Reducing the Complexity of the Sum-of-Squares Test for Stability of Delayed Linear Systems

Yashun Zhang, Matthew Peet, and Keqin Gu

***Abstract*—This paper considers the problem of reducing the computational complexity associated with the Sum-of-Squares approach to stability analysis of time-delay systems. Specifically, this paper considers systems with a large state-space but where delays affect only certain parts of the system. This yields a coefficient matrix of the delayed state with low rank— a common scenario in practice. The paper uses the general framework of coupled differential-difference equations with delays in feedback channels. This framework includes systems of both the neutral and retarded-type. The Sum-of-Squares method is used to search a Lyapunov-Krasovskii functional which is necessary and sufficient for stability of this class of systems. This paper shows how exploiting the structure of the new functional can yield dramatic improvements in computational complexity. Numerical examples are given to illustrate this improvement.**

***Index Terms*—Lyapunov-Krasovskii functional, time delay, semidefinite programming, sum-of-squares, complexity.**

I. Introduction

In this paper we consider stability of linear time-delay systems with fixed delays. The existence of a monotonically decreasing quadratic Lyapunov function is necessary and sufficient for stability of these systems [6], [9], [15]. As is customary, we refer to these Lyapunov functions as Lyapunov-Krasovskii functionals as the state-space is infinite dimensional. The problem of finding such a functional is considered computationally intractable. An obvious solution is to use simplified versions of the functional. Naturally, however, stability conditions derived in such a manner will be conservative [6]. A solution to this dilemma was proposed in [4] which used a "discretized" version of the Lyapunov-Krasovskii functional. The product was a series of sufficient conditions which appears to converge to necessity as the level of discretization is increased. The significance of this work is that it gives a quantifiable tradeoff between computational complexity and accuracy of the stability test. See Fig. 1 in the numerical example. In [20] and [21], the problem was approached using polynomials instead of discretized functionals. We refer to this result as the Sum-of-Squares (SOS) method. The advantage of the Sum-of-Squares approach is that it is easily generalized to nonlinear and uncertain systems [17]. It should be pointed out that it is possible to asymptotically approach the analytical limit of stability without the complete quadratic Lyapunov-Krasovskii functional. An interesting method that accomplishes this is the delay partitioning method described in [3]. In all of the above cases, the conditions are expressed using semidefinite programming (SDP) [14], [18]. A problem with both the discretized functional method and the Sum-of-Squares method is that the computational cost increases quickly for large systems with multiple delays.

In most practical systems, although the number of state variables is rather large, there are relatively few delayed elements and these delayed elements enter through low-rank coefficient matrices. Examples include a nuclear reactor model described in Equation (3.1) of Chapter 2 of [10]; chemostat models in microbiology described in Equation (5.4) in Chapter 2 of [10]; or any system with delayed feedback. However, this feature is not typically leveraged when deriving stability conditions. In this paper, we reformulate the standard model of time-delayed equations by using coupled differential-difference equations with a single delay in each feedback channel. The idea is that if the dimension of the feedback channel is substantially smaller than the number of states, then this formulation allows us to exploit this low-dimensional structure to potentially reduce the computational cost of stability analysis [7]. In addition, using coupled differential-difference equations allows us to address a larger class of systems that includes time delay systems of both retarded and neutral type.

Coupled differential-difference equations have been studied for some time. See references [2], [22] and [25]. Asymptotic stability analysis based on the input-to-state stability of the difference equations was given in [23]. This result was strengthened to uniform asymptotic stability and extended to the general coupled differential-functional equations in [8], which also considered the possibility of reducing the complexity of the discretized Lyapunov-Krasovskii functional method. A reformulation of coupled differential-difference equations with single independent delay in each channel was proposed in [7] with a discretized Lyapunov implementation in [11]. It is interesting to notice that this special form of coupled differential-difference equations has been known as the "Roesser's model", and studied earlier using frequency domain approaches [1], [12].

The purpose of this paper is to adapt the Sum-of-Squares approach to coupled differential-difference equations and a structured Lyapunov-Krasovskii functional. The goal is to realize a complexity reduction of several orders of magnitude in systems with low-dimensional delay channels. The paper is structured as follows. We begin by introducing the coupled formulation. This is accompanied by a necessary and sufficient quadratic Lyapunov result and some basics on Sum-of-Squares. In Section III, we adapt the Sum-of-Squares approach to positivity of the Lyapunov-Krasovskii functional introduced previously. In Section IV, we give the derivative of the Lyapunov-Krasovskii functional and apply our results to enforce negativity. In Section V, we combine our results to give an asymptotically exact, semidefinite-programming-based approach to stability of linear time-delay systems. Finally, we discuss computational complexity and use numerical examples to illustrate the advantages of the current approach.

A. Notation

$\mathbb{Z}$ denotes the set of positive integers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbb{R}^{p\times q}$ denotes the set of all $p \times q$ real matrices. $\mathbb{S}^n$ denotes all the $n \times n$ symmetric real matrices. For $X \in \mathbb{S}^n$, the notation $X \geq 0$ ($X > 0$) means that X is positive semidefinite (definite). I denotes the identity matrix with appropriate dimension. For $n \in \mathbb{Z}$ and a given positive real number r, we use $\mathcal{PC}(r, n)$ to denote the vector space of bounded functions $f : [-r, 0) \to \mathbb{R}^n$ which are right continuous everywhere, and continuous everywhere except possibly at a finite number of points in the interval. Unless otherwise stated, $\|f\|$ denotes either the 2-norm if $f \in \mathbb{R}^n$ or the L_2-norm if f is a square-integrable function. Sometimes we use the notation $\|f\|_{L_2}$ for additional clarity. For a given function y , if y is defined on $[t - r, t]$, we will use $y_{r,t}$ to denote the segment of y on this interval, but translated to the origin. Specifically, $y_{r,t}(s) = y(t+s)$, for $s \in [-r, 0)$. Through some abuse of notation, we will occasionally use $\phi = (\phi_1, \phi_2, \ldots, \phi_K) \in \mathcal{PC}$ to denote $\phi_i \in \mathcal{PC}(r_i, m_i), i = 1, 2, \ldots, K$, where r_i and m_i will be clear from context.

Y. Zhang is with the School of Automation, Nanjing University of Science and Technology, China (email: yashunzhang@gmail.com).

M. Peet is with the Department of Mechanical, Materials, and Aerospace Engineering, Illinois Institute of Technology, IL, USA (email: mpeet@iit.edu).

K. Gu is with the Department of Mechanical and Industrial Engineering, Southern Illinois University Edwardsville, IL, USA (kgu@siue.edu).

This work was completed while Y. Zhang was visiting Southern Illinois University Edwardsville.

II. PRELIMINARIES

A. Coupled Differential-Difference Equations

Consider a linear time-delay system described by the coupled differential-difference equation

$$\dot{x}(t) \;=\; Ax(t) + \sum_{j=1}^{K} B_j y_j(t - r_j), \tag{1}$$

$$y_i(t) \;=\; C_i x(t) + \sum_{j=1}^{K} D_{ij} y_j(t - r_j), \qquad i = 1, 2, \ldots K, \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $y_i(t) \in \mathbb{R}^{m_i}$, and K is the number of delay "channels". Without loss of generality, we assume the delays are in ascending order $0 < r_1 < \cdots < r_K$. The initial conditions are given by $x(0) = \psi \in \mathbb{R}^n$, and $y_{i_{r_i},0} = \phi_i \in \mathcal{PC}(r_i, m_i)$. For the sake of brevity, we use the notation $y_t := (y_{1_{r_1},t}, \cdots, y_{K_{r_K},t})$, so the state of the system at time t is $(x(t), y_t)$.

B. Quadratic Lyapunov-Krasovskii Functionals

The following is a necessary and sufficient condition for stability of a system defined by the coupled equations (1) and (2).

Theorem 1 ([7]): Suppose there exist $L_i \in \mathbb{S}^{m_i}$, $L_i > 0$, $i = 1, 2, \ldots K$ such that the following linear matrix inequality (LMI) is satisfied.

$$D^T U D - U < 0, \tag{3}$$

where $U = \mathrm{diag}(L_1, L_2, \ldots, L_K)$ and $D = [D_{ij}]_{K \times K}$ is the matrix formed by using the matrices D_{ij} as sub-blocks. Then the system described by (1) and (2) is uniformly asymptotically stable if and only if there exist an $\varepsilon > 0$, a matrix $P \in \mathbb{S}^n$, and matrix functions $Q_i(s) \in \mathbb{R}^{n \times m_i}$, $R_{ij}(s, \eta) = R_{ji}^T(\eta, s) \in \mathbb{R}^{m_i \times m_j}$, and $S_i(s) \in \mathbb{S}^{m_i}$ such that for any $\psi \in \mathbb{R}^n$ and $\phi = (\phi_1, \phi_2, \ldots, \phi_K) \in \mathcal{PC}$, we have that $V(\psi, \phi) \geq \varepsilon \psi^T \psi$ and $\dot{V}(\psi, \phi) \leq -\varepsilon \psi^T \psi$, where

$$\begin{aligned} V(\psi, \phi) \;&=\; \psi^T P \psi + \sum_{i=1}^{K} \int_{-r_i}^{0} \phi_i^T(s)\, S_i(s)\, \phi_i(s)\, ds \\ &\quad + \sum_{i=1}^{K} \sum_{j=1}^{K} \int_{-r_i}^{0} \int_{-r_j}^{0} \phi_i^T(s)\, R_{ij}(s, \eta)\, \phi_j(\eta)\, d\eta\, ds \\ &\quad + 2\, \psi^T \sum_{i=1}^{K} \int_{-r_i}^{0} Q_i(s)\, \phi_i(s)\, ds, \end{aligned} \tag{4}$$

$$\begin{aligned} \dot{V}(\psi, \phi) \;&=\; \frac{d}{dt} V(x(t), y_t)|_{x(t)=\psi, y_t=\phi} \\ &=\; \limsup_{h \to 0^+} \frac{1}{h} \left(V\left(x(t+h), y_{t+h}\right) - V(\psi, \phi) \right). \end{aligned}$$

C. Sum-of-Squares

Sum-of-Squares is a branch of polynomial computing which considers the positivity of polynomials. Although the question of polynomial positivity is NP-hard, testing whether a polynomial can be represented as the sum of squares of polynomials is computationally tractable. In this paper, we use the following notation. For a given nonnegative integer d, let Z_d be the vector of monomials

$$Z_d(s) := \left[\, 1 \;\; s \;\; s^2 \;\; \cdots \;\; s^d \,\right]^T.$$

Let $Z_{n,d} : \mathbb{R} \to \mathbb{R}^{n(d+1) \times n}$ be defined as

$$Z_{n,d}(s) = I_{n \times n} \otimes Z_d(s),$$

where $\otimes$ denotes the Kronecker product. A $n \times n$ symmetric polynomial matrix $G(s)$ is sum-of-squares if and only if it can be represented as $G(s) = Z_{n,d}^T(s)\, J\, Z_{n,d}(s)$ for some positive semidefinite matrix J. This is a convex constraint on the coefficients of G. We will denote the sum-of-squares constraint on the coefficients of a polynomial by

$$G \in \Sigma_{n,d} := \{G : \mathbb{R} \to \mathbb{S}^n | G(s) = Z_{n,d}^T(s)\, J\, Z_{n,d}(s), J \geq 0\}.$$

That a polynomial be sum-of-squares is sufficient for positivity. However, it is necessary only in special cases. Furthermore, positivity over compact semialgebraic subsets can be tested through a combination of sum-of-squares and Positivstellensatz results in a manner akin to the S-procedure. See [19] for details. Some early work on matrix sum-of-squares can be found in [26]. Through considerable abuse of notation, for a semialgebraic set H, we will allow "$G(s) \in \Sigma_{n,d}$ for $s \in H$" to denote conditions derived using a Positivstellensatz. Because of the variety and complexity of Positivstellensatz results, we do not list the conditions explicitly.

III. POSITIVITY CONDITIONS

The purpose of this paper is to use polynomial computing to construct a matrix P and continuous matrix-valued functions $Q_i(s)$, $S_i(s)$, and $R_{ij}(s, \eta)$ such that the conditions of Theorem 1 hold.

Now we consider the quadratic Lyapunov-Krasovskii functional given by Equation (4). In order to apply polynomial computing to positivity of this functional, we need to convert the functional positivity condition to conditions which can be enforced using SDP. To this end, we would like to apply Theorems 5 and 7 of [21]. Unfortunately, however, these theorems cannot be directly applied because the structure of the Lyapunov-Krasovskii functional given by Equation (4) is different from the structure of the functional in [21]. The first main technical result of this paper shows how Theorem 5 of [21] can be modified to take into account the structure of the functional (4).

Proposition 2: Suppose we are given a matrix $P \in \mathbb{S}^n$ and continuous matrix-valued functions $Q_i : [-r_i, 0] \to \mathbb{R}^{n \times m_i}$ and $S_i : [-r_i, 0] \to \mathbb{S}^{m_i}$, $i = 1, 2, \ldots, K$. Then the following two statements are equivalent.

1) There exists an $\varepsilon > 0$ such that

$$\begin{aligned} &\psi^T P \psi + 2\, \psi^T \sum_{i=1}^{K} \int_{-r_i}^{0} Q_i(s)\, \phi_i(s)\, ds \\ &\quad + \sum_{i=1}^{K} \int_{-r_i}^{0} \phi_i^T(s)\, S_i(s)\, \phi_i(s)\, ds \geq \varepsilon \left(\|\psi\|^2 + \|\phi\|_{L_2}^2 \right) \end{aligned}$$

for all $\psi \in \mathbb{R}^n$ and continuous $\phi = (\phi_1, \phi_2, \ldots, \phi_K)$.

2) There exist continuous functions $T_i : [-r_i, 0] \to \mathbb{S}^n$ and an $\epsilon > 0$ such that

$$\begin{bmatrix} \frac{1}{\sum_{i=1}^{K} r_i} P & Q_i(s) \\ Q_i^T(s) & S_i(s) \end{bmatrix} + \begin{bmatrix} T_i(s) & 0 \\ 0 & 0 \end{bmatrix} \geq \epsilon I \qquad for\ all\ s \in [-r_i, 0],\ i = 1, 2, \ldots, K,$$

$$\sum_{i=1}^{K} \int_{-r_i}^{0} T_i(s)\, ds = 0.$$

Proof: We begin by introducing the changes of variables $\eta_i = s/r_i$. Then the left-hand side of the inequality of 1) is given by

$$V_1(\psi, \phi) := \int_{-1}^{0} \Phi^T(\eta)\, M(\eta)\, \Phi(\eta)\, d\eta,$$

where we relabeled all the $\eta_i \to \eta$ and have defined

$$\begin{aligned} \Phi(\eta) :\; &=\; \left[\psi^T \;\; \phi_1^T(r_1 \eta) \;\; \cdots \;\; \phi_K^T(r_K \eta) \right]^T, \\ M(\eta) :\; &=\; \begin{bmatrix} P & \bar{Q}(\eta) \\ \bar{Q}^T(\eta) & \bar{S}(\eta) \end{bmatrix}, \end{aligned}$$

$$\bar{Q}(\eta) : = \begin{bmatrix} r_1 Q_1(r_1\eta) & r_2 Q_2(r_2\eta) & \cdots & r_K Q_K(r_K\eta) \end{bmatrix},$$
$$\bar{S}(\eta) : = \operatorname{diag}\left(r_1 S_1(r_1\eta) \;\; r_2 S_2(r_2\eta) \;\; \cdots \;\; r_K S_K(r_K\eta) \right).$$

By Theorem 5 of [21], $V_1(\psi,\phi) \geq \varepsilon(\|\psi\|^2 + \|\phi\|^2_{L_2})$ is equivalent to the existence of an $\epsilon > 0$ and continuous function $Z(s) : [-1,0] \to \mathbb{S}^n$ such that

$$M(\eta) + \begin{bmatrix} Z(\eta) & 0 \\ 0 & 0 \end{bmatrix} \geq \epsilon I \quad \text{for } \eta \in [-1,0]$$

where $\int_{-1}^{0} Z(\eta)\, d\eta = 0$. Now, by applying Proposition 2 of [5] inductively, positivity of $M(\eta) + \begin{bmatrix} Z(\eta) & 0 \\ 0 & 0 \end{bmatrix}$ is equivalent to the existence of an $\epsilon > 0$ and some continuous functions L_i such that

$$\begin{bmatrix} \frac{r_1}{\sum_{i=1}^K r_i} P + Z(\eta) - \sum_{i=2}^K L_i(\eta) & r_1 Q_1(r_1\eta) \\ r_1 Q_1^T(r_1\eta) & r_1 S_1(r_1\eta) \end{bmatrix} \geq \epsilon I$$

and

$$\begin{bmatrix} \frac{r_i}{\sum_{i=1}^K r_i} P + L_i(\eta) & r_i Q_i(r_i\eta) \\ r_i Q_i^T(r_i\eta) & r_i S_i(r_i\eta) \end{bmatrix} \geq \epsilon I$$

for $\eta \in [-1,0]$ and $i = 2,\ldots,K$. Now, if we define $T_i(\eta) = \frac{1}{r_i} L_i(\eta/r_i)$ for $i = 2,\ldots,K$ and

$$T_1(\eta) = \frac{1}{r_1}\left(Z(\eta/r_1) - \sum_{i=2}^K L_i(\eta/r_1) \right),$$

then by changing variables back to $s_i = \eta/r_i$

$$\sum_{i=1}^K \int_{-r_i}^0 T_i(\eta)\, d\eta \;=\; \int_{-1}^0 \left(Z(s_1) - \sum_{i=2}^K L_i(s_1) \right) ds_1 + \sum_{i=2}^K \int_{-1}^0 L_i(s_i)\, ds_i = 0.$$

Furthermore, by the same change of variables, the inequalities become

$$r_i \begin{bmatrix} \frac{1}{\sum_{i=1}^K r_i} P + T_i(s_i) & Q_i(s_i) \\ Q_i^T(s_i) & S_i(s_i) \end{bmatrix} > 0, \quad \text{for } s_i \in [-r_i,0] \text{ and } i = 1,2,\ldots,K,$$

which is equivalent to Statement 2) of the proposition. Thus we have that 1) $\Rightarrow$ 2). Furthermore, if we start by letting $L_i = r_i T_i(r_i\eta)$, then all steps can be reversed to show 2) $\Rightarrow$ 1). ∎

To represent the conditions of Proposition 2 using SDP, we can require our functions to be polynomial and strengthen the positivity conditions "> 0" by using the Sum-of-Squares condition "$\in \Sigma_{m,d}$" for some $m, d > 0$. Such conditions may be conservative due to the choice of d, but can be enforced using SDP - the critical point. We now present the second significant result of this paper, which shows how the concepts in Theorem 7 of [21] can be applied to our new Lyapunov-Krasovskii functional.

Proposition 3: Let $m = \sum_{j=1}^K m_j$. Suppose $R_{ij} : [-r_i,0] \times [-r_j,0] \to \mathbb{R}^{m_i \times m_j}$ are polynomial matrices of degree d and $R_{ij}(s,\eta) = R_{ji}^T(\eta,s)$ for $i,j = 1,2,\ldots,K$. Then

$$V_2(\phi) := \sum_{i=1}^K \sum_{j=1}^K \int_{-r_i}^0 \int_{-r_j}^0 \phi_i^T(s)\, R_{ij}(s,\eta)\, \phi_j(\eta)\, d\eta\, ds \geq 0$$

for all $\phi = (\phi_1,\phi_2,\ldots,\phi_K) \in \mathcal{PC}$ if and only if $R \in \Gamma_{m,d}$, where

$$R(s,\eta) = \begin{bmatrix} R_{11}(r_1 s, r_1\eta) & \cdots & R_{1K}(r_1 s, r_K\eta) \\ \vdots & \ddots & \vdots \\ R_{K1}(r_K s, r_1\eta) & \cdots & R_{KK}(r_K s, r_K\eta) \end{bmatrix},$$

and

$$\Gamma_{m,d} = \left\{ Z_{m,d}^T(s)\, L\, Z_{m,d}(\eta) \,\middle|\, L \in \mathbb{S}^{m(d+1)},\, L \geq 0 \right\}.$$

Proof: Introduce the following new variables: $s = r_i\omega_i$, $\eta = r_j\theta_j$, $i,j = 1,2,\ldots,K$. Then, $V_2(\phi)$ may be expressed as

$$\begin{aligned} V_2(\phi) \;&=\; \sum_{i=1}^K \sum_{j=1}^K r_i r_j \int_{-1}^0 \int_{-1}^0 \phi_i^T(r_i\omega_i)\, R_{ij}(r_i\omega_i, r_j\theta_j) \\ &\quad \times \phi_j(r_j\theta_j)\, d\omega_i\, d\theta_j \\ &= \int_{-1}^0 \int_{-1}^0 \Phi^T(s)\, R(s,\eta)\, \Phi(\eta)\, d\eta\, ds, \end{aligned}$$

where we have unified the variables and where

$$\Phi(s) = \begin{bmatrix} r_1\phi_1^T(r_1 s) & r_2\phi_2^T(r_2 s) & \cdots & r_K\phi_K^T(r_K s) \end{bmatrix}^T.$$

Now Theorem 7 of [21] states that since R is a polynomial of degree d, positivity of V_2 is equivalent to $R \in \Gamma_{m,d}$. ∎

Note that the conditions of Proposition 3 are SDP constraints on the coefficients of the polynomials R_{ij}. Therefore, in combination with Proposition 2, this result forms the basis of a Sum-of-Squares/SDP approach to optimization of Lyapunov-Krasovskii functionals of the form given by Equation (4). This is expanded upon in the following two sections.

IV. The Lyapunov-Krasovskii Derivative Condition

In this section, we obtain a condition for negativity of the derivative of the Lyapunov-Krasovskii functional presented as (4). Through some manipulation, it can be shown that this derivative may be expressed as follows.

$$\begin{aligned} \dot{V}(\psi,\phi) \;&=\; \sum_{i=1}^K \int_{-r_i}^0 z_i^T(s)\, F_i(s)\, z_i(s)\, ds \\ &\quad + \sum_{i=1}^K \sum_{j=1}^K \int_{-r_i}^0 \int_{-r_j}^0 \phi_i^T(s) E_{ij}(s,\eta) \phi_j(\eta) d\eta ds, \end{aligned} \tag{5}$$

where

$$z_i(s) = \begin{bmatrix} \psi^T & \phi_1^T(-r_1) & \cdots & \phi_K^T(-r_K) & \phi_i^T(s) \end{bmatrix}^T,$$

and

$$F_i(s) = \begin{bmatrix} \frac{1}{\sum_{j=1}^K r_j} F_{11} & \frac{1}{\sum_{j=1}^K r_j} F_{12} & F_{13i}(s) \\ *^T & \frac{1}{\sum_{j=1}^K r_j} F_{22} & F_{23i}(s) \\ *^T & *^T(s) & F_{33i}(s) \end{bmatrix},$$

$$E_{ij}(s,\eta) = -\frac{\partial R_{ij}(s,\eta)}{\partial s} - \frac{\partial R_{ij}(s,\eta)}{\partial \eta},$$

$$F_{11} = \sum_{j=1}^K \left[Q_j(0)\, C_j + C_j^T\, Q_j^T(0) + C_j^T\, S_j(0)\, C_j \right] + PA + A^T P,$$

$$F_{12} = \begin{bmatrix} G_1 & \cdots & G_K \end{bmatrix},$$

$$G_j = \sum_{k=1}^K [\, Q_k(0)\, D_{kj} + C_k^T\, S_k(0)\, D_{kj} \,] + PB_j - Q_j(-r_j) \quad \text{for } j = 1,2,\ldots,K,$$

$$F_{22} = \sum_{i=1}^{K} \begin{bmatrix} D_{i1}^T S_i(0) D_{i1} & \cdots & D_{i1}^T S_i(0) D_{iK} \\ \vdots & \ddots & \vdots \\ D_{iK}^T S_i(0) D_{i1} & \cdots & D_{iK}^T S_i(0) D_{iK} \end{bmatrix} - \operatorname{diag}\left(S_1(-r_1) \;\; S_2(-r_2) \;\; \cdots \;\; S_K(-r_K) \right),$$

$$F_{13i}(s) = A^T Q_i(s) + \sum_{j=1}^{K} C_j^T R_{ij}^T(s,0) - \frac{dQ_i(s)}{ds},$$

$$F_{23i}(s) = \left[H_{i1}^T(s) \;\; \cdots \;\; H_{iK}^T(s) \right]^T,$$

$$H_{ij}(s) = B_j^T Q_i(s) + \sum_{k=1}^{K} D_{kj}^T R_{ik}^T(s,0) - R_{ij}^T(s,-r_j) \quad \text{for } j = 1,2,\ldots,K,$$

$$F_{33i}(s) = -\frac{dS_i(s)}{ds}.$$

Notice that $\dot V$ has the same form as V. Therefore, Propositions 2 and 3 may be used to obtain conditions for negativity of $\dot V$ that are suitable for implementation via polynomial optimization. This is summarized in the following proposition.

Proposition 4: $\dot V(\psi,\phi) \le -\epsilon \|\psi\|^2$ for some $\epsilon > 0$ if there exist matrix functions $W_i : [-r_i, 0] \to \mathbb{S}^{n+m}$, $i = 1,2,\ldots,K$, with $m = \sum_{j=1}^{K} m_j$, such that

$$-F_i(s) + \begin{bmatrix} W_i(s) & 0 \\ 0 & 0 \end{bmatrix} \in \Sigma_{n+m+m_i,d} \quad \text{for } s \in [-r_i,0] \text{ and for } i = 1,2,\ldots,K,$$

$$\sum_{i=1}^{K} \int_{-r_i}^{0} W_i(s)\, ds = 0,$$

and $-E \in \Gamma_{m,d}$, where

$$E(s,\eta) = \begin{bmatrix} E_{11}(r_1 s, r_1 \eta) & \cdots & E_{1K}(r_1 s, r_K \eta) \\ \vdots & \ddots & \vdots \\ E_{K1}(r_K s, r_1 \eta) & \cdots & E_{KK}(r_K s, r_K \eta) \end{bmatrix}.$$

Proof: The negativity condition may be written as

$$-\dot V(\psi,\phi) = -V_{d1}(\psi,\phi) - V_{d2}(\psi,\phi) \ge \epsilon \|\psi\|^2,$$

where V_{d1} and V_{d2} are the first and second parts of Equation (5). If the conditions of the proposition are satisfied, then Proposition 2 implies that there exists some $\epsilon > 0$ such that $-V_{d1}(\psi,\phi) \ge \epsilon \|\psi\|^2$. Likewise, Proposition 3 may be used to show that $-V_{d2}(\psi,\phi) \ge 0$. Thus the negativity condition is satisfied. ■

Note that if Q_i, S_i, and R_{ij} are polynomials, then the map from the coefficients of these polynomials to those of E_{ij} and F_i will be linear, easily represented using packages such as SOSTOOLS [24].

V. Stability Conditions

In this section, we summarize the results of this paper by giving conditions for stability in a form which can be implemented using a combination of Sum-of-Squares and SDP.

Theorem 5: Let $m = \sum_{j=1}^{K} m_j$. The coupled delay-differential system described by Equations (1) and (2) is asymptotically stable if there exist a matrix $P \in \mathbb{S}^n$, and polynomial matrices $Q_i : [-r_i, 0] \to \mathbb{R}^{n \times m_i}$, $S_i : [-r_i, 0] \to \mathbb{S}^{m_i}$, $T_i : [-r_i, 0] \to \mathbb{S}^n$, $R_{ij} : [-r_i, 0] \times [-r_j, 0] \to \mathbb{R}^{m_i \times m_j}$, $W_i : [-r_i, 0] \to \mathbb{S}^{n+m}$, $i,j = 1,2,\ldots,K$ such that

$$\begin{bmatrix} \frac{1}{\sum_{i=1}^{K} r_i} P & Q_i(s) \\ Q_i^T(s) & S_i(s) \end{bmatrix} + \begin{bmatrix} T_i(s) & 0 \\ 0 & 0 \end{bmatrix} \in \Sigma_{n+m_i,d} \quad \text{for } s \in [-r_i,0] \text{ and } i = 1,2,\ldots,K,$$

$$-F_i(s) + \begin{bmatrix} W_i(s) & 0 \\ 0 & 0 \end{bmatrix} \in \Sigma_{n+m+m_i,d} \quad \text{for } s \in [-r_i,0] \text{ and } i = 1,2,\ldots,K,$$

$$\sum_{i=1}^{K} \int_{-r_i}^{0} T_i(s)\, ds = 0,$$

$$\sum_{i=1}^{K} \int_{-r_i}^{0} W_i(s)\, ds = 0,$$

$$R \in \Gamma_{m,d},$$

$$-E \in \Gamma_{m,d},$$

where R and E are the composite matrix functions defined by the blocks R_{ij} and E_{ij}, respectively and where the functions F_i and E_{ij} are as defined in the previous section.

Proof: We first show that the conditions of the theorem imply that the system satisfies the regularity conditions of the Lyapunov theorem. That is, the condition (3) is satisfied. Observe that the second inequality and the definition of F implies that $-F_{33i}(s) = \frac{\partial S_i(s)}{\partial s} > 0$, which in turn implies that $S_i(0) \ge S_i(-r_i)$ for all $i = 1,2,\ldots,K$. Now define $L_i = S_i(0)$, as per the condition (3). Then we have that the corresponding $U = \operatorname{diag}\left(S_1(0) \;\; S_2(0) \;\; \cdots \;\; S_K(0) \right)$. Now, it can be shown that the conditions of the theorem imply that $F_{22} < 0$. Therefore by definition

$$\begin{aligned} U - D^T U D &\ge \operatorname{diag}\left(S_1(-r_1) \;\; S_2(-r_2) \;\; \cdots \;\; S_K(-r_K) \right) \\ &\quad - D^T U D \\ &= -F_{22} > 0, \end{aligned}$$

where the first inequality holds because $S_i(0) \ge S_i(-r_i)$. Finally, since the first inequality condition of the theorem implies $S_i(s) > 0$ for all s, we have that $U > 0$, which means that condition (3) is satisfied. Therefore, the technical conditions of Theorem 1 are satisfied and stability can be established by positivity of the relevant Lyapunov function and negativity of its derivative. As discussed previously, positivity of the functional is established via Propositions 2 and 3. Negativity of the derivative is established in the same manner. ■

The result in Theorem 5 can be extended to uncertain systems and time-varying delay systems. In particular the uncertain parameters in the uncertain system matrices or time-varying delays can be included in the variables of polynomial matrices of polynomial Lyapunov-Krasovskii functionals, as given in [16].

VI. Discussions and Examples

A. Complexity Discussion

We first compare the computational complexity of the conditions associated with Theorem 5 and the computational complexity of the conditions associated with Theorem 11 of [21]. Consider the case of a single delay. The number of decision variables in the SDP associated with Theorem 5 is of the order $O(n,m,d) = (n+md)^2$, where n is the dimension of $x(t)$, d is the degree of the polynomials, and m is the dimension of $y(t)$. For the SDP problem associated with the previous formulation described in [21], the number of decision variables is of the order $(n(d+1))^2$. For a large d, it can be estimated that the new method offers a reduction in the number of decision variables of order $O\left(\frac{m}{n}\right)^2$. For example, when $n = 6$ and $m = 1$, the number of decision variables is reduced by 97.3%. Since the worst-case complexity of SDP is roughly proportional to q^3 for Cholesky factorization, where q is the number of decision variables, for the case of $n = 6$ and $m = 1$, we have a complexity reduction of approximately 10^{-5} or five orders of magnitude. Note that as $\lim \frac{m}{n} \to 0$, the complexity of the SDP test approached the complexity of solving the Lyapunov inequality for an undelayed system.

B. Examples

In this section, we give two numerical examples. All calculation has been implemented by using MATLAB 7 with SOSTOOLS and LMI Lab on a laptop PC with Intel Duo Core Processor T2250 (1.73GHz) and 1.5 GB RAM.

1) Example 1: We first consider the case of the system with delayed state feedback of the form

$$\dot{x}(t) = \begin{bmatrix} 0 & .5 & 0 & 0 & 0 & 0 \\ -.5 & -.5 & 0 & 0 & 0 & 0 \\ 0 & 1 & .1 & 1 & 0 & 0 \\ 0 & 0 & -2 & .2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -.9 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ -.5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_1(t - \frac{r}{\sqrt{2}}) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -2 & 0 \\ -1 & -1.45 \end{bmatrix} u_2(t-r)$$

where the state-feedback controller is

$$u_1(t) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} x(t),$$
$$u_2(t) = \begin{bmatrix} -.2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} x(t).$$

The closed-loop system has 6 states and two delay channels with a combined dimension of 3. It can be verified by calculating imaginary poles that the system is asymptotically stable for $r \in [r_{\min}, r_{\max}]$, where $r_{\min} = 0.64963$ and $r_{\max} = 1.75515$. In Fig. 1, we use a log-log plot to illustrate the accuracy of the stability test for this example vs. the computation time required to run the algorithm using SeDuMi for several different values of the degree. The computational time only includes the time taken to run the SDP. Time taken to define the SDP is not included. The largest stable delay $\hat{r}_M$ for a given degree is calculated using a bisection method and compared to the analytical value. This value is defined as the accuracy of the algorithm. The results are compared to another asymptotically exact test - the discretized functional method [11] - which was also adapted to the new coupled differential-difference formulation to reduce computational cost. Also included is the previous version of the SOS test [21]. Ideally, many other stability results would be included in this plot. However, because tests by other authors in the literature did not use the same experimental regime as our tests (different equipment, etc.), it would not be appropriate to include them. However, if it were possible, some interesting points of comparison would include [3] and [13].

The numerical results of Fig. 1 indicate an approximately 2 orders of magnitude reduction in computational complexity compared to the old SOS formulation. This is roughly in-line with the theoretical complexity analysis. Perhaps the most interesting feature of Fig. 1 is that both asymptotically exact methods using the coupled differential-difference formulation have roughly the same convergence rate.

2) Example 2: For the second example, we consider the system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{305}{256} & -\frac{7}{8} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{5} \end{bmatrix} y_1(t - r_1) + \begin{bmatrix} 0 \\ -\frac{4}{5} \end{bmatrix} y_2(t - r_2),$$
$$y_1(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t),$$
$$y_2(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t),$$

which is equivalent to the system discussed in the example of [13], where a stability test based on pseudo-delay substitution and positive polynomials was used. For different ratios 2 and 1/3 of r_2/r_1, the analytical solutions $r_{2\max}$ of the maximum allowable r_2 are 7.5278 and 1.3213, respectively. The estimate of $r_{2\max}$ is denoted as $\hat{r}_{2M}$. A bisection process is used and an initial interval containing $\hat{r}_{2M}$ of length 2 is subdivided 16 times. Our algorithm is also compared to the methods in [11] and [13]. The results are listed in Table I.

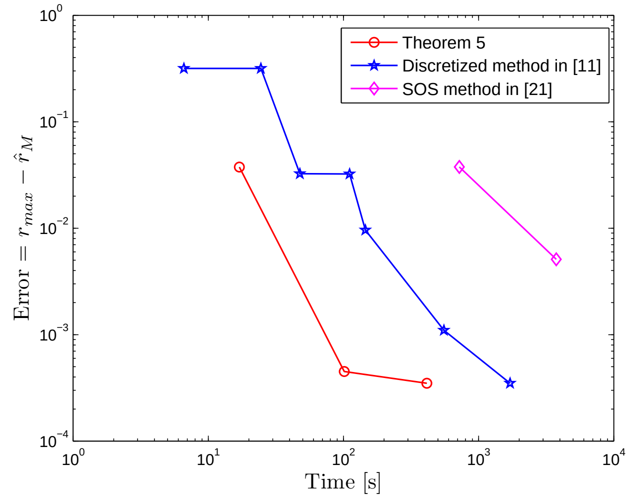

Fig. 1. Log-Log plot of accuracy vs. computation time: From left, the '∘'-points denote the cases of monomial degree $d = 1, 2, 3$; the '⋆'-points denote the cases with gridding $N = [1, 1], [1, 2], [2, 2], [2, 3], [3, 3], [4, 4], [5, 5]$; the '⋄'-points denote the cases with monomial degree $d = 1, 2$.

TABLE I
COMPARISON OF DIFFERENT STABILITY CONDITIONS FOR EXAMPLE 2

r_2/r_1	Methods	$\hat{r}_{2M}/r_{2\max}$ [%]	Time [s]
2	Theorem 5 ($d = 2$)	98.82	143.6
	[11] ($N = [2, 2]$)	97.66	248.2
	[13] (LP)	27.83	315.0
	[13] (SOS)	26.07	915.9
1/3	Theorem 5 ($d = 2$)	99.29	130.2
	[11] ($N = [3, 1]$)	99.82	134.7
	[13] (LP)	52.86	315.0
	[13] (SOS)	49.51	915.9

In the table, d is the monomial degree, N is the gridding of discretized Lyapunov functional method, and LP is linear programming. The computation time includes the time taken to define the problem associated with SOS and LMI.

VII. CONCLUSION

In this paper, it was shown that the complexity of the Sum-of-Squares/SDP conditions for stability analysis of linear time-delay systems can be reduced by several orders of magnitude using a coupled differential-difference formulation. The reduction of complexity is particularly large in the case where the delayed system is high-dimensional with relatively few delays in relatively few channels—a situation which arises commonly in practice.

REFERENCES

[1] P. Agathoklis and S. Foda, "Stability and the matrix Lyapunov equation for delay differential systems," *Int. J. Control,* vol. 49, no. 2, pp. 417–432, 1989.

[2] E. Fridman, "Stability of linear descriptor systems with delay: a Lyapunov-based approach," *J. Math. Anal. Applicat.* vol. 273, no. 1, pp. 24–44, 2002.

[3] F. Gouaisbaut and D. Peaucelle, “Delay-dependent stability analysis of linear time delay systems,” in The sixth IFAC workshop on time-delay systems, L’Aquila, Italy, July, 2006.
[4] K. Gu, “Discretized LMI set in the stability problem of linear uncertain time-delay systems,” *Int. J. Control,* vol. 68, no. 4, pp. 923–934, 1997.
[5] K. Gu, “A further refinement of discretized Lyapunov functional method for the stability of time-delay systems,” *Int. J. Control,* vol. 74, no. 10, pp. 967–976, 2001.
[6] K. Gu, V. L. Kharitonov, and J. Chen, *Stability of Time-Delay Systems*. Boston, MA: Birkhäuser, 2003.
[7] K. Gu, “Stability problem of systems with multiple delay channels,” *Automatica,* vol. 46, no. 4, pp. 743–751, 2010.
[8] K. Gu and Y. Liu, “Lyapunov-Krasovskii functional for uniform stability of coupled differential-functional equations,” *Automatica,* vol. 45, no. 3, pp. 798–804, 2009.
[9] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations*. New York: Springer-Verlag, 1993.
[10] V. Kolmanovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*. Dordrecht, The Netherlands: Kluwer Academic Publishers, 1999.
[11] H. Li and K. Gu, “Discretized Lyapunov-Krasovskii functional for coupled differential-difference equations with multiple delay channels,” *Automatica,* vol. 46, no. 5, pp. 902–909, 2010.
[12] J. J. Loiseau and D. Brethé, “The use of 2-D systems theory for the control of time-delay systems,” *JESA, European Journal of Automatic Systems*, vol. 31, no. 6, pp. 1043–1058, 1997.
[13] U. Münz, C. Ebenbauer, T. Haag, and F. Allgöwer, “Stability analysis of time-delay systems with incommensurate delays using positive polynomials,” *IEEE Trans. Automat. Control,* vol. 54, no. 5, pp. 1019–1024, 2009.
[14] Y. Nesterov, “Squared functional systems and optimization problems,” in *High Performance Optimization Methods,* H. Frenk, K. Roos, T. Terlaky, and S. Zhang, Eds. Dordrecht, The Netherlands: Kluwer Academic Publishers, 2000, ch. 17, pp. 405–440.
[15] S.-I. Niculescu, *Delay Effects on Stability: A Robust Control Approach*, ser. Lecture Notes in Control and Information Science. Heidelberg, Germany: Springer-Verlag, 2001, vol. 269.
[16] A. Papachristodoulou, M. M. Peet, and S.-I Niculescu, “Stability analysis of linear systems with time-varying delays: delay uncertainty and quenching,” in *Proc. IEEE Conf. Decision Control,* New Orleans, LA, USA, Dec. 2007, pp. 2117–2122.
[17] A. Papachristodoulou, M. M. Peet, and S. Lall. Analysis of polynomial systems with time delays via the sum of squares decomposition. *IEEE Trans. Automat. Control*, vol. 54, no. 5, pp. 1058–1064, 2009.
[18] P. A. Parrilo, “Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization,” Ph.D. dissertation, California Institute of Technology, Pasadena, CA, 2000.
[19] M. M. Peet. “Stability and control of functional differential equations,” Ph.D. dissertation, Stanford University, Stanford, CA, 2006.
[20] M. M. Peet and A. Papachristodoulou. “Positive forms and stability of linear time-delay systems,” in *Proc. IEEE Conf. Decision Control,* San Diego, CA, USA, Dec. 2006, pp. 187–193.
[21] M. M. Peet, A. Papachristodoulou, and S. Lall. “Positive forms and stability of linear time-delay systems,” *SIAM J. Control Optim.,* vol. 47, no. 6, pp. 3237–3258, 2009.
[22] P. Pepe and E. I. Verriest, “On the stability of coupled delay differential and continuous time difference equations,” *IEEE Trans. Automat. Control,* vol. 48, no. 8, pp. 1422–1427, 2003.
[23] P. Pepe, Z.-P Jiang, and E. Fridman, “A new Lyapunov-Krasovskii methodology for coupled delay differential and difference equations,” *Int. J. Control,* vol. 81, no. 1, pp. 107–115, 2007.
[24] S. Prajna, A. Papachristodoulou, and P. A. Parrilo, “Introducing SOSTOOLS: a general purpose sum of squares programming solver,” in *Proc. IEEE Conf. Decision Control,* Las Vegas, USA, Dec. 2002, pp. 741–746.
[25] V. Răsvan and S.-I Niculescu, “Oscillations in lossless propagation models: a Lyapunov-Krasovskii approach,” *IMA J. Math. Control Inform.,* vol. 19, no. 1-2, pp. 157–172, 2002.
[26] C. W. Scherer and C. W. J. Hol, “Matrix sums-of-squares relaxations for robust semi-definite programs,” *Mathematical Programming, Series B,* vol. 107, no. 1-2, pp. 189–211, 2006.