On the Conservatism of the Sum-of-Squares Method for Analysis of Time-Delayed Systems ⋆

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Abstract

In this paper, we present a converse Lyapunov result which shows that using polynomial Lyapunov functionals to prove the stability of linear time-delay systems is not conservative. This result motivates the sum-of-squares approach to stability analysis of linear time-delay systems. This main result is based on an extension of the Weierstrass approximation theorem to show that linear varieties of polynomials can be used to approximate linear varieties of the space of continuous functions.

Key words: Function Approximation, Polynomials, Time Delay, Generated Lyapunov Functions, Lyapunov Stability

1 Introduction

Time delays are a common mathematical modeling tool, with a ubiquitous presence in communication networks [1,6], industrial processes [9], and biological systems [4,10], among others. See [5,3] for sources containing a survey of applications of systems with delay.

It is well-known that stability of linear time-delay systems is equivalent to the existence of a "complete-quadratic"-type Lyapunov function. See Equation (1) for the form of the Lyapunov functional. In this paper, we do not give any new method of constructing these functionals. Rather, we examine the convergence properties of an existing method of stability analysis for time-delay systems - the sum-of-squares approach. The sum-of-squares methodology uses convex optimization of positive polynomial variables to construct Lyapunov-Krasovskii functionals. Applying the sum-of-squares methodology to time-delay systems relies on the assumption that for any exponentially stable linear system, there will exist a Lyapunov functional of the "complete-quadratic-type" and furthermore, that the functional is defined by *polynomials*.

The contribution of this paper is to present a converse Lyapunov result, showing that Lyapunov-Krasovskii functionals and the associated stability conditions can be assumed polynomial. Previous work by the authors has proven the existence of polynomial Lyapunov functions for ordinary nonlinear systems [7]. The problem of proving the existence of polynomial Lyapunov functionals for time-delayed systems are different, however. In particular, a significant technical contribution of this paper is to develop an extension of the Weierstrass approximation theorem when subject to affine constraints.

To analyze stability analysis of time-delay systems, consider the "Complete-Quadratic"-type Lyapunov functionals.

⋆ A preliminary version of this paper was presented at MTNS 2008. Corresponding author M. Peet.
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These have the following form.

$$V(x) = \int_{-h}^{0} \begin{bmatrix} x(0) \\ x(s) \end{bmatrix} M(s) \begin{bmatrix} x(0) \\ x(s) \end{bmatrix} ds + \int_{-h}^{0}\int_{-h}^{0} x(s)^T N(s,t) x(t) ds\, dt. \tag{1}$$

Neglecting the double-integral term, for a linear system with discrete delays, it was shown in [8] that this functional proves stability of a time-delay system if and only if there exist continuous functions T and Q such that

$$M(s) + \begin{bmatrix} T(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \text{for all } s \in [-h, 0], \tag{2}$$

$$-(L(M))(s) + \begin{bmatrix} Q(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \text{for all } s \in [-h, 0], \tag{3}$$

with

$$\int_{-h}^{0} T(s) ds = 0 \qquad \text{and} \qquad \int_{-h}^{0} Q(s) ds = 0.$$

Here L is a linear transformation defined by the dynamics. For example, if

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h),$$

then L is defined as

$$(L(M))(s) := \begin{bmatrix} A_0^T M_{11} + M_{11} A_0 & M_{11} A_1 & 0 \\ A_1^T M_{11} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4}$$

$$+ \begin{bmatrix} 0 & 0 & A_0^T M_{12}(s) \\ 0 & 0 & A_1^T M_{12}(s) \\ M_{21}(s) A_0 & M_{21}(s) A_1 & 0 \end{bmatrix} \tag{5}$$

$$+ \frac{1}{h} \begin{bmatrix} M_{12}(0) + M_{21}(0) + M_{22}(0) & -M_{12}(-h) & 0 \\ -M_{21}(-h) & -M_{22}(-h) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6}$$

$$+ \begin{bmatrix} 0 & 0 & -\dot{M}_{12}(s) \\ 0 & 0 & 0 \\ -\dot{M}_{21}(s) & 0 & -\dot{M}_{22}(s) \end{bmatrix}. \tag{7}$$

In this paper, we prove that the existence of continuous functions T, and Q, and a continuously differentiable function M which satisfy the above conditions implies the existence of polynomial functions M_p, T_p, and Q_p which also satisfy the conditions. We also address the question of the double-integral term. This work shows that using polynomial Lyapunov-Krasovskii functionals is not conservative when proving stability of time-delay systems via sum-of-squares.

This paper is organized as follows. In Section 2, we present notation and give a statement of the Weierstrass approximation theorem. In Section 3, we give some background on the stability of time-delay systems. In section 4, the SOS method for stability analysis of time-delay systems is presented. In Section 5, we present the first new technical contribution of the paper, an extension of Weierstrass to linear varieties. In Section 6, we extend this work to derivatives of functions of a single variable. In Section 7 we show how the theorem can be applied to the problem of optimization of polynomials. In Sections 8, we present the main results of the paper in the case of a single delay. In Section 9, we extend our results the the case of multiple delays. Finally, the paper concludes in Section 10.

2 Notation and Background

Most notation is standard. $\mathbb{R}$ is the real numbers. $\mathbb{R}^{n\times m}$ is the real matrices of dimension n by m. $\mathbb{S}^n \subset \mathbb{R}^{n\times n}$ is the subspace of symmetric matrices. For $M \in \mathbb{S}$, $M \succeq 0$ denotes positive semidefinite. $\mathcal{C}[X,Y]$ is the Banach space of functions $f : X \to Y$ with norm

$$\|f\|_\infty = \sup_{s\in X} \|f(s)\|_Y. \tag{8}$$

$\mathcal{C}^1[X,Y]$ is the subspace of $\mathcal{C}[X,Y]$ of functions continuously differentiable on X. For $Y = \mathbb{R}^{n\times m}$, we use

$$\|\cdot\|_Y = \bar{\sigma}(F(s)), \tag{9}$$

where $\bar{\sigma}(F)$ denotes the maximum singular value norm.

The following is a statement of the Weierstrass approximation theorem.

Theorem 1 *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and $X \subset \mathbb{R}^n$ is compact. Then there exists a sequence of polynomials which converges to f uniformly in X.*

3 Background on Time-Delay Systems

Consider linear systems with discrete delays. Specifically, we are interested in systems of the form

$$\dot{x}(t) = \sum_{i=0}^{k} A_i x(t-h_i), \tag{10}$$

where the trajectory $x : [-h,\infty) \to \mathbb{R}^n$. In the simplest case we are given information about the the delays $0 = h_0 < h_1 < \cdots < h_k = h$ and the matrices $A_0,\ldots,A_k \in \mathbb{R}^{n\times n}$ and we would like to determine whether the system is stable.

For these types of systems, the boundary conditions are specified by a given function $\phi : [-h,0] \to \mathbb{R}^n$ and the constraint

$$x(t) = \phi(t) \quad \text{for all } t \in [-h,0]. \tag{11}$$

Let $\phi \in C[-h,0]$. Then there exists a unique function x satisfying (10) and (11). The system is called ***exponentially stable*** if there exists $\sigma > 0$ and $a \geq 0$ such that for every initial condition $\phi \in C[-h,0]$ the corresponding solution x satisfies

$$\|x(t)\| \leq a e^{-\sigma t}\|\phi\| \quad \text{for all } t \geq 0. \tag{12}$$

We write the solution as an explicit function of the initial conditions using the map $G : C[-h,0] \to \Omega[-h,\infty)$, defined by

$$(G\phi)(t) = x(t) \quad \text{for all } t \geq -h, \tag{13}$$

where x is the unique solution of (10) and (11) corresponding to initial condition ϕ. Also for $s \geq 0$ define the *flow map* $\Gamma_s : C[-h,0] \to C[-h,0]$ by

$$\Gamma_s\phi = H_s G\phi, \tag{14}$$

which maps the state of the system x_t to the state at a later time $x_{t+s} = \Gamma_s x_t$.

3.1 Lyapunov Functionals

Suppose $V : C[-h,0] \to \mathbb{R}$. We use the notion of derivative as follows. Define the ***Upper Dini derivative*** of V with respect to Γ by

$$\dot{V}(\phi) = \limsup_{r\to 0^+} \frac{1}{r}\big(V(\Gamma_r\phi) - V(\phi)\big). \tag{15}$$

We will use the notation $\dot{V}$ for both the Dini derivative and the usual derivative, and state explicitly which we mean if it is not clear from context. We will consider the set X of quadratic functions, where $V \in X$ if there exist piecewise continuous functions $M : [-h, 0) \to \mathbb{S}^{2n}$ and $N : [-h, 0) \times [-h, 0) \to \mathbb{R}^{n\times n}$ such that

$$V(\phi) = \int_{-h}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds + \int_{-h}^{0} \int_{-h}^{0} \phi(s)^T N(s,t)\phi(t)\, ds\, dt. \tag{16}$$

The following important result shows that for linear systems with delay, the system is exponentially stable if and only if there exists a quadratic Lyapunov function.

Theorem 2 *The linear system defined by equations (10) and (11) is exponentially stable if and only if there exists a Dini-differentiable function $V \in X$ and $\varepsilon > 0$ such that for all $\phi \in C[-h, 0]$*

$$\begin{aligned} V(\phi) &\geq \varepsilon \|\phi(0)\|^2 \\ \dot{V}(\phi) &\leq -\varepsilon \|\phi(0)\|^2. \end{aligned} \tag{17}$$

Further $V \in X$ may be chosen such that the corresponding functions M and N of equation (16) have the following smoothness property: $M(s)$ and $N(s,t)$ are bounded and continuous at all $s, t \in [-h, 0]$ except possibly points h_i.

4 The SOS Stability Test

Suppose we wish to prove stability by constructing functionals of the form of Equation (16) using polynomial optimization. The derivative of a quadratic functional $V \in X$ has a similar structure to V and is also defined by matrix functions which are linear transformations of M and N. In [8], a necessary and sufficient condition was given for positivity of the first part of the functional. A version of this is as follows.

Theorem 3 *Suppose $M : [-h, 0] \to \mathbb{S}^{n+m}$ is continuous except possibly at points h_i and is bounded. Then the following are equivalent.*

(i) There exists an $\epsilon > 0$ so that for all $c \in \mathbb{R}^n$ and continuous $y : [-h, 0] \to \mathbb{R}^m$,

$$\int_{-h}^{0} \begin{bmatrix} c \\ y(t) \end{bmatrix}^T M(t) \begin{bmatrix} c \\ y(t) \end{bmatrix} dt \geq \epsilon \|y\|_{L_2} \tag{18}$$

(ii) There exist an $\eta > 0$ and a function $T : [-h, 0] \to \mathbb{S}^n$, continuous except possibly at points h_i, which is bounded and satisfies

$$M(t) + \begin{bmatrix} T(t) & 0 \\ 0 & -\eta I \end{bmatrix} \geq 0 \quad \textit{for all } t \in [-h, 0] \tag{19}$$

$$\textit{and} \quad \int_{-h}^{0} T(t)\, dt = 0. \tag{20}$$

This theorem converts positivity of an integral to *pointwise* positivity of a function with a linear constraint. If we assume M and T are polynomial, pointwise positivity is equivalent to a sum-of-squares constraint. The constraint that T integrates to zero is a bounded linear constraint.

It was also shown in this paper that if N is polynomial,

$$\int_{-h}^{0} \int_{-h}^{0} x(s)^T N(s,t) x(t) ds\, dt \geq 0$$

if and only if $N(s,t) = Z(s)^T Q Z(t)$ for some $Q \geq 0$ where $Z(s)$ is a basis of monomials. The condition that the derivative of the Lyapunov function be negative has a similar structure. For details on formulating the semidefinite program, we refer to [8].

5 Approximation on Linear Varieties

In this section, we show that a continuous solution of the conditions in Theorem 3 imply the existence of a polynomial solution to the same conditions. This is achieved through an extension to the Weierstrass approximation theorem to linear varieties of the Banach space $\mathcal{C}[X, \mathbb{R}^{n\times m}]$.

Theorem 4 *Let X be a compact subset of $\mathbb{R}^n$ and $L_i : \mathcal{C}(X, \mathbb{R}^{n\times o}) \to \mathbb{R}^{p\times q}$ be bounded linear operators for $i = 1, \ldots, k$. Then for any $f \in \mathcal{C}(X, \mathbb{R}^{n\times o})$ and $\delta > 0$, there exists polynomial r such that $\|f - r\|_\infty \le \delta$ and $L_i r = L_i f$ for $i = 1, \ldots, k$.*

PROOF. First we note that by adding trivial constraints, we can assume without loss of generality that $L_i : \mathcal{C}(X, \mathbb{R}^{n\times o}) \to \mathbb{R}$. i.e. The L_i map to the real numbers.

Proceed by induction. Suppose that the proposition is true for $k = m - 1$. If $L_m = 0$, then let r be as given by the proposition for $m - 1$. In this case $L_m r = L_m f = 0$. Otherwise, there exists some $g \in \mathcal{C}(X, \mathbb{R}^{n\times o})$ such that $L_m g = c > 0$. Let β be a uniform bound for operators L_i in $i = 1, \ldots, k$. Rescale g so that $\|g\|_\infty = \delta/4$. Let $\gamma = \min\{\frac{\delta}{4}, \frac{c}{\beta}\}$.

By assuming that the proposition is true for $k = m - 1$, we assume there exists some polynomial p such that

$$L_i f = L_i p \text{ for } i = 1, \ldots, m-1 \quad \text{and} \quad \|f + g - p\|_\infty \le \gamma. \tag{21}$$

Therefore

$$\|f - p\|_\infty \le \|f + g - p\|_\infty + \|g\|_\infty \le \frac{\delta}{4} + \frac{\delta}{4} \le \delta/2. \tag{22}$$

Furthermore,

$$\begin{aligned} L_m p &= L_m f + L_m g - L_m (f + g - p) && (23)\\ &\ge L_m f + c - \beta\gamma \ge L_m f. && (24) \end{aligned}$$

By similar logic, there also exists a polynomial b with $\|f - b\|_\infty \le \delta/2$, $L_m b \le L_m f$ and $L_i f = L_i b$ for $i = 1, \ldots, m-1$. Now since $L_m p \ge L_m f$ and $L_m b \le L_m f$, there exists some $\lambda \in [0, 1]$ such that $\lambda L_m p + (1 - \lambda) L_m b = Lf$. Now let $r = \lambda p + (1 - \lambda) b$. Then r is polynomial,

$$L_m r = \lambda L_m p + (1 - \lambda) L_m b = L_m f, \tag{25}$$

$$\begin{aligned} L_i r &= \lambda L_i p + (1 - \lambda) L_i b && (26)\\ &= \lambda L_i f + (1 - \lambda) L_i f = L_i f \quad \text{for } i = 1, \ldots, m-1, && (27) \end{aligned}$$

and

$$\begin{aligned} \|f - r\|_\infty &= \|\lambda(f - p) + (1 - \lambda)(f - b)\|_\infty && (28)\\ &\le \lambda \|f - p\|_\infty + (1 - \lambda)\|f - b\|_\infty && (29)\\ &\le \delta/2 + \delta/2 = \delta. && (30) \end{aligned}$$

Therefore, if the proposition is true for $k = m - 1$, it is also true for $k = m$. Now add the trivial constraint $L_1 = 0$. Then the proposition is true for $k = 1$ by the Weierstrass Approximation Theorem.

We now proceed to develop a number of extensions to the main result.

6 Approximation of Derivatives on Linear Varieties

Because the derivative of the Lyapunov function contains the derivative of the functions M and N, we must extend the previous work to approximating the derivatives of functions.

Proposition 1 *Let $L_i, K_i : \mathcal{C}([a,b],\mathbb{R}^{n\times m}) \to \mathbb{R}^{p\times q}$ be bounded linear operators. Then for any $f \in \mathcal{C}^1([a,b],\mathbb{R}^{n\times m})$ and $\delta > 0$, there exists a polynomial r such that $\|f-r\|_\infty \le \delta$, $\|\dot f - \dot r\|_\infty \le \delta$, $L_i r = L_i f$ for $i = 1,\ldots,k$, and $K_i \dot r = K_i \dot f$ for $i = 1,\ldots,l$.*

PROOF. Define the bounded linear operators $G_i : \mathcal{C}([a,b],\mathbb{R}^{n\times m}) \to \mathbb{R}^{p\times q}$ by

$$G_i z := L_i(\phi z), \quad (\phi z)(s) := \int_a^s z(t)dt. \tag{31}$$

By Theorem 4, there exists a polynomial v such that $\|\dot f - v\|_\infty \le \delta/(b-a)$, $G_i \dot f = G_i v$ for $i = 1,\ldots,k$, and $K_i \dot f = K_i v$ for $i = 1,\ldots,m$. Let

$$r(s) = f(a) + \int_a^s v(t)dt. \tag{32}$$

Then r is polynomial, $\dot r = v$,

$$\|\dot f - \dot r\|_\infty = \|\dot f - v\|_\infty \le \delta/(b-a), \tag{33}$$

and

$$\begin{aligned} \|f-r\|_\infty &= \sup_{s\in[a,b]} \| \int_a^s \dot f(t) - v(t)dt\| && (34)\\ &\le \sup_{t\in[a,b]} \|\dot f(t) - v(t)\|(b-a) && (35)\\ &= (b-a)\|\dot f - v\|_\infty \le \delta. && (36) \end{aligned}$$

Furthermore, since $f - r = \phi(\dot f - v)$,

$$\begin{aligned} L_i(f-r) &= L_i(\phi(\dot f - v)) && (37)\\ &= G_i(\dot f - v) = 0 \quad \text{for } i = 1,\ldots,k. && (38) \end{aligned}$$

7 Polynomial Solutions for Polynomial Optimization

In this section, we briefly examine what Theorem 4 means from the perspective of optimization over polynomials. In particular, consider the optimization problem (39) where $L_i, K_i : \mathcal{C}([a,b],\mathbb{R}^{n\times m}) \to \mathbb{R}^{o\times p}$ for $i = 0,\ldots,k_1$ and $G_i, H_i : \mathcal{C}([a,b],\mathbb{R}^{n\times m}) \to \mathcal{C}([a,b],\mathbb{S}^c)$ for $i = 1,\ldots,k_2$ are bounded linear operators and $E \in \mathcal{C}([a,b],\mathbb{S}^c)$.

$$\begin{aligned} &\max\ L_0 f + K_0 \dot f : \\ &C_i + L_i f + K_i \dot f = 0, && \text{for } i = 1,\ldots,k_1 \\ &(G_i f)(s) + (H_i \dot f)(s) \succ E(s) && \text{for } s \in [a,b],\ i = 1,\ldots,k_2. \end{aligned} \tag{39}$$

The following theorem is used directly in the proof of Theorem 6.

Theorem 5 *Consider optimization problem 39. Suppose $f \in \mathcal{C}^1([a,b],\mathbb{R}^{n\times m})$ is feasible with objective value c. Then there exists a feasible polynomial with objective value c.*

PROOF. Suppose f is feasible with objective value c. Let β be a uniform bound for the K_i and L_i. By feasibility of f, there exists a $\gamma > 0$ such that

$$L_i f(s) + K_i \dot{f}(s) \succ \gamma + E(s) \text{ for } s \in [a,b],\ i = 1, \ldots, k_2. \tag{40}$$

Let $\delta = \frac{\gamma}{4\beta}$. By theorem 1, there exists a polynomial q with $\|f - q\|_\infty \le \delta$ and $\|\dot{f} - \dot{q}\|_\infty \le \delta$ such that

$$L_0 q + K_0 \dot{q} = L_0 f + K_0 \dot{f} = c \tag{41}$$
$$L_i q + K_i \dot{q} = L_i f + K_i \dot{f} = C_i, \quad \text{for } i = 1, \ldots, m \tag{42}$$
$$\tag{43}$$

Then we have,

$$L_i q(s) + K_i \dot{q}(s) \tag{44}$$
$$= L_i f(s) + K_i \dot{f}(s) + L_i(q - f)(s) + K_i(\dot{q} - \dot{f})(s) \tag{45}$$
$$\succeq \gamma E(s) - \beta\|f - q\|_\infty - \beta\|\dot{f} - \dot{q}\|_\infty \tag{46}$$
$$\succeq \gamma - \gamma/2 - E(s) \succ E(s) \tag{47}$$

Thus q is feasible with objective value c

If the derivative condition is not included, then Theorem 5 is easily extended from intervals to arbitrary compact $X \subset \mathbb{R}^n$. For $X \subset \mathbb{R}^n$, the derivative can also be included in a manner similar to the work in [7]. Note that polynomials exactly solve the optimization problem. While counter-intuitive, this conclusion is motivated by the observation that the problem is, in a sense, finite-dimensional.

8 Polynomial Lyapunov-Krasovskii Functionals

We now wish to prove that we can use polynomial optimization to construct functionals of the form of Equation (16).

8.1 The Single Integral Term

The important question for the first case is whether the conditions in Theorem 3 can be expressed using polynomials. The following theorem answers this question in the case of a single delay. Recall that we defined the linear map $L : \mathcal{C}^1([-h,0], \mathbb{S}^{2n}) \to \mathcal{C}([-h,0], \mathbb{S}^{3n})$ as in Equation (7) in the introduction.

Theorem 6 *Suppose there exist continuous functions $M \in \mathcal{C}^1([-h,0], \mathbb{S}^{2n})$, $Q \in \mathcal{C}([-h,0], \mathbb{S}^{2n})$ and $T \in \mathcal{C}([-h,0], \mathbb{S}^n)$ such that*

$$M(s) + \begin{bmatrix} T(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \textit{for all } s \in [-h, 0], \tag{48}$$

$$-(L(M))(s) + \begin{bmatrix} Q(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \textit{for all } s \in [-h, 0], \tag{49}$$

$$\int_{-h}^{0} T(s) ds = 0 \quad \textit{and} \quad \int_{-h}^{0} Q(s) ds = 0. \tag{50}$$

Then there exist polynomials M_p, Q_p, and T_p, of the same dimension, which satisfy

$$M_p(s) + \begin{bmatrix} T_p(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \textit{for all } s \in [-h, 0], \tag{51}$$

$$-(L(B))(s) + \begin{bmatrix} Q_p(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \textit{for all } s \in [-h, 0], \tag{52}$$

$$\int_{-h}^{0} T_p(s)ds = 0 \quad \textit{and} \quad \int_{-h}^{0} Q_p(s)ds = 0. \tag{53}$$

PROOF. The theorem is expressed as an optimization problem and thus the proof follows immediately from theorem 5.

8.2 The Double-Integral Term

In this section, we briefly prove using a slight manipulation of existing theory that we can also assume that the second part of the Lyapunov functional is also defined by polynomials.

Lemma 2 *For any continuously differentiable N and any $\gamma > 0$, there exists a polynomial N_p such that for any $\phi \in \mathcal{C}([-h, 0], \mathbb{R}^n)$,*

$$\left| \int_{-\tau}^{0} \int_{-\tau}^{0} \phi(s) \left(N(s,t) - N_p(s,t)\right) \phi(t) ds dt \right| \leq \gamma \|\phi\|_{L_2}^2 \tag{54}$$

PROOF. By the Weierstrass approximation theorem, if N is continuously differentiable, then for any $\gamma > 0$, there exists some polynomial $N_p(s,t)$ such that $\|N - N_p\|_\infty \leq \gamma/\tau^2$. Now, it can be shown using Cauchy-Schwartz that

$$\left| \int_{-\tau} \int_{-\tau} \phi(s)(P(s,t) - N(s,t))\phi(t) \right| \tag{55}$$

$$\leq \int_{-\tau} \int_{-\tau} \|P(s,t) - N(s,t)\| ds dt \|\phi\|_{L_2} \tag{56}$$

$$\leq \gamma \|\phi\|_{L_2} \tag{57}$$

See [11] for a proof.

Note: It is worth considering the interpretation of Lemma 2. The “complete-quadratic” converse Lyapunov result presented earlier states that stability implies the existence of a functional whose double-integral term uses a continuous function N (at least for a single delay). However, as Lemma 2 shows, functionals which use a polynomial N can approximate functionals which use a continuous N to arbitrary precision in $\|\phi\|_{L_2}$. Furthermore, as we can decompose any polynomial $N(s,t) = S_1(s)S_2(t)$, it follows that one can never achieve strict positivity of the double integral term using a polynomial N. The conclusion, then, is that the double-integral term of the converse Lyapunov functional will never be strictly positive.

8.3 Combined Theorem

The combined result in this subsection reflects the discussion in the the previous subsection on the limits of polynomial kernels in the double-integral term. To include the double-integral term in the derivative, we redefine the linear map as

$$(R(M,N))(s) := \begin{bmatrix} A_0^T M_{11} + M_{11}A_0 & M_{11}A_1 & 0 \\ A_1^T M_{11} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{58}$$

$$+ \begin{bmatrix} 0 & 0 & A_0^T M_{12}(s) \\ 0 & 0 & A_1^T M_{12}(s) \\ M_{21}(s)A_0 & M_{21}(s)A_1 & 0 \end{bmatrix} \tag{59}$$

$$+ \frac{1}{h} \begin{bmatrix} M_{12}(0) + M_{21}(0) + M_{22}(0) & -M_{12}(-h) & 0 \\ -M_{21}(-h) & -M_{22}(-h) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{60}$$

$$+ \begin{bmatrix} 0 & 0 & -\dot{M}_{12}(s) \\ 0 & 0 & 0 \\ -\dot{M}_{21}(s) & 0 & -\dot{M}_{22}(s) \end{bmatrix} + \begin{bmatrix} 0 & 0 & N(0,s) \\ 0 & 0 & -N(-\tau,s) \\ N(s,0) & -N(s,-\tau) & 0 \end{bmatrix} \tag{61}$$

$$\tag{62}$$

Theorem 7 *There exist continuously differentiable functions M and N such that for any $\phi \in \mathcal{C}([-h,0],\mathbb{R}^n)$,*

$$\int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds > \gamma_1 \left(\|\phi(0)\|^2 + \|\phi\|_{L_2}^2 \right) \tag{63}$$

$$\int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds + \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s)N(s,t)\phi(t)ds\,dt > \gamma_2 \left(\|\phi(0)\|^2 + \|\phi\|_{L_2}^2 \right) \tag{64}$$

and

$$\int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(s) \end{bmatrix}^T (R(M,N))(s) \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(s) \end{bmatrix} ds < -\beta_1 \left(\|\phi(0)\|^2 + \|\phi\|_{L_2}^2 \right) \tag{65}$$

$$\int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(s) \end{bmatrix}^T (R(M,N))(s) \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(s) \end{bmatrix} + \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s) \left(\frac{d}{ds} + \frac{d}{dt} \right) N(s,t)\phi(t) < -\beta_2 \left(\|\phi(0)\|^2 + \|\phi\|_{L_2}^2 \right) \tag{66}$$

where $\gamma, \beta > 0$ if and only if there exist polynomials M_p, N_p, T_p and Q_p such that

$$M_p(s) + \begin{bmatrix} T_p(s) & 0 \\ 0 & 0 \end{bmatrix} \succ \gamma_1 I \quad \text{for all } s \in [-h,0], \tag{67}$$

$$-(R(M_p,N_p))(s) + \begin{bmatrix} Q_p(s) & 0 \\ 0 & 0 \end{bmatrix} \succ \beta_1 I \quad \text{for all } s \in [-h,0], \tag{68}$$

$$\int_{-h}^{0} T_p(s)ds = 0 \quad \text{and} \quad \int_{-h}^{0} Q_p(s)ds = 0. \tag{69}$$

and

$$\int_{-\tau}^{0} \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix} M_p(s) \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix} ds + \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s)N_p(s,t)\phi(t)ds\,dt > \gamma_2\left(\|\phi(0)\|^2 + \|\phi\|_{L_2}^2\right) \tag{70}$$

$$\int_{-\tau}^{0} \begin{bmatrix}\phi(0)\\ \phi(-\tau)\\ \phi(s)\end{bmatrix}^T (R(M_p,N_p))(s) \begin{bmatrix}\phi(0)\\ \phi(-\tau)\\ \phi(s)\end{bmatrix} + \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s)\left(\frac{d}{ds}+\frac{d}{dt}\right) N_p(s,t)\phi(t) < -\beta_2\left(\|\phi(0)\|^2 + \|\phi\|_{L_2}^2\right). \tag{71}$$

PROOF. The proof is simple. By Theorem 3, the conditions

$$\int_{-\tau}^{0} \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix} M(s) \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix} ds > \gamma_1\left(\|\phi(0)\|^2 + \|\phi\|_{L_2}^2\right) \tag{72}$$

$$\int_{-\tau}^{0} \begin{bmatrix}\phi(0)\\ \phi(-\tau)\\ \phi(s)\end{bmatrix}^T (R(M,N))(s) \begin{bmatrix}\phi(0)\\ \phi(-\tau)\\ \phi(s)\end{bmatrix} ds < -\beta_1\left(\|\phi(0)\|^2 + \|\phi\|_{L_2}^2\right) \tag{73}$$

are equivalent to

$$M(s) + \begin{bmatrix}T(s) & 0\\ 0 & 0\end{bmatrix} \succ \gamma_1 I \quad \text{for all } s\in[-h,0], \tag{74}$$

$$-(R(M,N))(s) + \begin{bmatrix}Q(s) & 0\\ 0 & 0\end{bmatrix} \succ \beta_1 I \quad \text{for all } s\in[-h,0], \tag{75}$$

$$\int_{-h}^{0} T(s)ds = 0 \quad \text{and} \quad \int_{-h}^{0} Q(s)ds = 0 \tag{76}$$

for some continuous Q,T. By Theorem 6, Lemma 2 and Theorem 5 of [7], for any $\eta > 0$ this is equivalent to the existence of polynomials M_p, N_p, T_p and Q_p such that $\|N_p - N\|_\infty < \eta$, $\|\left(\frac{d}{ds}+\frac{d}{dt}\right)(N_p - N)\|_\infty < \eta$, $\|M_p - M\|_\infty < \eta$, $\|L(M_p,N_p) - L(M,N)\|_\infty < \eta$ and

$$M_p(s) + \begin{bmatrix}T_p(s) & 0\\ 0 & 0\end{bmatrix} \succ \gamma_1 I \quad \text{for all } s\in[-h,0], \tag{77}$$

$$-(R(M_p,N_p))(s) + \begin{bmatrix}Q_p(s) & 0\\ 0 & 0\end{bmatrix} \succ \beta_1 I \quad \text{for all } s\in[-h,0], \tag{78}$$

$$\int_{-h}^{0} T_p(s)ds = 0 \quad \text{and} \quad \int_{-h}^{0} Q_p(s)ds = 0. \tag{79}$$

Finally, since $\|N_p - N\|_\infty < \eta$, $\|\left(\frac{d}{ds}+\frac{d}{dt}\right)(N_p - N)\|_\infty < \eta$, $\|M_p - M\|_\infty < \eta$ and $\|L(M_p, N_p) - L(M,N)\|_\infty < \eta$, we conclude that for η sufficiently small that

$$\int_{-\tau}^{0} \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix} M_p(s) \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix} ds + \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s)N_p(s,t)\phi(t) ds\, dt > \gamma_2 \left(\|\phi(0)\|^2 + \|\phi\|_{L_2}^2\right) \tag{80}$$

$$\int_{-\tau}^{0} \begin{bmatrix}\phi(0)\\ \phi(-\tau)\\ \phi(s)\end{bmatrix}^T (R(M_p, N_p))(s) \begin{bmatrix}\phi(0)\\ \phi(-\tau)\\ \phi(s)\end{bmatrix} + \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s)\left(\frac{d}{ds}+\frac{d}{dt}\right) N_p(s,t)\phi(t) < -\beta_2 \left(\|\phi(0)\|^2 + \|\phi\|_{L_2}^2\right). \tag{81}$$

Note that the above theorem could have been expressed more elegantly using strict positivity of the double integral term. However, the result would be vacuous, as one can never achieve strict positivity of this term using a continuous N.

9 Multiple Delays

In the case of multiple delays, the existence results of the previous section still hold. However, the presentation of the results becomes more complex. For this reason, we will restrict ourselves to presenting only the core information. First we suppose that there are multiple points of discrete delay $h = h_k, \dots, h_0 = 0$. We define the jump values of M at the discontinuities (h_i) as follows.

$$\Delta M(h_i) := \tag{82}$$

$$\lim_{t\to(-h_i)_+} M(t) - \lim_{t\to(-h_i)_-} M(t) \quad \text{for} \quad i = 1, \dots, k-1 \tag{83}$$

Definition 3 *Define the map L by $D = L(M)$ if for all $t \in [-h, 0]$ we have*

$$D_{11} = A_0^T M_{11} + M_{11} A_0 \tag{84}$$

$$+ \frac{1}{h}\big(M_{12}(0) + M_{21}(0) + M_{22}(0)\big), \tag{85}$$

$$D_{12} = \begin{bmatrix} M_{11}A_1 & \cdots & M_{11}A_{k-1}\end{bmatrix} \tag{86}$$

$$- \begin{bmatrix} \Delta M_{12}(h_1) & \cdots & \Delta M_{12}(h_{k-1})\end{bmatrix}, \tag{87}$$

$$D_{13} = \frac{1}{h}\big(M_{11}A_k - M_{12}(-h)\big), \tag{88}$$

$$D_{22} = -\frac{1}{h}\operatorname{diag}\big(\Delta M_{22}(h_1), \cdots, \Delta M_{22}(h_{k-1})\big) \tag{89}$$

$$D_{23} = 0, \quad D_{33} = -\frac{1}{h} M_{22}(-h), \tag{90}$$

$$D_{14}(t) = A_0^T M_{12}(t) - \dot{M}_{12}(t) \tag{91}$$

$$D_{24}(t) = \begin{bmatrix} A_1^T M_{12}(t) \\ \vdots \\ A_{k-1}^T M_{12}(t)\end{bmatrix}, \tag{92}$$

$$D_{34}(t) = A_k^T M_{12}(t), \quad D_{44}(t) = -\dot{M}_{22}(t). \tag{93}$$

Here M is partitioned according to

$$M(t) = \begin{bmatrix} M_{11} & M_{12}(t) \\ M_{21}(t) & M_{22}(t)\end{bmatrix}, \tag{94}$$

where $M_{11} \in \mathbb{S}^n$ and D is partitioned according to

$$D(t) = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14}(t) \\ D_{21} & D_{22} & D_{23} & D_{24}(t) \\ D_{31} & D_{32} & D_{33} & D_{34}(t) \\ D_{41}(t) & D_{42}(t) & D_{43}(t) & D_{44}(t) \end{bmatrix}. \tag{95}$$

Theorem 8 *Suppose there exist functions $M : [-h, 0] \to \mathbb{S}^{2n}$, $Q : [-h, 0] \to \mathbb{S}^{2n}$ and $T : [-h, 0] \to \mathbb{S}^{nk+2}$ which are piecewise continuous with possible jumps at h_i. Suppose that that*

$$M(s) + \begin{bmatrix} T(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \textit{for all } s \in [-h, 0], \tag{96}$$

$$-(L(M))(s) + \begin{bmatrix} Q(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \textit{for all } s \in [-h, 0], \tag{97}$$

$$\int_{-h}^{0} T(s)ds = 0 \quad \textit{and} \quad \int_{-h}^{0} Q(s)ds = 0. \tag{98}$$

Then there exist B, C, and D of the same dimension as M, T, and Q, respectively, and defined by polynomials on the intervals $[-h_i, -h_{i-1}]$ with jumps at the h_i, and which satisfy

$$B(s) + \begin{bmatrix} C(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \textit{for all } s \in [-h, 0], \tag{99}$$

$$-(L(B))(s) + \begin{bmatrix} D(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \textit{for all } s \in [-h, 0] \tag{100}$$

$$\int_{-h}^{0} C(s)ds = 0 \quad \textit{and} \quad \int_{-h}^{0} D(s)ds = 0. \tag{101}$$

PROOF. Let M_i, T_i, and Q_i be defined by the functions M, T, and Q restricted to the interval $[-h_i, -h_{i-1}]$ with right and left-hand limits defined by continuity. Then M_i, T_i, and Q_i are continuous. Then by scaling and application of Theorem 5, there exist polynomials B_i, T_i, and Q_i for $i = 1, \ldots, k$ such that

$$B_i(s) + \begin{bmatrix} C_i(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \text{for all } s \in [-h_i, -h_{i-1}], \tag{102}$$

$$-(L(B))(s) + \begin{bmatrix} D_i(s) & 0 \\ 0 & 0 \end{bmatrix} \succ 0 \quad \text{for all } s \in [-h_i, -h_{i-1}], \tag{103}$$

for $i = 1, \ldots, k$, and

$$\sum_{i=1}^{k} \int_{-h_i}^{-h_{i-1}} C_i(s)ds = 0, \qquad \sum_{i=1}^{k} \int_{-h_i}^{-h_{i-1}} D_i(s)ds = 0, \tag{104}$$

where B is the function which is defined by B_i on the interval $[-h_i, -h_{i-1}]$ for $i = 1, \ldots, k$. If we similarly define C and D by C_i and D_i, respectively, on the interval $[-h_i, -h_{i-1}]$, then we recover the conditions of the theorem.

10 Conclusion

In this paper, we have shown that polynomials can be used without conservatism in the construction of the complete-quadratic Lyapunov functional, a particularly important tool in the analysis of time-delay systems. As part of this work, we have extended the Weierstrass approximation theorem to affine subspaces of the set of continuous functions. Note as well that Theorem 5 may be used in an opposite sense. That is, this theorem can be used to prove that if polynomials M_p and N_p approximate continuous functions M and N to sufficient accuracy, then if M and N prove exponential stability, then M_p and N_p will also prove exponential stability. This interpretation is relevant to solving Lyapunov equalities. See [2] for work in this area.

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