0$$

for all continuous  $\phi : [-r, 0] \rightarrow \mathbb{R}^n$  if and only if

$$N \in \Gamma_{n,d} := \{Z_{n,d}^T(s) L Z_{n,d}(\eta) \mid L \in \mathbb{S}^{n(d+1)}, L \geq 0\}.$$

The third result is from Proposition 2 of [3]. This is a technical lemma used in the proof of Proposition 6.

*Lemma 5 ([3]):* Suppose that  $P, Q_1, Q_2, R_1$ , and  $R_2$  are continuous functions of compatible dimensions defined on a compact interval,  $I$ . Then

$$\begin{bmatrix} 2P(s) & Q_1(s) & Q_2(s) \\ Q_1^T(s) & R_1(s) & 0 \\ Q_2^T(s) & 0 & R_2(s) \end{bmatrix} > 0$$

for all  $s \in I$  if and only if there exists some continuous function  $T$  such that for all  $s \in I$ ,

$$\begin{bmatrix} P(s) + T(s) & Q_1(s) \\ Q_1^T(s) & R_1(s) \end{bmatrix} > 0, \quad \begin{bmatrix} P(s) - T(s) & Q_2(s) \\ Q_2^T(s) & R_2(s) \end{bmatrix} > 0.$$

Now we consider the quadratic Lyapunov-Krasovskii functional given by Equation (3). In order to apply polynomial computing to positivity of this functional, we need to convert the functional positivity condition to conditions which can be enforced using SDP. To this end, we would like to apply Lemmas 3 and 4 as was done in [16]. Unfortunately, however, these lemmas cannot be directly applied to the structure of the Lyapunov-Krasovskii functional given by Equation (3). The first main technical result of this paper shows how Lemma 3 can be modified to take into account the structure of Functional (3).

*Proposition 6:* Suppose we are given a matrix  $P \in \mathbb{S}^n$  and continuous matrix-valued functions  $Q_i : [-r_i, 0] \rightarrow \mathbb{R}^{n \times m_i}$  and  $S_i : [-r_i, 0] \rightarrow \mathbb{S}^{m_i}, i = 1, 2, \dots, K$ . Then the following two statements are equivalent.

1) There exists an  $\epsilon > 0$  such that

$$\begin{aligned} & \psi^T P \psi + 2\psi^T \sum_{i=1}^K \int_{-r_i}^0 Q_i(s) \phi_i(s) ds \\ & + \sum_{i=1}^K \int_{-r_i}^0 \phi_i^T(s) S_i(s) \phi_i(s) ds \geq \epsilon (\|\psi\|^2 + \|\phi\|_{L_2}^2) \end{aligned}$$

for all  $\psi \in \mathbb{R}^n$  and continuous  $\phi = (\phi_1, \phi_2, \dots, \phi_K)$ .

2) There exist continuous functions  $T_i : [-r_i, 0] \rightarrow \mathbb{S}^n$ , and an  $\epsilon > 0$  such that

$$\begin{bmatrix} \frac{1}{\sum_{i=1}^K r_i} P & Q_i(s) \\ Q_i^T(s) & S_i(s) \end{bmatrix} + \begin{bmatrix} T_i(s) & 0 \\ 0 & 0 \end{bmatrix} \geq \epsilon I$$

for all  $s \in [-r_i, 0], i = 1, 2, \dots, K$ ,

$$\sum_{i=1}^K \int_{-r_i}^0 T_i(s) ds = 0.$$

*Proof:* We begin by introducing the changes of variables  $\eta_i = s/r_i$ . Then the left-hand side of the inequality of 1) is given by

$$V_1(\psi, \phi) := \int_{-1}^0 \Phi^T(\eta) M(\eta) \Phi(\eta) d\eta,$$

where we relabeled all the  $\eta_i \rightarrow \eta$  and have defined

$$\begin{aligned} \Phi(\eta) & := [\psi^T \quad \phi_1^T(r_1\eta) \quad \dots \quad \phi_K^T(r_K\eta)]^T \\ M(\eta) & := \begin{bmatrix} P & r_1 Q_1(r_1\eta) & \dots & r_K Q_K(r_K\eta) \\ r_1 Q_1(r_1\eta)^T & r_1 S_1(r_1\eta) & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ r_K Q_K(r_K\eta)^T & 0 & 0 & r_K S_K(r_K\eta) \end{bmatrix} \end{aligned}$$

By Lemma 3,  $V_1(\psi, \phi) \geq \epsilon (\|\psi\|^2 + \|\phi\|_{L_2}^2)$  is equivalent to the existence of an  $\epsilon > 0$  and continuous function  $Z(s) : [-1, 0] \rightarrow \mathbb{S}^n$  such that

$$M(\eta) + \begin{bmatrix} Z(\eta) & 0 \\ 0 & 0 \end{bmatrix} \geq \epsilon I \quad \text{for } \eta \in [-1, 0]$$

where  $\int_{-1}^0 Z(\eta) d\eta = 0$ . Now, by applying Lemma 5

inductively, positivity of  $M(\eta) + \begin{bmatrix} Z(\eta) & 0 \\ 0 & 0 \end{bmatrix}$  is equivalent to the existence of an  $\epsilon > 0$  and some continuous functions  $L_i$  such that

$$\begin{aligned} & \begin{bmatrix} \frac{r_i}{\sum_{i=1}^K r_i} P + Z(\eta) - \sum_{i=2}^K L_i(\eta) & r_1 Q_1(r_1\eta) \\ r_1 Q_1^T(r_1\eta) & r_1 S_1(r_1\eta) \end{bmatrix} \geq \epsilon I \\ & \text{and } \begin{bmatrix} \frac{r_i}{\sum_{i=1}^K r_i} P + L_i(\eta) & r_i Q_i(r_i\eta) \\ r_i Q_i^T(r_i\eta) & r_i S_i(r_i\eta) \end{bmatrix} \geq \epsilon I \end{aligned}$$

for  $\eta \in [-1, 0]$  and  $i = 2, \dots, K$ . Now, if we define  $T_i(\eta) = \frac{1}{r_i} L_i(\eta/r_i)$  for  $i = 2, \dots, K$  and

$$T_1(\eta) = \frac{1}{r_1} \left( Z(\eta/r_1) - \sum_{i=2}^K L_i(\eta/r_1) \right),$$

then by changing variables back to  $s_i = \eta/r_i$

$$\sum_{i=1}^K \int_{-r_i}^0 T_i(\eta) d\eta = \int_{-1}^0 \left( Z(s_1) - \sum_{i=2}^K L_i(s_1) \right) ds_1 + \sum_{i=2}^K \int_{-1}^0 L_i(s_i) ds_i = 0.$$

Furthermore, by the same change of variables, the inequalities become

$$r_i \begin{bmatrix} \frac{1}{\sum_{i=1}^K r_i} P + T_i(s_i) & Q_1(s_i) \\ Q_i^T(s_i) & r_i S_i(s_i) \end{bmatrix} > 0$$

for  $s_i \in [-r_i, 0]$ ,  $i = 1, \dots, K$ ,

which is equivalent to Statement 2) of the Proposition. Thus we have that 1)  $\Rightarrow$  2). Furthermore, if we start by letting  $L_i = r_i T_i(r_i \eta)$ , then all steps can be reversed to show 2)  $\Rightarrow$  1). This completes the proof.  $\blacksquare$

To represent the conditions of Proposition 6 using SDP, we can require our functions to be polynomial and strengthen the positivity conditions “ $> 0$ ” by using the Sum-of-Squares condition “ $\in \Sigma_{m,d}$ ” for some  $m, d > 0$ . Such conditions may be conservative due the choice of  $d$ , but can be enforced using SDP - the critical point. We now present the second significant result of this paper, which shows how the concepts in Lemma 4 can be applied to our new Lyapunov functional.

*Proposition 7:* Let  $m = \sum_{j=1}^K m_j$ . Suppose  $R_{ij} : [-r_i, 0] \times [-r_j, 0] \rightarrow \mathbb{R}^{m_i \times m_j}$  are polynomial matrices of degree  $d$ . Then

$$V_2(\phi) := \sum_{i=1}^K \sum_{j=1}^K \int_{-r_i}^0 \int_{-r_j}^0 \phi_i^T(s) R_{ij}(s, \eta) \phi_j(\eta) d\eta ds \geq 0$$

for all  $\phi = (\phi_1, \phi_2, \dots, \phi_K) \in \mathcal{PC}$  if and only if  $R \in \Gamma_{m,d}$ , where

$$R(s, \eta) = \begin{bmatrix} R_{11}(r_1 s, r_1 \eta) & \cdots & R_{1K}(r_1 s, r_K \eta) \\ \vdots & \ddots & \vdots \\ R_{K1}(r_K s, r_1 \eta) & \cdots & R_{KK}(r_K s, r_K \eta) \end{bmatrix},$$

and

$$\Gamma_{m,d} = \left\{ Z_{m,d}^T(s) L Z_{m,d}(\eta) \mid L \in \mathbb{S}^{m(d+1)}, L \geq 0 \right\}.$$

*Proof:* Introduce the following new variables

$$s = r_i \omega_i, \quad \eta = r_j \theta_j, \quad i, j = 1, \dots, K.$$

Then,  $V_2(\phi)$  may be expressed as

$$\begin{aligned} V_2(\phi) &= \sum_{i=1}^K \sum_{j=1}^K r_i r_j \int_{-1}^0 \int_{-1}^0 \phi_i^T(r_i \omega_i) \\ &\quad \times R_{ij}(r_i \omega_i, r_j \theta_j) \phi_j(r_j \theta_j) d\omega_i d\theta_j \\ &= \int_{-1}^0 \int_{-1}^0 \Phi^T(s) R(s, \eta) \Phi(\eta) d\eta ds, \end{aligned}$$

where we have unified the variables and where

$$\Phi(s) = [r_1 \phi_1^T(r_1 s) \quad r_2 \phi_2^T(r_2 s) \quad \cdots \quad r_K \phi_K^T(r_K s)]^T.$$

Now Lemma 4 states that since  $R$  is a polynomial of degree  $d$ , positivity of  $V_2$  is equivalent to  $R \in \Gamma_{m,d}$ .  $\blacksquare$

Note that the conditions of Proposition 7 are SDP constraints on the coefficients of the polynomials  $R_{ij}$ . Therefore, in combination with Proposition 6, this result forms the basis of a Sum-of-Squares/SDP approach to optimization of Lyapunov-Krasovskii functionals of the form given by Equation (3). This is expanded upon in the following two sections.

#### IV. THE LYAPUNOV-KRASOVSKII DERIVATIVE CONDITION

In this section, we obtain a condition for negativity of the derivative of the Lyapunov-Krasovskii functional presented as (3). Through some manipulation, it can be shown that this derivative may be expressed as follows.

$$\begin{aligned} \dot{V}(\psi, \phi) &= \sum_{i=1}^K \sum_{j=1}^K \int_{-r_i}^0 \int_{-r_j}^0 \phi_i^T(s) E_{ij}(s, \eta) \phi_j(\eta) d\eta ds \\ &\quad + \sum_{i=1}^K \int_{-r_i}^0 z_i^T(s) F_i(s) z_i(s) ds, \end{aligned} \quad (4)$$

where

$$z_i(s) = [\psi^T \quad \phi_1^T(-r_1) \quad \cdots \quad \phi_K^T(-r_K) \quad \phi_i^T(s)]^T,$$

and

$$E_{ij}(s, \eta) = -\frac{\partial R_{ij}(s, \eta)}{\partial s} - \frac{\partial R_{ij}(s, \eta)}{\partial \eta},$$

$$F_i(s) = \begin{bmatrix} \frac{1}{\sum_{j=1}^K r_j} F_{11} & \frac{1}{\sum_{j=1}^K r_j} F_{12} & F_{13i}(s) \\ \frac{1}{\sum_{j=1}^K r_j} F_{12}^T & \frac{1}{\sum_{j=1}^K r_j} F_{22} & F_{23i}(s) \\ F_{13i}^T(s) & F_{23i}^T(s) & F_{33i}(s) \end{bmatrix},$$

$$F_{11} = PA + A^T P + \sum_{j=1}^K [Q_j(0) C_j$$

$$+ C_j^T Q_j^T(0) + C_j^T S_j(0) C_j],$$

$$F_{12} = [G_1 \quad \cdots \quad G_K],$$

$$G_j = \sum_{k=1}^K [Q_k(0) D_{kj} + C_k^T S_k(0) D_{kj}] + P B_j$$

$$- Q_j(-r_j), \quad \text{for } j = 1, 2, \dots, K,$$

$$F_{22} = \sum_{i=1}^K \begin{bmatrix} D_{i1}^T S_i(0) D_{i1} & \cdots & D_{i1}^T S_i(0) D_{iK} \\ \vdots & \ddots & \vdots \\ D_{iK}^T S_i(0) D_{i1} & \cdots & D_{iK}^T S_i(0) D_{iK} \end{bmatrix}$$

$$- \text{diag}(S_1(-r_1) \quad S_2(-r_2) \quad \cdots \quad S_K(-r_K)),$$

$$F_{13i}(s) = A^T Q_i(s) + \sum_{j=1}^K C_j^T R_{ij}^T(s, 0) - \frac{dQ_i(s)}{ds},$$

$$F_{23i}(s) = [H_{i1}^T(s) \quad \cdots \quad H_{iK}^T(s)]^T,$$

$$H_{ij}(s) = B_j^T Q_i(s) + \sum_{k=1}^K D_{kj}^T R_{ik}^T(s, 0) - R_{ij}^T(s, -r_j),$$

$$F_{33i}(s) = -\frac{dS_i(s)}{ds}.$$

Notice that  $\dot{V}$  has the same form as  $V$ . Therefore, Propositions 6 and 7 may be used to obtain conditions for negativity of  $\dot{V}$  that are suitable for implementation via polynomial optimization. This is summarized in the following Proposition.

*Proposition 8:*  $\dot{V}(\psi, \phi) \leq -\epsilon\|\psi\|^2$  for some  $\epsilon > 0$  if there exist matrix functions  $W_i : [-r_i, 0] \rightarrow \mathbb{S}^{n+m}$ ,  $i = 1, 2, \dots, K$ , with  $m = \sum_{j=1}^K m_j$ , such that

$$\begin{aligned} -F_i(s) + \begin{bmatrix} W_i(s) & 0 \\ 0 & 0 \end{bmatrix} &\in \Sigma_{n+m+m_i, d} \\ \text{for } s \in [-r_i, 0], \quad i &= 1, 2, \dots, K, \\ \sum_{i=1}^K \int_{-r_i}^0 W_i(s) ds &= 0, \end{aligned}$$

and  $-E \in \Gamma_{m, d}$ , where

$$E(s, \eta) = \begin{bmatrix} E_{11}(r_1 s, r_1 \eta) & \cdots & E_{1K}(r_1 s, r_K \eta) \\ \vdots & \ddots & \vdots \\ E_{K1}(r_K s, r_1 \eta) & \cdots & E_{KK}(r_K s, r_K \eta) \end{bmatrix}.$$

*Proof:* The negativity condition may be written as

$$-\dot{V}(\psi, \phi) = -V_{d1}(\psi, \phi) - V_{d2}(\psi, \phi) \geq \epsilon\|\psi\|^2,$$

where  $V_{d1}$  and  $V_{d2}$  are the first and second parts of Equation (4). If the conditions of the theorem are satisfied, then Proposition 6 implies that there exists some  $\epsilon > 0$  such that  $-V_{d2}(\psi, \phi) \geq \epsilon\|\psi\|^2$ . Likewise, Proposition 7 may be used to show that  $-V_{d1}(\psi, \phi) \geq 0$ . Thus the negativity condition is satisfied. ■

Note that if  $Q_i$ ,  $S_i$ , and  $R_{ij}$  are polynomial, then the map from the coefficients of these polynomials to those of  $E_{ij}$  and  $F_i$  will be linear, easily represented using packages such as SOSTools.

## V. STABILITY CONDITIONS

In this section, we summarize the results of this paper by giving conditions for stability in a form which can be implemented using a combination of Sum-of-Squares and SDP.

*Theorem 9:* Let  $m = \sum_{j=1}^K m_j$ . The coupled delay-differential system described by Equations (1) and (2) is asymptotically stable if there exist a matrix  $P \in \mathbb{S}^n$ , and polynomial matrices  $Q_i : [-r_i, 0] \rightarrow \mathbb{R}^{n \times m_i}$ ,  $S_i : [-r_i, 0] \rightarrow \mathbb{S}^{m_i}$ ,  $T_i : [-r_i, 0] \rightarrow \mathbb{S}^n$ ,  $R_{ij} : [-r_i, 0] \times [-r_j, 0] \rightarrow \mathbb{R}^{m_i \times m_j}$ ,  $W_i : [-r_i, 0] \rightarrow \mathbb{S}^{n+m}$ ,  $i, j = 1, 2, \dots, K$  such that

$$\begin{aligned} \begin{bmatrix} \frac{1}{\sum_{i=1}^K r_i} P & Q_i(s) \\ Q_i^T(s) & S_i(s) \end{bmatrix} + \begin{bmatrix} T_i(s) & 0 \\ 0 & 0 \end{bmatrix} &\in \Sigma_{n+m_i, d} \\ \text{for } s \in [-r_i, 0], \quad i &= 1, \dots, K, \\ -F_i(s) + \begin{bmatrix} W_i(s) & 0 \\ 0 & 0 \end{bmatrix} &\in \Sigma_{n+m+m_i, d} \\ \text{for } s \in [-r_i, 0], \quad i &= 1, \dots, K, \\ \sum_{i=1}^K \int_{-r_i}^0 T_i(s) ds &= 0, \quad \sum_{i=1}^K \int_{-r_i}^0 W_i(s) ds = 0, \\ R &\in \Gamma_{m, d}, \quad -E \in \Gamma_{m, d}, \end{aligned}$$

where  $R$  and  $E$  are the composite matrix functions defined by the blocks  $R_{ij}$  and  $E_{ij}$ , respectively and where the functions  $F_i$  and  $E_{ij}$  are as defined in the previous section.

*Proof:* We first show that the conditions of the theorem imply that the system satisfies the regularity conditions of the Lyapunov theorem. That is, the conditions of Proposition 1 are satisfied. Observe that the second inequality and the definition of  $F$  implies that

$$-F_{33i}(s) = \frac{\partial S_i(s)}{\partial s} > 0,$$

which in turn implies that  $S_i(0) \geq S_i(-r_i)$  for all  $i = 1, \dots, K$ . Now define  $L_i = S_i(0)$ , as per Proposition 1. Then we have that the corresponding

$$U = \text{diag} ( S_1(0) \ S_2(0) \ \cdots \ S_K(0) ).$$

Now, it can be shown that the conditions of the theorem imply that  $F_{22} < 0$ . Therefore by definition

$$\begin{aligned} U - D^T U D &\geq \text{diag} ( S_1(-r_1) \ S_2(-r_2) \ \cdots \ S_K(-r_K) ) \\ -D^T U D &= -F_{22} > 0, \end{aligned}$$

where the first inequality holds because  $S_i(0) \geq S_i(-r_i)$ . Finally, since the first inequality condition of the theorem implies  $S_i(s) > 0$  for all  $s$ , we have that  $U > 0$ , which means that all conditions of Proposition 1 are satisfied. Therefore, the technical conditions of Theorem 2 are satisfied and stability can be established by positivity of the relevant Lyapunov function and negativity of its derivative. As discussed previously, positivity of the functional is established via Propositions 6 and 7. Negativity of the derivative is established in the same manner. ■

The result in Theorem 9 can be extended to uncertain systems and time-varying delay systems. In particular the uncertain parameters in the uncertain system matrices or time-varying delays can be included in the variables of polynomial matrices of polynomial Lyapunov-Krasovskii functionals, as given in [11].

### A. Complexity Analysis

It can be shown that the computing complexity of the SDP can be reduced substantially by our method when state variables are rather large but delay components are scalar or low dimensional, i.e.  $m_i \ll n$ .

For instance, consider the case with a single delay. The number of decision variables in the SDP associated with Theorem 9 is of the order  $O(n, m, d) = (n + md)^2$ , where  $n$  is the dimension of  $x(t)$ ,  $d$  is the degree of the polynomials, and  $m$  is the dimension of  $y(t)$ . For the SDP problem associated with the previous formulation described in [16], the number of decision variables is of the order described above with  $m$  replaced by  $n$ . Since the worst-case complexity of SDP is roughly proportional to  $q^3$ , where  $q$  is the number of decision variables, it can be concluded that the complexity of the SDP can be reduced by several orders of magnitude when the state variables are large but the delay components are scalar or low dimensional.

## VI. NUMERICAL EXAMPLES

In this section, we present several numerical examples to illustrate how the computational burden of the Sum of Square method is reduced.

**Example 1** Consider the system

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y(t-r), \\ y(t) &= [1 \ 0] x(t).\end{aligned}$$

This system was studied in [16]. The bounds  $r_{\min}$ ,  $r_{\max}$  and the number of decision variables  $q$  in SDP for different degree  $d$  of monomials are shown below.

$d$	Methods	$r_{\min}$	$r_{\max}$	$q$
1	Theorem 11 in [16]	.10017	1.6249	102
	Theorem 9	.10017	1.62	42
2	Theorem 11 in [16]	.10017	1.7172	176
	Theorem 9	.10017	1.71557	63
3	Theorem 11 in [16]	.10017	1.71785	274
	Theorem 9	.10017	1.71783	90

**Example 2** Consider the system

$$\begin{aligned}\dot{x}(t) &= A_0 x(t) + B_1 y_1 \left( t - \frac{r}{\sqrt{2}} \right) + B_2 y_2 (t-r), \\ y_1(t) &= C_1 x(t), \\ y_2(t) &= C_2 x(t),\end{aligned}$$

where

$$\begin{aligned}A_0 &= \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.9 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 0 & 0 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1.45 & 0 & 0 \end{bmatrix}^T, \\ B_2 &= [0 \ -0.5 \ 1 \ 0 \ 0 \ 0]^T, \\ C_1 &= \begin{bmatrix} 0.2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad C_2 = [0 \ 1 \ 0 \ 0 \ 0 \ 0].\end{aligned}$$

By using Theorem 11 in [16] and Theorem 9, the maximum delay  $r_{\max}$  and the number of decision variables  $q$  in SDP for different degree of monomials  $d$  are included in the following table.

$d$	Methods	$r_{\max}$	$q$
1	Theorem 11 in [16]	2.4	2808
	Theorem 9	2.6	452
2	Theorem 11 in [16]	*	4908
	Theorem 9	2.7	615

We remark that for degree 2, the old formulation resulted in a SDP at the very edge of tractability for a desktop computer. In the Sum of Square method, the worst-case computing complexity is  $q^3$ . Therefore, the results in Example 1 and Example 2 show that our method dramatically reduced the computational complexity by several orders of magnitude for similar accuracy.

## VII. CONCLUSION

In this paper, it was shown that the complexity of the Sum-of-Square/SDP conditions for Lyapunov-Krasovskii functional based stability analysis of linear time-delay systems can be substantially reduced through a coupled differential-difference equation based formulation. The reduction of complexity is particularly large in the case where the delayed system is high-dimensional with relatively few delays in relatively few channels—a situation which arises commonly in practice.

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