Designing Observer-Based Controllers for PDE Systems: A Heat-Conducting Rod With Point Observation and Boundary Control

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Abstract— **In this paper, we propose a method to synthesize infinite-dimensional observer-based controllers for partial-differential systems. To illustrate the approach, we use a one-dimensional model of heat conduction with point observation and boundary control. Our method uses *Sum-of-Squares* optimization to solve linear operator inequalities in an infinite-dimensional Hilbert space. We use the semigroup framework and an operator version of the separation theorem and the Lyapunov inequality. We implement our method using SOSTOOLS and SeDuMi. We simulate the effects of our controller using Matlab.**

I. INTRODUCTION

Some physical systems have dynamics which are best modeled using partial differential equations (PDE). In this paper we address the problem of controller synthesis for one such system. Specifically, we propose a method of controller synthesis for a one-dimensional heat conducting rod with point observation and boundary control. Often, controller design for infinite-dimensional systems is done using an *early lumping* approach [20], [19]. In this approach the governing PDE is discretized in the non-temporal variables to obtain a finite-dimensional model using, for example, finite-difference and finite-element methods [11]. Then, controller synthesis techniques are applied to the approximate finite-dimensional model [8]. The advantage of this approach is that one can utilize results from the well researched area of finite-dimensional controller synthesis. Unfortunately, in this approach some of the system dynamics are ignored. These unmodeled dynamics may lead to unsatisfactory performance [2]. In our approach, we use *Sum-of Squares* polynomials and convex optimization to construct both infinite-dimensional observers and controllers without discretization. Thus we synthesize observer based controllers for the infinite-dimensional model and not it's finite-dimensional approximation.

Some results on the boundary control of heat conducting systems can be found in [13], [3] and [4]. References on the more general problem of boundary control of systems governed by parabolic equations include [1], [12] and [17].

Conceptually, our approach is based on the semigroup framework [9]. Semigroup theory is a rigorous treatment of state-space for infinite-dimensional systems. We use this state-space framework to express the observer and controller design problems separately by use of a separation theorem involving a transformation of variables. We then express the observer/controller design problems as linear operator inequalities. We reformulate these inequalities using positive polynomial variables to parameterize the space of positive operators on a relevant Hilbert space. Because Sum-of-Squares allows us to optimize over the space of positive polynomials, we are able to solve these reformulated operator inequalities.

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II. PRELIMINARIES

A. Sum-of-Squares polynomials

As the name implies, a *Sum-of-Squares*(SOS) polynomial is one which can be represented as a sum of squared polynomials. That is, a polynomial $p(x), x \in \mathbb{R}^n$ is SOS if there exist polynomials $p_i(x), i = 1, \cdots, N$ such that

$$p(x) = \sum_{i=1}^{N} (p_i(x))^2.$$

A sufficient condition for a polynomial to be non-negative is that it be a SOS polynomial.

To check whether a polynomial is SOS we employ the following theorem.

Theorem 1: A polynomial $p(x), x \in \mathbb{R}^n$ of degree $2d$ is sum-of-squares if and only if there exists a symmetric positive semi-definite (PSD) matrix $Q \succeq 0$ such that

$$p(x) = z(x)^T Q z(x), \tag{1}$$

where $z(x)$ is a vector of monomials of degree d or less. If we can find a symmetric PSD matrix Q such that (1) is satisfied then the polynomial $p(x)$ is SOS and hence non-negative.

Therefore to find a symmetric PSD matrix Q satisfying (1) we have to find the components of Q which satisfy a number of linear constraints and a linear matrix inequality (to ensure it's semi-definiteness). Hence, the problem of checking whether a polynomial is SOS can be formulated as a convex feasibility problem.

What this implies is that even though the question of polynomial positivity is NP-hard [7], the problem of checking whether a polynomial is SOS can be solved using Semi-definite programming (SDP). It is generally accepted that SDPs can be solved in polynomial time by employing Interior-point methods [14]. This renders the problem of performing a search for a SOS decomposition for a polynomial of the form in (1) tractable.

B. Semigroup theory

Semigroup theory is utilized for studying infinite-dimensional systems in the time domain using the state space architecture. Some important references on the subject are [9], [6], [18] and [5]. In this subsection we will outline concepts pertinent to the work presented in the paper.

Definition 1: A *strongly continuous semigroup* or a C_0-semigroup is an operator-valued function $T(t):\mathbb{R}^+\to\mathscr{L}(Z)$ that satisfies the following properties:

$$\begin{aligned}&T(t+s)=T(t)T(s) \text{ for } t,s\geq 0;\\&T(0)=I;\\&\|T(t)z_0-z_0\|\to 0 \text{ as } t\to 0^+ \text{ for all } z_0\in Z,\end{aligned}$$

where Z is a Hilbert space, $\mathscr{L}(Z)$ is the Banach space of linear and bounded operators mapping Z back to itself, $\|\cdot\|$ is the norm induced by the inner product defined on Z and I is the identity operator on Z.

Associated with a C_0-semigroup is an operator known as it's infinitesimal generator.

Definition 2: The infinitesimal generator A of a C_0-semigroup $T(t)$ on a Hilbert space Z is defined by

$$Az=\lim_{t\to 0^+}\frac{1}{t}(T(t)-I)z.$$

The subset of Z on which this limit exists is the domain of the infinitesimal generator and is denoted by $D(A)\subset Z$.

Now, consider the following differential equation defined on a Hilbert space Z:

$$\dot{z}(t)=Az(t), t\geq 0,\ z(0)=z_0\in D(A)\subset Z \tag{2}$$

and suppose A is the generator of a C_0-semigroup $T(t)$. Then the solution of this differential equation is given by

$$z(t)=T(t)z_0. \tag{3}$$

On the other hand, let's consider an ordinary differential equation (ODE) defined on $\mathbb{R}^n, n\in\mathbb{N}$,

$$\dot{x}(t)=Bx(t), t\geq 0,\ x(0)=x_0\in\mathbb{R}^n. \tag{4}$$

Here $B\in\mathbb{R}^{n\times n}$ is a matrix. The solution to the ODE is given by

$$x(t)=e^{Bt}x_0. \tag{5}$$

On comparing (3) and (5) we see that the C_0-semigroup $T(t)$ can be thought of as a generalization of the evolution operator e^{Bt}.

As in the case of ODEs, the solution of (4), $x(t)$, converges to zero exponentially fast as $t\to\infty$ if and only if the matrix B is Hurwitz. Similarly, the solution of the differential equation (2) on the Hilbert space Z, $z(t)$, converges to zero exponentially fast if and only if the operator A is the infinitesimal generator of an exponentially stable C_0-semigroup. To see this, we refer to the following definition.

Definition 3: A C_0-semigroup $T(t)$ on a Hilbert space Z is exponentially stable if there exist positive constants M and α such that

$$\|T(t)\|\leq Me^{-\alpha t} \text{ for } t\geq 0.$$

Here $\|\cdot\|$ is the induced operator norm.
Now suppose that the operator A in (2) is the infinitesimal generator of an exponentially stable C_0-semigroup $T(t)$, then the solution is

$$z(t)=T(t)z_0.$$

Since $T(t)$ is assumed to be an exponentially stable C_0-semigroup, we get

$$\|z(t)\|=\|T(t)z_0\|\leq\|T(t)\|\|z_0\|\leq Me^{-\alpha t}\|z_0\|.$$

Therefore $\|z(t)\|\to 0$ exponentially fast as $t\to\infty$ which implies that $z(t)\to 0$, a.e. exponentially fast as $t\to\infty$.

We will be using the following operator version of the Lyapunov inequality to check if an operator on a Hilbert space is the infinitesimal generator of an exponentially stable semigroup.

Theorem 2: Suppose that A is the infinitesimal generator of a C_0-semigroup $T(t)$ on a Hilbert space Z. Then $T(t)$ is exponentially stable if and only if there exists a positive operator $P\in\mathscr{L}(Z)$ such that

$$\langle Az,Pz\rangle+\langle Pz,Az\rangle<0 \text{ for all } z\in D(A), z\neq 0,$$

where $\langle\cdot,\cdot\rangle$ is the inner product defined on Z.

The condition in Theorem 2 implies that for every $z\in D(A)$ there exits a positive scalar ε such that

$$\langle Az,Pz\rangle+\langle Pz,Az\rangle=-\varepsilon\langle z,z\rangle.$$

Then the proof follows from [9, Theorem 5.1.3].

III. PROBLEM SETUP

In this section we will explicitly outline the problem this paper aims to solve.

Consider the following one-dimensional heat conducting rod:

$$\begin{aligned}w_t(z,t)&=w_{zz}(z,t), && z\in(0,1), t>0\\w(0,t)&=0, && t\geq 0\\w_z(1,t)&=u(t), && t\geq 0\\w(z,0)&=w_0(z), && z\in[0,1]\\y(t)&=w(1,t), && t\geq 0,\end{aligned}$$

with boundary control, $u(t)\in\mathbb{R}$, and point observation at the right end. Here $z\in[0,1]$ is the spatial variable and $w(z,t)$ is the temperature of the rod and is fixed at $z=0$.

The governing PDE for the system can be expressed as a differential equation on the Hilbert space $L_2(0,1)$ as follows

$$\begin{aligned}&\dot{x}(t)=Ax(t)+Bu(t), x(0)\in D(A)\\&y(t)=Cx(t).\end{aligned}$$

We will denote this linear system by $\Sigma(A,B,C)$. Here

$$Ah=h'' \text{ for } h\in D(A) \tag{6}$$

is the infinitesimal generator of a C_0-semigroup. $D(A)$ denotes the domain of the operator A and is defined as

$$\begin{aligned}D(A)=\{h\in L_2(0,1): h,h' \text{ abs. continuous },\\ h''\in L_2(0,1), h(0)=h'(1)=0\}.\end{aligned}$$

The state space is the Hilbert space $L_2(0,1)$ with the state $x(t) = w(\cdot,t) \in L_2(0,1), t \geq 0$. Since the state space is the function space $L_2(0,1)$, the system in question is infinite-dimensional. Note that throughout the paper a single prime and a double prime denotes a single derivative and a double derivative with respect to the spatial variable respectively. The input operator B maps $\mathbb{R}$ to $L_2(0,1)$ and is defined as

$$(Bu(t))(z) = \delta_1(z)u(t), \tag{7}$$

where δ_1 is the Dirac delta function centered at $z = 1$. We use the Dirac delta function for it's *sifting* property. That is, for a delta function centered at α in some interval $[a,b]$ and a function $f \in L_2[a,b]$ we have the property that $\int_0^1 \delta_\alpha(z) f(z) dz = f(\alpha)$.

The output operator C maps $L_2(0,1)$ to $\mathbb{R}$ and is defined as

$$Cx(t) = \langle x(t), \delta_1 \rangle = (x(t))(1), \tag{8}$$

where $\langle \cdot, \cdot \rangle$ is the inner product defined on the space $L_2(0,1)$.

The operator C ensures that we only receive a partial knowledge of the state. Specifically, we can only measure the temperature of the the rod at the right end. To synthesize controllers for such systems we need to design an observer. An observer is a system which estimates the state of the plant we wish to control by using the output from the plant. We utilize the state estimate provided by the observer to design stabilizing controllers for the plant.

We will design a Luenberger observer-based controller, denoted by $\Sigma_o(A,B,C,L,F)$, of the form

$$\begin{aligned} \dot{\hat{x}}(t) &= (A + LC)\hat{x}(t) + Bu(t) - Ly(t), \\ u(t) &= F\hat{x}(t), \end{aligned}$$

where $\hat{x}(t) \in L_2(0,1)$ is the observer state and the operators $L \in \mathscr{L}(\mathbb{R}, L_2(0,1))$ and $F \in \mathscr{L}(L_2(0,1), \mathbb{R})$ have to be constructed such that the input $u(t) = F\hat{x}(t)$ stabilizes the plant $\Sigma(A,B,C)$. For this purpose we will employ the following theorem.

Theorem 3 (Separation theorem): Consider the linear system $\Sigma(A,B,C)$ and assume that it is exponentially stabilizable and exponentially detectable. If $F \in \mathscr{L}(L_2(0,1), \mathbb{R})$ and $L \in \mathscr{L}(\mathbb{R}, L_2(0,1))$ are such that $A + BF$ and $A + LC$ generate exponentially stable C_0-semigroups, then the observer based controller $\Sigma_o(A,B,C,L,F)$ stabilizes the system to be controlled.

See [9, Theorem 5.3.3] for the proof. Here the term *exponentially stabilizable* means that there exists some $F \in \mathscr{L}(L_2(0,1), \mathbb{R})$ such that $A + BF$ generates an exponentially stable C_0-semigroup. Similarly, *exponentially detectable* implies that there exists some $L \in \mathscr{L}(\mathbb{R}, L_2(0,1))$ such that $A + LC$ generates an exponentially stable C_0-semigroup.

In this paper we will use Sum-of Squares polynomials and convex optimization to construct operators F and L satisfying Theorem 3 thus ensuring that the observer based control input stabilizes the system in question.

Note that throughout the paper we will assume that for any $F \in \mathscr{L}(L_2(0,1), \mathbb{R})$ and $L \in \mathscr{L}(\mathbb{R}, L_2(0,1))$, the operators $A + BF$ and $A + LC$ are the infinitesimal generators of C_0-semigroups for the operators A, B and C given in (6), (7) and (8) respectively. We make this strong assumption since the goal of this paper is to illustrate a method of controller synthesis. For applications however, one must ensure that $A + BF$ and $A + LC$ generate C_0-semigroups. For this purpose, the *Hille-Yosida* theorem [10, Section 7.4.2] can be employed. Additionally, one could also refer to [16].

IV. MAIN RESULTS

We saw in the *Separation theorem* that for the linear system $\Sigma(A,B,C)$ if there exist operators $F \in \mathscr{L}(L_2(0,1), \mathbb{R})$ and $L \in \mathscr{L}(\mathbb{R}, L_2(0,1))$ such that $A + BF$ and $A + LC$ generate exponentially stable C_0-semigroups, then the observer-based controller $\Sigma_o(A,B,C,L,F)$ stabilizes the system. In this section we provide results which are employed to construct the desired operators so that we may synthesize observer based controllers for the heat conducting rod.

The following theorems reformulate the operator inequalities associated with Theorem 2 as polynomial feasibility problems.

Theorem 4: Let $(Ax)(z) = x''(z)$ for all $x \in D(A)$ and $Cx(t) = \langle x(t), \delta_1 \rangle = (x(t))(1)$. Suppose there exist polynomials $M_o(z)$ and $N_o(z)$ defined on $z \in [0,1]$ such that

$$M_o(z) > 0 \text{ for all } z \in [0,1], \tag{9}$$

and

$$\begin{bmatrix} \frac{M_o''(z)}{2} & \frac{N_o(z)}{2} \\ \frac{N_o(z)}{2} & -\frac{M_0'(1)}{2} \end{bmatrix} \prec 0 \text{ for all } z \in [0,1], \tag{10}$$

Then the operator $A + LC$ is the infinitesimal generator of an exponentially stable C_0-semigroup where the operator $L \in \mathscr{L}(\mathbb{R}, L_2(0,1))$ is defined as

$$La = M_o^{-1}(z) N_o(z) a \text{ for any } a \in \mathbb{R}.$$

Proof:

To show that $A + LC$ generates an exponentially stable C_0-semigroup, we will have to show that there exists a positive operator $P \in \mathscr{L}(L_2(0,1))$ such that P and $A + LC$ satisfy

$$\langle (A + LC)x, Px \rangle + \langle Px, (A + LC)x \rangle < 0 \text{ for all } z \in D(A), z \neq 0.$$

where

$$\begin{aligned} D(A) = \{ h \in L_2(0,1) : h, h' \text{ abs. continuous}, \\ h'' \in L_2(0,1), h(0) = h'(1) = 0 \}. \end{aligned}$$

Let $P \in \mathscr{L}(L_2(0,1))$ be given by

$$(Px)(z) = M_o(z)x(z).$$

Since M_o a strictly positive scalar polynomial, P is linear, positive and self-adjoint. Additionally, since the polynomial $M_o(z)$ is strictly positive on the compact set $[0,1]$, $M_o(x)^{-1}$ is continuous and thus the operator P is bounded. Therefore $P \in \mathscr{L}(L_2(0,1))$. Now,

$$\langle (A + LC)x, Px \rangle = \langle Ax, Px \rangle + \langle LCx, Px \rangle \text{ for all } x \in D(A).$$

Since $P = P^*$,

$$\langle (A + LC)x, Px \rangle = \langle Ax, Px \rangle + \langle PLCx, x \rangle.$$

Recall that

$$(Px)(z) = M_o(z)x(z),$$
$$La = M_o^{-1}(z)N_o(z)a \text{ for any } a \in \mathbb{R}$$

and

$$Cx(t) = \langle x(t), \delta_1 \rangle = (x(t))(1)$$

hence

$$((PLC)(x))(z) = (PL)x(1) = N_o(z)x(1).$$

Therefore we get

$$\begin{aligned}\langle (A+LC)x, Px\rangle &= \langle Ax, Px\rangle + \langle PLCx, x\rangle \\ &= \int_0^1 x(z)M_o(z)x''(z)dz + \int_0^1 x(1)N_o(z)x(z)dz.\end{aligned}$$

Applying integration by parts on the first term of above equation and using the fact that $x(0) = x'(1) = 0$ for all $x \in D(A)$ we get

$$\begin{aligned}&\langle (A+LC)x, Px\rangle \\ &= \int_0^1 \left(-x'(z)M_o(z)x'(z) - \frac{M_o'(1)}{2}x^2(1)\right)dz \\ &\quad + \int_0^1 \left(x(z)\frac{M_o''(z)}{2}x(z) + x(1)N_o(z)x(z)\right)dz \\ &= \int_0^1 \begin{bmatrix} x(z) \\ x(1) \end{bmatrix}^T \begin{bmatrix} \frac{M_o''(z)}{2} & \frac{N_o(z)}{2} \\ \frac{N_o(z)}{2} & -\frac{M_0'(1)}{2} \end{bmatrix} \begin{bmatrix} x(z) \\ x(1) \end{bmatrix} dz \\ &\qquad - \int_0^1 M_o(z)x'(z)^2 dz.\end{aligned}$$

By assumption, $M_o(z) > 0$ and

$$\begin{bmatrix} \frac{M_o''(z)}{2} & \frac{N_o(z)}{2} \\ \frac{N_o(z)}{2} & -\frac{M_0'(1)}{2} \end{bmatrix} \prec 0$$

for all $z \in [0,1]$. We conclude that

$$\langle (A+LC)x, Px\rangle < 0, \text{ for all } x \in D(A), x \neq 0.$$

Note that the field over which our state space $L_2(0,1)$ is defined is the set of reals $\mathbb{R}$. Hence

$$\langle (A+LC)x, Px\rangle = \langle Px, (A+LC)x\rangle.$$

Therefore

$$\langle (A+LC)x, Px\rangle + \langle Px, (A+LC)x\rangle < 0,$$

for all $x \in D(A), x \neq 0$. Additionally, note that the operator L is linear and bounded. From Theorem 2 we conclude that $A+LC$ generates an exponentially stable C_0-semigroup. ■

Theorem 5: Let $(Ax)(z) = x''(z)$ for all $x \in D(A)$ and $(Bu)(z) = \delta_1(z)u$ for $u \in \mathbb{R}$. Suppose there exist polynomials $M_c(z), R_1(z)$ and $R_2(z)$ such that

$$M_c(z) > 0, \text{ for all } z \in [0,1], \tag{11}$$
$$\begin{bmatrix} \frac{M_c''(z)}{2} & \frac{R_1(z)}{2} \\ \frac{R_1(z)}{2} & R_2(1) \end{bmatrix} \prec 0, \text{ for all } z \in [0,1] \tag{12}$$

Then $A+BF$ is the generator of an exponentially stable semigroup where the operator $F \in \mathscr{L}(L_2(0,1), \mathbb{R})$ is defined as

$$(Fx)(z) = \int_0^1 \left(K_1(z)x(z) + \frac{\partial}{\partial z}(K_2(z)x(z))\right)dz,$$

where

$$K_1(z) = M_c^{-1}(z)R_1(z),$$
$$K_2(z) = M_c^{-1}(z)R_2(z).$$

Proof: Suppose the hypotheses of the theorem hold true. We begin by defining the positive operator $P_c \in \mathscr{L}(L_2(0,1))$ as

$$(P_c x)(z) = M_c^{-1}(z)x(z),$$

which is well defined since $M_c(z) > 0$ on the compact set $[0,1]$ and is positive and self-adjoint.

Now,

$$\langle (A+BF)x, P_c x\rangle = \langle Ax, P_c x\rangle + \langle BFx, P_c x\rangle$$

for $x \in D(A)$. We will evaluate the two terms of the above expression separately.

$$\langle Ax, P_c x\rangle = \int_0^1 \frac{\partial^2 x(z)}{\partial z^2} M_c^{-1}(z)x(z)dz.$$

We now define a new variable

$$y(z) = x(z)M_c^{-1}(z),$$

which is well defined as $M_c(z) > 0$.

Then

$$\begin{aligned}\langle Ax, P_c x\rangle &= \int_0^1 \frac{\partial^2 (M_c(z)M_c^{-1}(z)x(z))}{\partial z^2} M_c^{-1}(z)x(z)dz \\ &= \int_0^1 \frac{\partial^2 (M_c(z)y(z))}{\partial z^2} y(z)dz.\end{aligned}$$

Expanding and applying integration by parts we get

$$\begin{aligned}&\langle Ax, P_c x\rangle \\ &= \int_0^1 \left[\frac{M_c''(z)}{2}[y(z)]^2 - M_c(z)[y'(z)]^2\right]dz \\ &\quad + \int_0^1 \left[M_c'(1)[y(1)]^2 + y'(1)M_c(1)y(1)\right]dz,\end{aligned} \tag{13}$$

where we have used the fact that

$$y(z) = x(z)M_c^{-1}(z)$$

and $x(0) = 0$, hence

$$y(0) = 0.$$

We have

$$\begin{aligned}x(z) &= M_c(z)y(z), \\ x'(z) &= M_c'(z)y(z) + M_c(z)y'(z), \\ x'(1) &= M_c'(1)y(1) + M_c(1)y'(1).\end{aligned}$$

Since $x'(1) = 0$,

$$\begin{aligned}M_c(1)y'(1) + M_c'(1)y(1) &= 0, \\ M_c(1)y'(1) &= -M_c'(1)y(1).\end{aligned}$$

Substituting $M_c(1)y'(1) = -M_c'(1)y(1)$ in the last term of (13) we get

$$\begin{aligned}
&\langle Ax, P_c x\rangle \\
&= \int_0^1 \left[\frac{M_c''(z)}{2}[y(z)]^2 - M_c(z)[y'(z)]^2\right] dz \\
&+ \int_0^1 \left[M_c'(1)[y(1)]^2 - M_c'(1)[y(1)]^2\right] dz \\
&= \int_0^1 \left[\frac{M_c''(z)}{2}[y(z)]^2 - M_c(z)[y'(z)]^2\right] dz.
\end{aligned}$$

Now, we have

$$\begin{aligned}
(Fx)(z) &= \int_0^1 \left(K_1(s)x(z) + \frac{\partial}{\partial z}\left(K_2(z)x(z)\right)\right) dz \\
&= \int_0^1 \left(R_1(z)y(z) + \frac{\partial}{\partial s}\left(R_2(z)y(z)\right)\right) dz \\
&= \int_0^1 R_1(z)y(z)dz + R_2(1)y(1) - R_2(0)y(0) \\
&= \int_0^1 R_1(z)y(z)dz + R_2(1)y(1)
\end{aligned}$$

Since Fx is a constant, we have

$$\begin{aligned}
&\langle BFx, P_c x\rangle = \langle BFx, y\rangle \\
&= \int_0^1 \delta_1(z)Fxy(z)dz \\
&= y(1)Fx \\
&= \int_0^1 \left(y(1)R_1(z)y(z) + y(1)R_2(1)y(1)\right) dz.
\end{aligned}$$

Adding equations the expressions for $\langle Ax, P_c x\rangle$ and $\langle BFx, P_c x\rangle$, we get

$$\begin{aligned}
&\langle (A+BF)x, P_c x\rangle \\
&= \int_0^1 \left(\frac{M_c''(z)}{2}[y(z)]^2 - M_c(z)[y'(z)]^2\right) dz \\
&\qquad + \int_0^1 \left(y(1)R_1(z)y(z) + R_2(1)y^2(1)\right) dz \\
&= \int_0^1 \begin{bmatrix} y(z) \\ y(1) \end{bmatrix} \begin{bmatrix} \frac{M_c''(z)}{2} & \frac{R_1(z)}{2} \\ \frac{R_1(z)}{2} & R_2(1) \end{bmatrix} \begin{bmatrix} y(z) \\ y(1) \end{bmatrix} dz - \int_0^1 M_c(z)y'(z)^2 dz
\end{aligned}$$

Since $M_c(z) > 0$ and

$$\begin{bmatrix} \frac{M_c''(z)}{2} & \frac{R_1(z)}{2} \\ \frac{R_1(z)}{2} & R_2(1) \end{bmatrix} \prec 0$$

for all $z \in [0,1]$, we conclude that

$$\langle (A+BF)x, P_c x\rangle < 0, \text{ for all } x \in D(A), x \neq 0.$$

Since our state space $L_2(0,1)$ is defined over the field of reals $\mathbb{R}$,

$$\langle (A+BF)x, P_c x\rangle = \langle P_c x, (A+BF)x\rangle.$$

Hence,

$$\langle (A+BF)x, P_c x\rangle + \langle P_c x, (A+BF)x\rangle < 0,$$

for all $x \in D(A), x \neq 0$. Note that it is easy to show that the operator F is linear and bounded.

From Theorem 2 we conclude that $A + BF$ is the generator of an exponentially stable semigroup. ■

V. IMPLEMENTATION AND SIMULATION

As shown in the previous section, in order to construct a stabilizing controller for the system $\Sigma(A,B,C)$ we have to search for polynomials $M_o(z), N_o(z), M_c(z), R_1(z)$ and $R_2(z)$ such that the conditions (9), (10), (11) and (12) are satisfied. We perform this search using SOSTOOLS [15], a freely available MATLAB toolbox which employs semi-definite programming (SDP) for implementing algorithms over the set of SOS polynomials. Note that this is a feasibility search and such polynomials will not exist if the system can not be stabilized.

Once we obtain the observer based controller using SOSTOOLS we would like to simulate the controlled plant. However, due to the absence of analog computers we use finite-difference methods to approximate the controlled model with a finite-dimensional model. We then use SIMULINK to simulate the finite-dimensional approximation. Note that the finite-dimensional approximation is used for simulation only and not for synthesis.

We present the numerical solution of the controlled plant in the following figures.

Figure 1 shows the desired steady state reference profile $w_{ref}(z) \in D(A)$ towards which we want the controller to drive the system state. Figure 2(a) shows the system state $(x(t))(z) = w(z,t)$, Figure 2(b) shows the observer state $(\hat{x}(t))(z) = w_{obs}(z,t)$ and Figure 2(c) presents the control tracking error $w(z,t) - w_{ref}(z)$. Additionally, Figure 2(d) shows the control input $u(t)$.

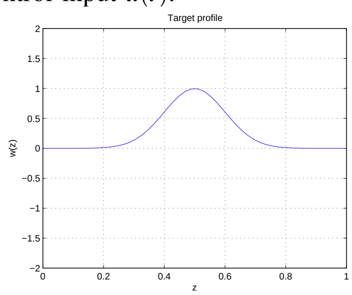
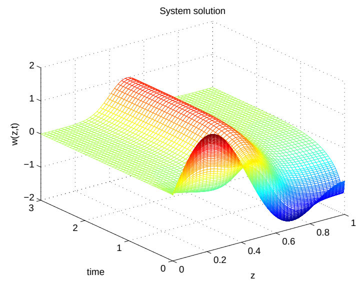


Fig. 1: Desired steady state target profile(reference profile), $w_{ref}(z) \in D(A)$

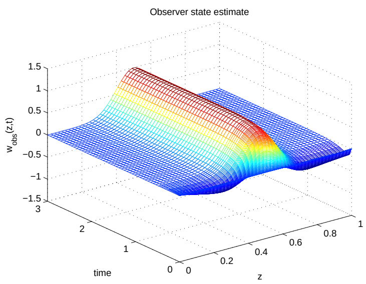
VI. CONCLUSIONS AND FUTURE WORK

A. Conclusions

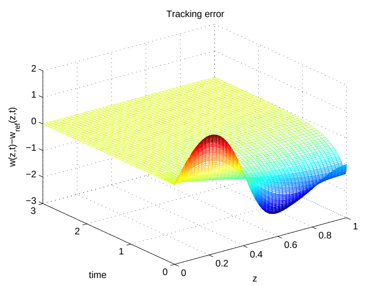
In this paper we presented a method to design observer-based controllers for an infinite-dimensional system. We used *Sum-of-Squares* polynomials and convex optimization to achieve the desired goals. The process outlined in the paper may be used for other infinite-dimensional systems as well. However, it must be noted that controller synthesis, using the presented method, is application-specific and hence will vary from system to system. This is because a different system will have different dynamics and the transformations will lead to different (9), (10), (11) and (12).



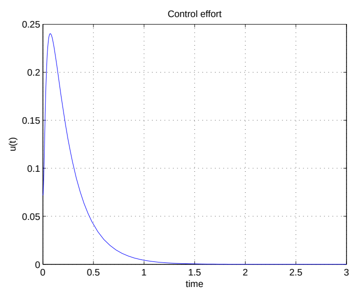
(a) Solution of the controlled plant $(x(t))(z) = w(z,t)$



(b) Solution of the observer $(\hat{x}(t))(z) = w_{obs}(z,t)$



(c) Tracking error $w(z,t) - w_{ref}(z)$



(d) Control effort $u(t)$

Fig. 2: System and Observer solutions

B. *Future work*

The next logical step in this research would be to construct linear operator-inequality-type conditions to calculate the H_∞-norm of the controlled plant so that we can do a performance analysis for the controllers. We would also like to design methods to synthesize optimal controllers with respect to the H_∞-norm.

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