Accelerating Convergence of Sum-of-Square Stability Analysis of Coupled Differential-Difference Equations

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Abstract: This article aims at accelerating the convergence of Lyapunov-Krasovskii stability analysis of coupled differential-difference equations using sum-of-square formulation. Under the assumption that the single integral and double integral terms are both positive definite, a necessary and sufficient condition for the quadratic integral expression is obtained. This result is applied to the Lyapunov-Krasovskii functional and derivative conditions. The method is less conservative than the previous method with identical order of polynomials. The effectiveness of this method is illustrated by numerical examples.

Keywords: Delay, Sum-of-Square, Lyapunov-Krasovskii functional

NOTATION

$\mathbb{R}$ denotes the set of reals. $\mathbb{R}^n$ and $\mathbb{R}^{p\times q}$ denote the sets of n-vectors and $p\times q$ matrices with real elements. $\mathbb{S}^n$ denotes the set of the $n\times n$ symmetric real matrices. For $X\in\mathbb{S}^n$, the notation $X\geq 0$ ($X>0$) means that X is positive semidefinite (definite). I denotes the identity matrix. For a given constant $r>0$ and positive integer n, $\mathbb{PC}(r,n)$ denotes the set of bounded functions $f:[-r,0)\to\mathbb{R}^n$ that are right continuous everywhere with possibly a finite number of discontinuous points. For a given function y defined in an interval $\mathbb{I}\supset[t-r,t)$, $y_{r,t}$ is a function defined on $[-r,0)$ by the relation $y_{r,t}(s)=y(t+s)$. The short-hand notation $\phi=(\phi_1,\phi_2,\ldots,\phi_K)\in\mathbb{PC}$ is used to denote $\phi_i\in\mathbb{PC}(r_i,m_i), i=1,2,\ldots,K$.

1. INTRODUCTION

Stability analysis of linear time-delay systems has been an active area of research. A point of interest is to construct polynomial-time algorithms for the stability analysis based on complete quadratic Lyapunov-Krasovskii functionals. A numerically implementable and asymptotically accurate Lyapunov-Krasovskii functional method, known as the discretized Lyapunov functional approach, was presented in proposed in Gu (2001). In this method, the matrix functions are restricted to be piecewise linear, and the resulting stability conditions are in the form of linear matrix inequalities (LMI). An alternative asymptotically accurate numerical method was the Sum-of-Square (SOS) method presented in Peet and Papachristodoulou (2006), wherein the parameters of the quadratic Lyapunov-Krasovskii functional are restricted to be polynomials by using SOS decomposition together with semidefinite programming (SDP).

While most state-space formulation of time-delay systems have been in the form of differential-difference equations, substantial attention has been paid to the research on coupled differential-difference equations. See, e.g. Fridman (2002), Pepe and Verriest (2003), Rasvan and Niculescu (2002), and Rasvan (2006). This formulation includes as special cases the traditional differential-difference equations of both retarded and neutral type, as well as some singular systems with time delays.

The stability analysis of coupled differential-difference equations based on the assumption of input-to-state stability of the difference equation was presented in Pepe, Jiang and Fridman (2007). The condition was strengthened to uniform asymptotic stability and extended to the general coupled differential-functional equations in Gu and Liu (2009), where the possibility of drastic reduction of computational time for the single delay case was illustrated for discretized Lyapunov-Krasovskii functional method. A standard form of coupled differential-difference equations with one independent delay in each channel was proposed in Gu (2010) to model practical systems with multiple delays through a process of "pulling out delays". It was shown that systems with multiple delays in some channels can be easily transformed to such a standard form. In view of the fact that delay elements in most practical systems are low-dimensional, the computational cost of Lyapunov-Krasovskii functional based stability analysis may be drastically reduced. Indeed, several orders of magnitude of reduction of computational cost was observed for both discretized Lyapunov-Krasovskii functional method presented in Li and Gu (2010) and the SOS method presented in Zhang, Peet and Gu (2010) that uses recent results in Peet and Papachristodoulou (2009).

In this article, a necessary and sufficient condition for positivity of quadratic integral expression is obtained under the assumption that the single integral and double integral terms are both positive definite. This result is applied to the Lyapunov-Krasovskii functional and derivative conditions. On the basis of the result, the SOS formulation for stability analysis of coupled differential-difference equations with multiple delays are derived, and the conditions are less conservative than the previous method when the order of polynomials is low. Some numerical examples are presented to illustrate the effectiveness of the method.

2. PRELIMINARIES

Consider a linear time-delay system described by the following coupled differential-difference equations

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{K} B_j y_j(t - r_j), \quad (1)$$

$$y_i(t) = C_i x(t) + \sum_{j=1}^{K} D_{ij} y_j(t - r_j),\ i = 1, 2, \ldots K, \quad (2)$$

where $x(t) \in \mathbb{R}^n$, $y_i(t) \in \mathbb{R}^{m_i}$, and K is the number of delay channels. The initial conditions are given by

$$x(0) = \psi \in \mathbb{R}^n, \quad \text{and} \quad y_{i r_i,0} = \phi_i \in \mathbb{PC}(r_i, m_i).$$

The state of the system at time t is $(x(t), y_t)$, where $y_t := (y_{1r_1,t}, y_{2r_2,t}, \cdots, y_{Kr_K,t}) \in \mathbb{PC}$. It is also convenient to write

$$m = \sum_{i=1}^{K} m_i. \quad (3)$$

A necessary and sufficient condition for the stability of the system in the form of quadratic Lyapunov-Krasovskii functional is given in Gu (2010), and is presented below with appropriate adaptation of notation.

Theorem 1. Suppose there exist $U_i \in \mathbb{S}^{m_i}$, $U_i > 0$, $i = 1, 2, \ldots K$ such that

$$D^T U D - U < 0, \quad (4)$$

where

$$U = \mathrm{diag}(U_1, U_2, \ldots, U_K), \quad (5)$$

and

$$D = \begin{bmatrix} D_{11} & D_{12} & \cdots & D_{1K} \\ D_{21} & D_{22} & \cdots & D_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ D_{K1} & D_{K2} & \cdots & D_{KK} \end{bmatrix}. \quad (6)$$

Then the system described by (1) and (2) is exponentially stable if and only if there exist a $P \in \mathbb{S}^n$, and absolutely continuous matrix functions $Q_i(s) \in \mathbb{R}^{n \times m_i}$, $R_{ij}(s, \eta) = R_{ji}^T(\eta, s) \in \mathbb{R}^{m_i \times m_j}$, and $S_i(s) \in \mathbb{S}^{m_i}$, such that the following functional

$$\begin{aligned} V(\psi, \phi) = {} & \psi^T P \psi + 2\psi^T \sum_{i=1}^{K} \int_{-r_i}^{0} Q_i(s)\phi_i(s) ds \\ & + \sum_{i=1}^{K}\sum_{j=1}^{K} \int_{-r_i}^{0}\int_{-r_j}^{0} \phi_i^T(s) R_{ij}(s, \eta)\phi_j(\eta) ds d\eta \\ & + \sum_{i=1}^{K} \int_{-r_i}^{0} \phi_i^T(s) S_i(s) \phi_i(s) ds, \end{aligned} \quad (7)$$

satisfies

$$V(\psi, \phi) \geq \varepsilon \psi^T \psi, \quad (8)$$

$$\dot{V}(\psi, \phi) \leq -\varepsilon \psi^T \psi, \quad (9)$$

for all $\psi \in \mathbb{R}^n$, $\phi = (\phi_1, \phi_2, \ldots, \phi_K) \in \mathbb{PC}$ and some $\varepsilon > 0$.

In Theorem 1, $\dot{V}$ is the derivative of V along the system trajectory, and can be expressed as

$$\begin{aligned} \dot{V}(\psi, \phi) = {} & \sum_{i=1}^{K}\sum_{j=1}^{K} \int_{-r_i}^{0}\int_{-r_j}^{0} \phi_i^T(s) E_{ij}(s, \eta)\phi_j(\eta) d\eta ds \\ & + \sum_{i=1}^{K} \int_{-r_i}^{0} z_i^T(s) F_i(s) z_i(s) ds, \end{aligned} \quad (10)$$

where

$$z_i(s) = \left[\ \psi^T \ \ \phi_1^T(-r_1) \ \cdots \ \phi_K^T(-r_K) \ \ \phi_i^T(s)\ \right]^T,$$

and

$$E_{ij}(s, \eta) = -\frac{\partial R_{ij}(s, \eta)}{\partial s} - \frac{\partial R_{ij}(s, \eta)}{\partial \eta},$$

$$F_i(s) = \begin{bmatrix} \frac{1}{\sum_{j=1}^{K} r_j} F_{11} & \frac{1}{\sum_{j=1}^{K} r_j} F_{12} & F_{13i}(s) \\ \frac{1}{\sum_{j=1}^{K} r_j} F_{12}^T & \frac{1}{\sum_{j=1}^{K} r_j} F_{22} & F_{23i}(s) \\ F_{13i}^T(s) & F_{23i}^T(s) & F_{33i}(s) \end{bmatrix},$$

$$F_{11} = PA + A^T P + \sum_{j=1}^{K} \left[Q_j(0) C_j + C_j^T Q_j^T(0) + C_j^T S_j(0) C_j \right],$$

$$F_{12} = [\, G_1 \ G_2 \ \cdots \ G_K \,],$$

$$G_j = \sum_{k=1}^{K} [\, Q_k(0) D_{kj} + C_k^T S_k(0) D_{kj} \,] + P B_j - Q_j(-r_j),$$

$$F_{22} = D^T \hat{S}_0 D - \hat{S}_r,$$

$$\hat{S}_0 = \mathrm{diag}\,(\, S_1(0) \ S_2(0) \ \cdots \ S_K(0) \,),$$

$$\hat{S}_r = \mathrm{diag}\,(\, S_1(-r_1) \ S_2(-r_2) \ \cdots \ S_K(-r_K) \,),$$

$$F_{13i}(s) = A^T Q_i(s) + \sum_{j=1}^{K} C_j^T R_{ij}^T(s, 0) - \frac{dQ_i(s)}{ds},$$

$$F_{23i}(s) = \left[\ H_{i1}^T(s) \ H_{i2}^T(s) \ \cdots \ H_{iK}^T(s)\ \right]^T,$$

$$H_{ij}(s) = B_j^T Q_i(s) + \sum_{k=1}^{K} D_{kj}^T R_{ik}^T(s, 0) - R_{ij}^T(s, -r_j),$$

$$F_{33i}(s) = -\frac{dS_i(s)}{ds}.$$

3. JOINT POSITIVITY

This section presents a necessary and sufficient condition for positivity of the Lyapunov-Krasovskii functional under the assumption that the single integral and double integral terms are both positive definite. Unlike the previous results in Peet and Papachristodoulou (2009) and Zhang, Peet and Gu (2010), wherein positivity is enforced on the two individual parts of the quadratic functional, we enforce joint positivity on the entire quadratic functional by introducing some new variables. Some concepts from the elementary linear operator theory is needed. The readers are referred to Kato (1966) and Kolmogorov and Fomin (1975) for background in such theory.

Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces over $\mathbb{R}$. Let $\langle\cdot,\cdot\rangle$ denotes the inner product on both $\mathbb{X}$ and $\mathbb{Y}$. However, they may not necessarily be Hilbert spaces as the norms may not necessarily be defined by the inner products. The set of all bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$ is denoted as $\mathbb{L}(\mathbb{X},\mathbb{Y})$. Let $\mathcal{A}\in\mathbb{L}(\mathbb{X},\mathbb{X})$ and $\mathcal{B}\in\mathbb{L}(\mathbb{X},\mathbb{Y})$. Then,

(1) The *adjoint operator* $\mathcal{B}^*$ of $\mathcal{B}$ is defined as $\mathcal{B}^*\in\mathbb{L}(\mathbb{Y},\mathbb{X})$ that satisfies $\langle\mathcal{B}^*y,x\rangle=\langle y,\mathcal{B}x\rangle$ for all $x\in\mathbb{X}$ and $y\in\mathbb{Y}$. When $\mathbb{Y}=\mathbb{X}$, then $\mathcal{B}$ is said to be self-adjoint if $\mathcal{B}^*=\mathcal{B}$.
(2) If $\mathcal{A}\in\mathbb{L}(\mathbb{X},\mathbb{X})$, and $\mathcal{A}$ is self-adjoint, then $\mathcal{A}$ is *positive* if $\langle x,\mathcal{A}x\rangle\geq 0$ for all $x\in\mathbb{X}$. It is *coercive* if there exists an $\varepsilon>0$ such that $\langle x,\mathcal{A}x\rangle\geq\varepsilon\langle x,x\rangle$ for all $x\in\mathbb{X}$. If $\mathcal{A}$ is coercive, then $\mathcal{A}^{-1}$ exists, is bounded, self-adjoint, and coercive.
(3) $\mathcal{B}$ is *bounded* if there exists a constant $M>0$ such that $\|Px\|\leq M\|x\|$ for all $x\in\mathbb{X}$.

The main idea of this paper is based on the following theorem.

Theorem 2. Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces over $\mathbb{R}$, with inner product $\langle\cdot,\cdot\rangle$ defined on both $\mathbb{X}$ and $\mathbb{Y}$. Let $\mathcal{P}\in\mathbb{L}(\mathbb{Y},\mathbb{Y})$, $\mathcal{Q}\in\mathbb{L}(\mathbb{X},\mathbb{Y})$, $\mathcal{S}\in\mathbb{L}(\mathbb{X},\mathbb{X})$, and $\mathcal{R}\in\mathbb{L}(\mathbb{X},\mathbb{X})$. Suppose $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{S}$ are self-adjoint, $\mathcal{R}$ is positive, and $\mathcal{S}$ is coercive. Then there exist $\mathcal{Q}^1\in\mathbb{L}(\mathbb{X},\mathbb{Y})$ and $\mathcal{P}^1=\mathcal{P}^{1*}\in\mathbb{L}(\mathbb{Y},\mathbb{Y})$, such that

$$\left\langle u,(\mathcal{P}-\mathcal{P}^1)u\right\rangle+2\left\langle u,\left(\mathcal{Q}-\mathcal{Q}^1\right)\phi\right\rangle+\langle\phi,\mathcal{S}\phi\rangle\geq 0,\quad(11)$$
$$\left\langle u',\mathcal{P}^1u'\right\rangle+2\left\langle u',\mathcal{Q}^1\phi'\right\rangle+\langle\phi',\mathcal{R}\phi'\rangle\geq 0,\quad(12)$$

are satisfied for all $u,u'\in\mathbb{Y}$ and $\phi,\phi'\in\mathbb{X}$ if and only if

$$\langle u,\mathcal{P}u\rangle+2\langle u,\mathcal{Q}\phi\rangle+\langle\phi,(\mathcal{S}+\mathcal{R})\,\phi\rangle\geq 0,\qquad(13)$$

for arbitrary $u\in\mathbb{Y}$ and $\phi\in\mathbb{X}$. The result is still valid if we restrict $\mathcal{Q}^1\in\mathbb{Q}\subset\mathbb{L}(\mathbb{X},\mathbb{Y})$, as long as $\mathbb{Q}$ contains the element $\mathcal{Q}\left(\mathcal{S}+\mathcal{R}\right)^{-1}\mathcal{R}$.

Proof. This is a consequence of Lemmas 7 and 8. ■

Let $\mathbb{X}=\mathbb{PC}$, $\mathbb{Y}=\mathbb{R}^n$. For $\psi,\chi\in\mathbb{R}^n$ and $\phi,\omega\in\mathbb{PC}$, define

$$\langle\psi,\chi\rangle=\psi^T\chi,\ \ |\psi|=\sqrt{\langle\psi,\psi\rangle},$$
$$\|\phi\|=\max_{1\leq i\leq K}\sup_{s\in[-r_i,0)}|\phi_i(s)|,$$
$$\langle\phi,\omega\rangle=\sum_{i=1}^{K}\int_{-r_i}^{0}\phi_i^T(s)\omega_i(s)ds.$$

Note that we have followed the convention that the norms in finite-dimensional spaces are denoted by $|\cdot|$. Notice also that the norm on $\mathbb{PC}$ is not from the inner product. The following proposition is a consequence of Theorem 2.

Proposition 3. Given matrix $P\in\mathbb{S}^n$, and absolutely continuous matrix functions $Q_i:[-r_i,0]\to\mathbb{R}^{n\times m_i}$, $S_i:[-r_i,0)\to\mathbb{S}^{m_i}$, and $R_{ij}:[-r_i,0)\times[-r_j,0)\to\mathbb{R}^{m_i\times m_j}$, $R_{ij}(s,\eta)=R_{ji}^T(\eta,s)$. Let S_i and R_{ij} satisfy

$$S_i(s)>\varepsilon I\text{ for some }\varepsilon>0\text{ and all }s\in[-r_i,0),\qquad(14)$$

and

$$\sum_{i=1}^{K}\sum_{j=1}^{K}\int_{-r_i}^{0}\int_{-r_j}^{0}\phi_i^T(s)R_{ij}(s,\eta)\phi_j(\eta)dsd\eta\geq 0,\qquad(15)$$

for all $\phi\in\mathbb{PC}$. Then, there exists an $\varepsilon>0$ such that the functional $V(\psi,\phi)$ defined in (7) satisfies (8) for all $\psi\in\mathbb{R}^n$, $\phi\in\mathbb{PC}$ if and only if there exist an $\epsilon>0$, a matrix $P^1\in\mathbb{S}^n$, and absolutely continuous matrix functions $Q_i^1:[-r_i,0]\to\mathbb{R}^{n\times m_i}$, $i=1,2,\ldots,K$, such that

$$\psi^T(P-P^1)\psi+2\psi^T\sum_{i=1}^{K}\int_{-r_i}^{0}\left[Q_i(s)-Q_i^1(s)\right]\phi_i(s)ds$$
$$+\sum_{i=1}^{K}\int_{-r_i}^{0}\phi_i^T(s)S_i(s)\phi_i(s)ds\geq\varepsilon\psi^T\psi,\qquad(16)$$

and

$$\psi^TP^1\psi+2\psi^T\sum_{i=1}^{K}\int_{-r_i}^{0}Q_i^1(s)\phi_i(s)ds$$
$$+\sum_{i=1}^{K}\sum_{j=1}^{K}\int_{-r_i}^{0}\int_{-r_j}^{0}\phi_i^T(s)R_{ij}(s,\eta)\phi_j(\eta)dsd\eta\geq 0,\ (17)$$

are satisfied for all $\psi\in\mathbb{R}^n$ and $\phi\in\mathbb{PC}$.

Proof. Define the bounded linear operators

$$\mathcal{P}\psi=(P-\varepsilon I)\psi,\qquad(18)$$
$$\mathcal{Q}\phi=\sum_{j=1}^{K}\int_{-r_j}^{0}Q_j(s)\phi_j(s)ds,\qquad(19)$$
$$\left(\mathcal{S}\phi\right)(s)=\begin{bmatrix}S_1(s)\phi_1(s)\\ S_2(s)\phi_2(s)\\ \vdots\\ S_K(s)\phi_K(s)\end{bmatrix},\ \mathcal{R}\phi=\begin{bmatrix}\mathcal{R}_1\phi\\ \mathcal{R}_2\phi\\ \vdots\\ \mathcal{R}_K\phi\end{bmatrix},\qquad(20)$$
$$(\mathcal{R}_i\phi)(s)=\sum_{j=1}^{K}\int_{-r_j}^{0}R_{ij}(s,\eta)\phi_j(\eta)d\eta.\qquad(21)$$

Then, the condition (8) may be expressed as (13). Theorem 2 indicates that this is equivalent to (11) and (12). The operator $\mathcal{P}^1\in\mathbb{L}(\mathbb{R}^n,\mathbb{R}^n)$ has the explicit expression

$$\mathcal{P}^1\psi=P^1\psi.\qquad(22)$$

By Riesz Representation Theorem (Kolmogorov and Fomin, 1975), $\mathcal{Q}^1\in\mathbb{L}(\mathbb{PC},\mathbb{R}^n)$ may be expressed as

$$\mathcal{Q}^1\phi=\sum_{j=1}^{K}\int_{-r_j}^{0}d[T_j(s)]\phi_j(s),\qquad(23)$$

where T_j is left continuous and of bounded variation in $[-r_j,0)$. The proof will be complete if we can show that

$$\mathcal{Q}^1=\mathcal{Q}\left(\mathcal{S}+\mathcal{R}\right)^{-1}\mathcal{R}\qquad(24)$$

may be expressed in the form of

$$\mathcal{Q}^1\phi = \sum_{j=1}^{K}\int_{-r_j}^{0} Q_j^1(s)\phi_j(s)ds, \tag{25}$$

where $Q_j^1(s)$ is absolutely continuous, because then $\mathbb{Q}$ may be defined as the set of linear operators that have the form of expression (25), and therefore (11) and (12) may be expressed as (16) and (17).

The equation (24) may be written as

$$\mathcal{Q}^1 = \mathcal{Q}\mathcal{S}^{-1}\mathcal{R} - \mathcal{Q}^1\mathcal{S}^{-1}\mathcal{R}. \tag{26}$$

Using (20), (21) and (23), after exchanging the order of integration and summation, we may write

$$\mathcal{Q}^1\mathcal{S}^{-1}\mathcal{R}\phi = \sum_{i=1}^{K}\int_{-r_i}^{0} W_i(\eta)\phi_i(\eta)d\eta, \tag{27}$$

where

$$W_i(\eta) = \sum_{j=1}^{K}\int_{-r_j}^{0} d[T(s)]S_j^{-1}(s)R_{ji}(s,\eta)$$

are absolutely continuous. A similar procedure applied to $\mathcal{Q}\mathcal{S}^{-1}\mathcal{R}$ allows us to conclude that $\mathcal{Q}^1$ may indeed be expressed in the form of (25) in view of (26). ■

Similarly, we may conclude the following proposition.

Proposition 4. Let the matrix and matrix functions be defined as in Proposition 3. Let

$$-F_{33i}(s) > \varepsilon I, \text{ for some } \varepsilon > 0 \text{ and all } s \in [-r_i, 0), \tag{28}$$

and

$$-\sum_{i=1}^{K}\sum_{j=1}^{K}\int_{-r_i}^{0}\int_{-r_j}^{0} \phi_i^T(s)E_{ij}(s,\eta)\phi_j(\eta)dsd\eta \geq 0 \tag{29}$$

be satisfied for all $\phi \in \mathbb{PC}$. Then, there exists an $\varepsilon > 0$ such that $\dot{V}(\psi,\phi)$ expressed in (10) satisfies (9) for all $\psi \in \mathbb{R}^n$, $\phi \in \mathbb{PC}$ if and only if there exist an $\epsilon > 0$, matrix $\bar{F}_{aa} \in \mathbb{S}^{n+m}$, and absolutely continuous matrix functions $\bar{F}_{abi} : [-r_i, 0] \to \mathbb{R}^{(n+m)\times m_i}$, $i = 1, 2, \ldots, K$, such that

$$-\sum_{i=1}^{K}\int_{-r_i}^{0} z_i^T(s)\left[F_i(s) - \bar{F}_i(s)\right]z_i(s)ds \geq \varepsilon\psi^T\psi, \tag{30}$$

$$-\sum_{i=1}^{K}\sum_{j=1}^{K}\int_{-r_i}^{0}\int_{-r_j}^{0} \phi_i^T(s)E_{ij}(s,\eta)\phi_j(\eta)dsd\eta - \sum_{i=1}^{K}\int_{-r_i}^{0} z_i^T(s)\bar{F}_i(s)z_i(s)ds \geq 0, \tag{31}$$

are satisfied for all $\psi \in \mathbb{R}^n$ and $\phi \in \mathbb{PC}$, where

$$z_i(s) = \left[\,\psi^T \;\; \phi_1^T(-r_1) \;\cdots\; \phi_K^T(-r_K) \;\; \phi_i^T(s)\,\right]^T, \tag{32}$$

$$\bar{F}_i(s) = \begin{bmatrix} \frac{1}{\sum_{j=1}^{K} r_j}\bar{F}_{aa} & \bar{F}_{abi}(s) \\ \bar{F}_{abi}^T(s) & 0 \end{bmatrix}. \tag{33}$$

In Zhang, Peet and Gu (2010), the special case of $P^1 = 0$ and $Q_i^1(s) = 0$ were used to enforce (8). The inclusion of arbitrary parameters P^1 and Q_i^1 makes stability conditions less restrictive and therefore reduces conservatism. Similarly, the condition for (9) here is also less restrictive than the counterpart in Zhang, Peet and Gu (2010).

4. SOS STABILITY CONDITIONS

For a given integer $d > 0$, let

$$Z_d(s) := \left[\,1 \;\; s \;\; s^2 \;\cdots\; s^d\,\right]^T,$$

and

$$Z_{n,d}(s) = I_{n\times n} \otimes Z_d(s),$$

where $\otimes$ denotes the Kronecker product. In the SOS formulation, a polynomial symmetric matrix of single variable $G(s)$ with order not exceeding $2d$ is expressed as a quadratic form of $Z_{n,d}(s)$,

$$G(s) = Z_{n,d}^T(s)\,J\,Z_{n,d}(s),\; J = J^T. \tag{34}$$

A bivariate polynomial matrix $\Pi(s,\eta) = \Pi^T(\eta,s)$ with the order of each variable not exceeding d is expressed as a bilinear form of $Z_{n,d}(s)$,

$$\Pi(s,\eta) = Z_{n,d}^T(s)\,L\,Z_{n,d}(\eta),\; L = L^T. \tag{35}$$

For single variable polynomial matrices, it is useful to define

$$\Sigma_{n,d,\mathbb{I}} = \left\{ G : \mathbb{R} \to \mathbb{S}^n \;\middle|\; \begin{array}{c} G(s) = Z_{n,d}^T(s)\,J\,Z_{n,d}(s) \\ J = J^T,\, G(s) \geq 0 \text{ for } s \in \mathbb{I} \end{array} \right\}, \tag{36}$$

where $\mathbb{I}$ is an interval of $\mathbb{R}$. For bivariate polynomial matrices, it is useful to define

$$\Gamma_{n,d} = \left\{ Z_{n,d}^T(s)\,L\,Z_{n,d}(\eta) \mid L \in \mathbb{S}^{n(d+1)}, L \geq 0 \right\}. \tag{37}$$

Given the polynomial matrices that depend linearly on some parameters, the software package SOSTOOLS (Prajna, Papachristodoulou and Parrilo, 2002) is available to carry out searches among the parameters for the satisfaction of conditions in the form of $G \in \Sigma_{n,d,\mathbb{I}}$ and $\Pi \in \Gamma_{n,d}$, as well as some other convex constraints.

By restricting the matrix functions to polynomials, the stability conditions can be rendered in a SOS format as stated in the following theorem.

Theorem 5. The system described by (1) and (2) is exponentially stable if there exist matrices $P, P^1 \in \mathbb{R}^{n\times n}$, $\bar{F}_{aa} \in \mathbb{S}^{n+m}$, polynomial matrices $Q_i(s)$, $Q_i^1(s) \in \mathbb{R}^{n\times m_i}$, $S_i(s) \in \mathbb{S}^{m_i}$, $T_i(s) \in \mathbb{S}^n$, $\bar{F}_{abi}(s) \in \mathbb{R}^{(n+m)\times m_i}$, $W_i(s) \in \mathbb{S}^{n+m}$, $i = 1, 2, \ldots, K$, and bivariate polynomial matrices $R_{ij}(\xi,\eta) = R_{ji}^T(\eta,\xi) \in \mathbb{R}^{m_i\times m_j}$, $i, j = 1, 2, \ldots, K$, such that the following conditions are satisfied,

$$\begin{bmatrix} P^1 & Q^1(s) \\ Q^{1T}(s) & R(s,\eta) \end{bmatrix} \in \Gamma_{n+m,d}, \tag{38}$$

$$\begin{bmatrix} \frac{-1}{\sum_{j=1}^{K} r_j}\bar{F}_{aa} & -\bar{F}_{ab}(s) \\ -\bar{F}_{ab}^T(s) & -E(s,\eta) \end{bmatrix} \in \Gamma_{n+2m,d}, \tag{39}$$

$$\begin{bmatrix} \frac{P - P^1}{\sum_{i=1}^{k} r_i} & Q_i(s) - Q_i^1(s) \\ Q_i^T(s) - Q_i^{1T}(s) & S_i(s) \end{bmatrix} + \begin{bmatrix} T_i(s) & 0 \\ 0 & 0 \end{bmatrix} \in \Sigma_{n+m_i,d,[-r_i,0)}, \quad i = 1, 2, \ldots, K, \tag{40}$$

$$\sum_{i=1}^{K}\int_{-r_i}^{0} T_i(s)ds = 0; \tag{41}$$

$$-F_i(s) + \bar{F}_i(s) + \begin{bmatrix} W_i(s) & 0 \\ 0 & 0 \end{bmatrix}$$
$$\in \Sigma_{n+m+m_i,d,[-r_i,0)}, \qquad i = 1, 2, \dots, K, \qquad (42)$$
$$\sum_{i=1}^{K} \int_{-r_i}^{0} W_i(s) ds = 0, \qquad (43)$$

where $\bar{F}_i(s)$ is defined in (33) and

$$Q^1(s) = \left[Q_1^1(r_1 s) \cdots Q_K^1(r_K s) \right],$$
$$R(s,\eta) = \begin{bmatrix} R_{11}(r_1 s, r_1\eta) & \cdots & R_{1K}(r_1 s, r_K\eta) \\ \vdots & \ddots & \vdots \\ R_{K1}(r_K s, r_1\eta) & \cdots & R_{KK}(r_K s, r_K\eta) \end{bmatrix},$$
$$\bar{F}_{ab}(s) = \left[\bar{F}_{ab1}(s) \cdots \bar{F}_{abK}(s) \right],$$
$$E(s,\eta) = \begin{bmatrix} E_{11}(r_1 s, r_1\eta) & \cdots & E_{1K}(r_1 s, r_K\eta) \\ \vdots & \ddots & \vdots \\ E_{K1}(r_K s, r_1\eta) & \cdots & E_{KK}(r_K s, r_K\eta) \end{bmatrix}.$$

Proof. From Theorem 1, Propositions 3 and 4, it is sufficient to show that (4), (14), (15), (16), (17), (28), (29), (30) and (31) are satisfied. Obviously, (14) is implied by (40). Using a similar proof to that of Proposition 6 in Zhang, Peet and Gu (2010), it can be shown that (16) is implied by (40). Similar to the proof of Proposition 7 in Zhang, Peet and Gu (2010), we may show that (38) implies (17) by introducing a integral variable transformation and an application of Theorem 7 of Peet and Papachristodoulou (2009). The inequality (15) is implied by (17).

The proofs of (28), (30), (31) and (29) are similar to those of (14), (16), (17) and (15).

It remains to be shown that (4) is satisfied. Notice that (42) implies

$$-F_{33i} = \frac{\partial S_i(s)}{ds} > 0, \qquad (44)$$

and (42) and (43) together imply

$$F_{22} < 0. \qquad (45)$$

As was shown in the first part of the proof of Theorem 9 in Zhang, Peet and Gu (2010), (44) and (45) imply (4). ■

5. NUMERICAL EXAMPLE AND OBSERVATION

The following numerical example is presented to illustrate the effectiveness of the method.

Example 6. Consider the system

$$\dot{x}(t) = \begin{bmatrix} -1 & -1 \\ 0.1 & -0.2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} y(t-r),$$
$$y(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t).$$

It can be easily calculated (for example, by using a frequency domain method) that the system is exponentially stable for $r \in [0, r_{\max})$, where $r_{\max} = 1.69413$. The value of $r_{\max}$ is estimated numerically using Theorem 5 (referred to as "joint positivity") and Theorem 9 of Zhang, Peet and Gu (2010) (referred to as individual positivity) through a bisection process. The estimated results corresponding to each method and different order of monomial d are listed in the following table. It is clear that the joint positivity accelerated convergence to the analytical stability limit.

Order of monomial d	0	1	2
Joint positivity	1.6887	1.6934	1.6938
Individual positivity	1.6674	1.6863	1.6869

It is interesting to note that such a quick convergence to analytical stability limit is much more difficult to achieve for systems of neutral type (corresponding to a nonzero D matrix) according to our numerical experiments. Indeed, according to Gu (2010), the theoretical value of R_{ij} may be nonsmooth everywhere in $[-r_i, 0)$, although it is absolutely continuous. It was reported in Li and Gu (2010) that a quick convergence is still possible for discretized Lyapunov-Krasovskii functional method for some systems of neutral type when the theoretical value of R_{ij} contains only a finite number of nonsmooth points.

6. CONCLUSION

In this paper, it was shown that the convergence of sum-of-square stability analysis of coupled differential-difference equations can be accelerated through enforcing joint positivity on the entire Lyapunov-Krasovskii functional. The method is less conservative than the previous method with identical order of polynomials.

APPENDIX

This appendix provides two lemmas that are instrumental to the proof of Theorem 2.

Lemma 7. Let $\mathbb{X}$ and $\mathbb{Y}$ Banach spaces with inner products, and $\mathcal{P} \in \mathbb{L}(\mathbb{Y},\mathbb{Y})$, $\mathcal{Q}^1, \mathcal{Q}^2 \in \mathbb{L}(\mathbb{X},\mathbb{Y})$, $\mathcal{S}, \mathcal{R} \in \mathbb{L}(\mathbb{X},\mathbb{X})$. Suppose $\mathcal{P}$ and $\mathcal{R}$ are self-adjoint, and $\mathcal{S}$ is self-adjoint and coercive. Then there exist a $\mathcal{P}^1 = \mathcal{P}^{1*} \in \mathbb{L}(\mathbb{Y},\mathbb{Y})$ such that

$$\left\langle u_1, (\mathcal{P} - \mathcal{P}^1) u_1 \right\rangle + 2 \left\langle u_1, \mathcal{Q}^2 \phi_S \right\rangle + \left\langle \phi_S, \mathcal{S}\phi_S \right\rangle \geq 0, \qquad (46)$$
$$\left\langle u_2, \mathcal{P}^1 u_2 \right\rangle + 2 \left\langle u_2, \mathcal{Q}^1 \phi_R \right\rangle + \left\langle \phi_R, \mathcal{R}\phi_R \right\rangle \geq 0 \qquad (47)$$

are satisfied for all $u_1, u_2 \in \mathbb{Y}$ and $\phi_S, \phi_R \in \mathbb{X}$, if and only if

$$\langle u, \mathcal{P}u \rangle + 2 \left\langle u, \mathcal{Q}^2 \phi_S' \right\rangle + \left\langle \phi_S', \mathcal{S}\phi_S' \right\rangle + 2 \left\langle u, \mathcal{Q}^1 \phi_R' \right\rangle + \left\langle \phi_R', \mathcal{R}\phi_R' \right\rangle \geq 0 \qquad (48)$$

is satisfied for all $u \in \mathbb{Y}$ and $\phi_S', \phi_R' \in \mathbb{X}$.

Proof. Necessity is obvious because the left hand side of (48) may be obtained by adding up the left hand side of (46) and (47) and constraining $u_1 = u_2$.

For sufficiency, suppose that (48) is satisfied for all $u \in \mathbb{Y}$ and $\phi_S', \phi_R' \in \mathbb{X}$. Then let

$$\mathcal{P}^1 = \mathcal{P} - \mathcal{Q}^2 \mathcal{S}^{-1} \mathcal{Q}^{2*}. \qquad (49)$$

Because $\mathcal{S}$ is coercive, $\mathcal{P}^1$ is well-defined, self-adjoint and bounded. We will show that both (46) and (47) are satisfied with this $\mathcal{P}^1$. The inequality (46) can be easily verified by observing

$$\text{Left hand side of (46)}$$
$$= \left\langle \left(\phi_S + \mathcal{S}^{-1}\mathcal{Q}^{2*}u_1\right), \mathcal{S}\left(\phi_S + \mathcal{S}^{-1}\mathcal{Q}^{2*}u_1\right)\right\rangle,$$

which is nonnegative. To show (47), we notice

$$\begin{aligned}
&\text{Left hand side of (47)}\\
&= \langle u_2, \mathcal{P}u_2\rangle + 2\left\langle u_2, \mathcal{Q}^2\left(-\mathcal{S}^{-1}\mathcal{Q}^{2*}u_2\right)\right\rangle\\
&\quad + \left\langle \left(-\mathcal{S}^{-1}\mathcal{Q}^{2*}u_2\right), \mathcal{S}\left(-\mathcal{S}^{-1}\mathcal{Q}^{2*}u_2\right)\right\rangle\\
&\quad +2\left\langle u_2, \mathcal{Q}^1\phi_R\right\rangle + \langle \phi_R, \mathcal{R}\phi_R\rangle, \qquad (50)
\end{aligned}$$

which again is nonnegative according to (48). ■

Lemma 8. Let $\mathcal{P} \in \mathbb{L}(\mathbb{Y}, \mathbb{Y})$, $\mathcal{Q} \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, $\mathcal{S}, \mathcal{R} \in \mathbb{L}(\mathbb{X}, \mathbb{X})$. Suppose $\mathcal{S}$ is coercive, and $\mathcal{R}$ is positive. Then there exists a $\mathcal{Q}^1 \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ such that

$$\begin{aligned}
&\langle u, \mathcal{P}u\rangle + 2\left\langle u, \left(\mathcal{Q} - \mathcal{Q}^1\right)\phi_S\right\rangle + \langle \phi_S, \mathcal{S}\phi_S\rangle\\
&\qquad +2\left\langle u, \mathcal{Q}^1\phi_R\right\rangle + \langle \phi_R, \mathcal{R}\phi_R\rangle \geq 0 \qquad (51)
\end{aligned}$$

is satisfied for all $u \in \mathbb{Y}$ and $\phi_S, \phi_R \in \mathbb{X}$, if and only if

$$\langle u', \mathcal{P}u'\rangle + 2\langle u', \mathcal{Q}\phi\rangle + \langle \phi, (\mathcal{S}+\mathcal{R})\phi\rangle \geq 0. \qquad (52)$$

is satisfied for all $u' \in \mathbb{Y}$ and $\phi \in \mathbb{X}$. The result is still valid if we restrict $\mathcal{Q}^1 \in \mathbb{Q} \subset \mathbb{L}(\mathbb{X}, \mathbb{Y})$, as long as $\mathbb{Q}$ contains the element $\mathcal{Q}(\mathcal{S}+\mathcal{R})^{-1}\mathcal{R}$.

Proof. Necessity is obvious because (52) may be obtained from (51) with the constraint

$$\phi_S = \phi_R.$$

To show sufficiency, let (52) be satisfied for all $u' \in \mathbb{Y}$ and $\phi \in \mathbb{X}$. It is sufficient to show that (51) is satisfied for

$$\mathcal{Q}^1 = \mathcal{Q}(\mathcal{S}+\mathcal{R})^{-1}\mathcal{R}, \qquad (53)$$

which is well defined and bounded because $\mathcal{S}+\mathcal{R}$ is coercive. For this purpose, it is helpful to define

$$\mathcal{T} = \mathcal{R}(\mathcal{S}+\mathcal{R})^{-1}\mathcal{S}. \qquad (54)$$

It can also be easily shown that

$$\mathcal{S} - \mathcal{S}(\mathcal{S}+\mathcal{R})^{-1}\mathcal{S} = \mathcal{T}, \qquad (55)$$
$$\mathcal{R} - \mathcal{R}(\mathcal{S}+\mathcal{R})^{-1}\mathcal{R} = \mathcal{T}. \qquad (56)$$

With $\mathcal{Q}^1$ defined in (53), direct calculation gives

$$\begin{aligned}
&\text{Left hand side of (51)}\\
&= \langle u, \mathcal{P}u\rangle + 2\left\langle u, \mathcal{Q}(\mathcal{S}+\mathcal{R})^{-1}\mathcal{S}\phi_S\right\rangle + \langle \phi_S, \mathcal{S}\phi_S\rangle\\
&\quad +2\left\langle u, \mathcal{Q}(\mathcal{S}+\mathcal{R})^{-1}\mathcal{R}\phi_R\right\rangle + \langle \phi_R, \mathcal{R}\phi_R\rangle\\
&= I_1 + I_2, \qquad (57)
\end{aligned}$$

where,

$$\begin{aligned}
I_1 &= \langle u, \mathcal{P}u\rangle + 2\langle u, \mathcal{Q}\phi\rangle + \langle \phi, (\mathcal{S}+\mathcal{R})\phi\rangle, \qquad (58)\\
\phi &= (\mathcal{S}+\mathcal{R})^{-1}(\mathcal{S}\phi_S + \mathcal{R}\phi_R),\\
I_2 &= \langle \phi_S, \mathcal{T}\phi_S\rangle - 2\langle \phi_R, \mathcal{T}\phi_S\rangle + \langle \phi_R, \mathcal{T}\phi_R\rangle. \qquad (59)
\end{aligned}$$

In arriving at (57), the equations (54), (55) and (56) have been used. I_1 is nonnegative in view of (52). Also, from the definition, $\mathcal{T}$ can be shown as a positive operator as follows,

$$\begin{aligned}
\mathcal{T} &= \mathcal{S}(\mathcal{S}+\mathcal{R})^{-1}\left[(\mathcal{S}+\mathcal{R})\mathcal{S}^{-1}\mathcal{R}\right](\mathcal{S}+\mathcal{R})^{-1}\mathcal{S}\\
&= \mathcal{S}(\mathcal{S}+\mathcal{R})^{-1}(\mathcal{R}+\mathcal{R}\mathcal{S}^{-1}\mathcal{R})(\mathcal{S}+\mathcal{R})^{-1}\mathcal{S}.
\end{aligned}$$

Therefore,

$$I_2 = \langle(\phi_S - \phi_R), \mathcal{T}(\phi_S - \phi_R)\rangle \geq 0.$$

Thus (51) is satisfied. ■

REFERENCES

Fridman, E. (2002). Stability of linear descriptor systems with delay: a Lyapunov-based approach. *J. Math. Anal. Applicat.,* 273, 14–44.

Gu, K. (2001). A further refinement of discretized Lyapunov functional method for the stability of time-delay systems. *Int. J. Control,* 74, 967–976.

Gu, K. (2010). Stability problem of systems with multiple delay channels. *Automatica*, 46, 727-735.

Gu, K. and Liu, Y. (2009). Lyapunov-Krasovskii functional for uniform stability of coupled differential-functional equations. *Automatica,* 45, 798–804.

Kato, T. (1966). *Perturbation Theory of Linear Operators.* Springer-Verlag, New York.

Kolmogorov, A. N. and Fomin, S. V. (1975) *Introductory Real Analysis*, Translated and edited by R. A. Silverman, Dover, New York.

Li, H. and Gu, K. (2010). Discretized Lyapunov Krasovskii functional for coupled differential difference equations with multiple delay channels. *Automatica*, doi:10.1016/j.automatica.2010.02.007.

Pepe, P. and Verriest, E. I. (2003). On the stability of coupled delay differential and continuous time difference equations. *IEEE Trans. Automat. Control,* 48, 1422–1427.

Pepe, P. Jiang, Z.-P., and Fridman, E. (2007). A new Lyapunov-Krasovskii methodology for coupled delay differential and difference equations. *Int. J. Control,* 81, 107–115.

Peet, M. M. and Papachristodoulou, A. (2006). Positive forms and the stability of linear time-delay systems. In *Proc. 45th IEEE Conf. Decision Control,* San Diego, CA, USA, Dec. 13–15, 2006.

Peet, M. M. and Papachristodoulou, A. (2009). Positive forms and stability of linear time-delay systems. *SIAM J. Control Optim.,* 47, 3237–3258.

Prajna, S. Papachristodoulou, A. and Parrilo, P. A. (2002). Introducing SOSTOOLS: a general purpose sum of squares programming solver. In *Proc. 41th IEEE Conf. Decision Control,* Las Vegas, USA, Dec. 10–13, 2002.

Răsvan, V. and Niculescu, S.-I (2002). Oscillations in lossless propagation models: a Lyapunov-Krasovskii approach. *IMA J. Math. Control Inform.,* 19, 157–172.

Răsvan, V. (2006). Functional differential equations of lossless propagation and almost linear behavior. Plenary Lecture, *6th IFAC Workshop on Time-Delay Systems*, L'Aquila, Italy, July 10–12, 2006.

Zhang, Y., Peet, M. M., and Gu, K. (2010). Reducing the computational cost of the sum-of-squares stability test for time-delayed systems. In *2010 American Control Conference*, Baltimore, MD, June 30–July 2, 2010.