Stability Analysis of State-Dependent Delay Systems using Sum-of-Squares

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This paper develops an algorithm that can directly test the stability of the state-dependent delay system. Firstly, we construct a quadratic Lyapunov-Krasovskii functional for the system and deduct it's derivative along the system trajectory. Then we obtain a series LMIs by applying s-procedure on the Lyapunov-Krasovskii functional and it's derivative. Particularly, the stability of the system can be verified by examining the feasibility of these LMIs using semidefinite programming solver.

Nomenclature

$\mathbb{R}$	Set of the Real Numbers
$\mathbb{R}^+$	Set of the Positive Real Numbers
$\mathbb{R}^-$	Set of the Negative Real Numbers
$x(t)$	Solution of the State-Dependent Delay System, $x(t) \in R$
r_0	Maximum Value of the Delay Time
$\mathbb{C}$	Set of the Functions Mapping $[-r_0, 0]$ into $\mathbb{R}$
x_t	State of the State-Dependent Delay System Defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r_0, 0]$, $x_t \in \mathbb{C}$

I. INTRODUCTION

In this paper, we consider the local stability of following scalar state-dependent delay system:

$$\dot{x}(t) = ax(t - \tau(x(t))) \tag{1}$$

with state-dependent delay

$$\tau(x(t)) = \tau_0 + bx(t) \tag{2}$$

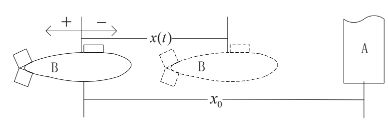

Figure 1. Submarine System

where $x(t) \in \mathbb{R}, a \in \mathbb{R}^-, \tau_0, b \in \mathbb{R}^+$. In order to ensure $\tau(x(t)) \geq 0$, we require $|x(t)| \leq \tau_0/(2b)$.

State-dependent delay exists in many practical systems, such as biological systems[1] and turning process.[2] Here we provide a specific example of state-dependent delay system, which is a submarine system briefly shown as Fig. 1. In this figure, $x_0 \in \mathbb{R}^+$ is the equilibrium distance between submarine B and rock A, and $x(t) \in \mathbb{R}$ is the displacement B deviates from the equilibrium position at time t. In order to fix B at the equilibrium position, the velocity of B is governed by the ordinary differential equation $\dot{x}(t) = ax(t)$ where $a \in \mathbb{R}^-$.

Generally speaking, B obtains its displacement information by measuring the time interval between the signal sent out by B and the signal reflected by A. More specifically, this time interval is determined by the equation $\tau(x(t)) = (2x_0 + 2x(t))/c$ under the assumption that the signal transmission speed is much faster than submarine speed, where c is the speed of signal transmission. In other words, there is a $\tau(x(t))$ time delay for B to acquire its displacement information. If we set $2x_0/c = \tau_0$ and set $2/c = b$, this yields $\tau(x(t)) = \tau_0 + bx(t)$. In this case, the velocity of B is strictly governed by the equation $\dot{x}(t) = ax(t - \tau(x(t)))$.

A number of studies have been done related to the state-dependent delay system. For example, Gyori concludes that under certain conditions the exponential stability of a state-dependent delay system is equal to the exponential stability of a corresponding fixed-time delay system,[3] Mallet-Paret investigates slowly oscillating periodic solutions of a singularly perturbed state-dependent delay equation,[4] Verriest derives a Riccati equation to verify the stability of the state-dependent delay system,[5] Humphries analyzes the bifurcation structures in the dynamics of the scalar state-dependent delay system.[6] However, there are very few published works on the stability analysis of the state-dependent delay system via LMI approach.

Motivated by above discussion, we propose an algorithm which can directly test the stability of the state-dependent delay system in this paper. Concretely speaking, the stability of the system is verified by testing the existence of the complete quadratic Lyapunov-Krasovskii functional that satisfies the Lyapunov-Krasovskii theorem. As is known to all, Lyapunov-Krasovskii theorem has been widely used for analyzing the stability of time-delay systems. Many useful results about this can be found in the book of Gu.[7] Regarding to verify the existence of the quadratic Lyapunov-Krasovskii functional, several methods, such as the discretized Lyapunov functional method[7] and Sum-of-Squares method, can be employed. Refer to the paper of Parrilo,[8] Lasere[9] and Chesi[10] for a detailed description of Sum-of-Squares Method. More specific studies about using Sum-of-Squares method for solving the stability problem of different types of time delay systems can be found in Peet and Papachristodoulou's papers[11][12].[13]

In our paper, we first construct the Lyapunov functional and derive the formula of it's derivative. Then we apply the s-procedure to the Lyapunov-Krasovskii functional and it's derivative, and derives a series LMIs. Especially, the feasibility of these LMIs is related to the stability of the system. That is to say if the LMIs is feasible, the existence of the Lyapunov-Krasovskii functional which satisfies the Lyapunov-Krasovskii theorem will be guaranteed and the state-dependent delay system will be stable. Finally, the validity of our algorithm is examined by MATLAB. We use SOSTOOLS[14] to construct the LMIs in MALTAB in this paper, we can also use GloptiPoly[15] or SOSOPT[16] toolboxes instead of SOSTOOLS.

This paper is organized as follows. In section II, we present some background material including the Lyapunov stability, Sum-of-Squares definition and s-procedure. Section III illustrates the detailed procedure of how to obtain the LMIs which are the stability conditions for the state-dependent delay system, this is the main results of our paper. We also present the numerical and simulation results of the stability of system (1) in this section. Section IV is the conclusions and future works worth to be done.

II. BACKGROUND

In this section, we briefly introduce the Lyapunov stability, Sum-of-Squares polynomial and s-procedure which are the essential theories for the main results of the paper.

II.A. Lyapunov Stablility

Definition 1(Stability Definition)[7]: Consider the state-dependent delay system

$$\dot{x}(t) = ax(t - \tau(x(t))), \tag{1}$$

the system is *stable* if given any $\epsilon > 0$, there exists a $\delta > 0$, such that for all initial conditions $\|\phi(t)\| \leq \delta$, $t \in [-r_0, 0]$, we have $\|x(t)\| \leq \epsilon$ for all $t > 0$. Furthermore, the system (1) is said to be *asymptotically stable* if there exists a $\delta_1 > 0$, such that for initial condition $\|\phi(t)\| \leq \delta_1$, $t \in [-r_0, 0]$, we have $\lim_{t\to+\infty} \|x(t)\| = 0$.

Sometimes, we concern the local stability of the system, in this case for a given positive real number r, the state is constrained in region Ω defined as[12]

$$\Omega := \{x_t \in \mathbb{C} : \|x_t\| \leq r\} \tag{3}$$

where

$$\|x_t\| = sup_{\theta\in[-r_0,0]} \|x(t + \theta)\|.$$

Theorem 1(Lyapunov-Krasovskii Theorem)[7]: The state-dependent delay system (1) is *asymptotically stable* if there exists a quadratic *Lyapunov-Krasovskii functional* $V(\phi)$ and some positive constants ϵ_1, ϵ_2 such that for all $t \in [0, +\infty)$

$$V(\phi) \geq \epsilon_1 \phi^T(0)\phi(0), \tag{4}$$

$$\dot{V}(\phi) = \dot{V}(x_t)|_{x_t=\phi} \leq -\epsilon_2 \phi^T(0)\phi(0), \tag{5}$$

here, $\dot{V}(x_t)$ is the derivative of Lyapunov-Krasovskii functional $V(x_t)$ along the trajectory of the state-dependent delay system (1).

II.B. Sum-of-Squares and S-Procedure

Definition 2(Sum-of-Squares Polynomial Definition): For a given multivarite polynomial $p(x), x \in \mathbb{R}^n$, if there exist some finite multivariate polynomials $h_i(x)$, for $i \in [1, 2, ..m]$, such that

$$p(x) = \sum_{i=1}^{m} h_i^2(x), \tag{6}$$

then $p(x)$ is *Sum-of-Squares polynomial*.

From the definition, we can easily infer that a polynomial is non-negative if it's Sum-of-Squares polynomial. We denote Σ as the set of Sum-of-Squares polynomials and Σ_{2d} as the set of Sum-of-Squares polynomials with degree of $2d$, where $d \in N$. We can test if a polynomial belongs to the set of Sum-of-Squares by following theorem:

Theorem 2: A multivariate polynomial $p(x) \in \Sigma_{2d}$ if and only if there exists a symmetric positive semidefinite matrix R such that

$$p(x) = Z_d^T(x) R Z_d(x) \tag{7}$$

where $Z_d(x)$ is a monomial vector in variable of x with degree d or less.

Based on above discussion, given a compact Semialgebraic set

$$Y := \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, 2, .., m\}, \tag{8}$$

we conclude that the main task of verifying the local stability of a system is to find the positive definite lyapunov function $V(x)$ with negative derivative for all $x \in Y$. However, verifying $V(x) \geq 0$ for all $x \in Y$ is an NP hard problem.[17] Notice that if there exists Sum-of-Squares polynomial s_i for all $i = 1, 2, ..., m$, such that

$$V(x) + \sum_{i=1}^{m} s_i(x) g_i(x) > 0, \tag{9}$$

then we can easily infer that $V(x)$ is positive for all $x \in Y$. Thus, we can verify the positivity of a function by testing the feasibility of equation (9) which is a LMI problem. This is called *s-procedure*.

III. MAIN RESULTS

In this section, we first set up the problem statement. Then we construct an appropriate Lyapunov-Krasovskii functional for the state-dependent delay system and derive its derivative. And we obtain the LMIs conditions for the positivity of the Lyapunov-Krasovskii functional and the negativity of it's derivative based on s-procedure. These conditions are the stability criteria for the state-dependent delay system. Finally, we evaluate our method through numerical and simulation results.

III.A. Problem Statement

Recall the scalar state-dependent delay system

$$\dot{x}(t) = a x(t - \tau(x(t))), \tag{1}$$

with state-dependent delay

$$\tau(x(t)) = \tau_0 + b x(t) \tag{10}$$

where $x(t) \in \mathbb{R}, a \in \mathbb{R}^-, \tau_0, b \in \mathbb{R}^+$. As we mentioned before, in order to ensure the positivity of $\tau(x(t))$, we require the solution $x(t)$ belongs to the set X defined by

$$X := \{x(t) \in \mathbb{R}^n : \|x(t)\| \leq \tau_0/(2b)\}.$$

If we choose the initial condition $\delta \leq \tau_0/(2b)$, then the state of the system is constrained in following set

$$\Omega := \{x_t \in \mathbb{C} : \|x_t\| \leq \tau_0/(2b)\}. \tag{11}$$

We will find an algorithm to test the local stability of the state-dependent delay system for all $x_t \in \Omega$.

III.B. Lyapunov-Krasovskii Functional and it's Derivative

We choose following Lyapunov-Krasovskii functional to test the stability of the system:

$$V(x_t) = V_1(x_t) + V_2(x_t). \tag{12}$$

$V_1(x_t)$ and $V_2(x_t)$ are defined as following separately

$$V_1(x_t) = \int_{-\tau(x(t))}^{0} v_1\big(x(t), x(t+\theta), \theta\big) d\theta, \tag{13}$$

$$V_2(x_t) = \int_{-\tau(x(t))}^{0} \int_{-\tau(x(t))}^{0} v_2\big(x(t+\theta), x(t+\xi), \theta, \xi\big) d\theta d\xi \tag{14}$$

where $v_1\big(x(t), x(t+\theta), \theta\big)$ and $v_2\big(x(t+\theta), x(t+\xi), \theta, \xi\big)$ are defined as

$$v_1(x_0, x_2, \theta) = z_d^T(x_0, x_2) P(\theta) z_d(x_0, x_2) \tag{15}$$

$$v_2(x_2, x_3, \theta, \xi) = z_d^T(x_2, \theta) M z_d(x_3, \xi) \tag{16}$$

here, $z_d(x)$ is a vector of monomials of variable x with degrees of 1 to d, $P(\theta)$ is a symmetric polynomial matrix with appropriate dimension and M is a semidefinite symmetric matrix with appropriate dimension.

By using Leibniz rule, we obtain the derivative of $V(x_t)$ along system (1)

$$\dot{V}(x_t) = V_3(x_t) - V_4(x_t) \tag{17}$$

where

$$V_3(x_t) = \int_{-\tau(x(t))}^{0} v_3\big(x(t), x\big(t-\tau(x(t))\big), x(t+\theta), \theta\big) d\theta, \tag{18}$$

$$V_4(x_t) = \int_{-\tau(x(t))}^{0} \int_{-\tau(x(t))}^{0} v_4\big(x(t+\theta), x(t+\xi), \theta, \xi\big) d\theta d\xi. \tag{19}$$

Here, in order to simplify the notation, we will use x_0 for $x(t)$, x_1 for $x(t-\tau(x(t)))$, x_2 for $x(t+\theta)$ and x_3 for $x(t+\xi)$ during the later part of the paper. Based on these notation, v_3 and v_4 are expressed as

$$\begin{aligned} v_3(x_0, x_1, x_2, \theta) = &\frac{1}{\tau(x_0)}(abx_1 - 1) v_1(x_0, x_1, -\tau(x_0)) + \frac{1}{\tau(x_0)} v_1(x_0, x_0, 0) + ax_1 \frac{\partial}{\partial x} v_1(x_0, x_2, \theta) \\ &- \frac{\partial}{\partial \theta} v_1(x_0, x_2, \theta) + (2abx_1 - 2) v_2(x_1, x_2, -\tau(x_0), \theta) + 2v_2(x_0, x_2, 0, \theta) \end{aligned} \tag{20}$$

$$v_4(x_2, x_3, \theta, \xi) = \frac{\partial}{\partial \theta} v_2(x_2, x_3, \theta, \xi) + \frac{\partial}{\partial \xi} v_2(x_2, x_3, \theta, \xi). \tag{21}$$

We have determined the form of the quadratic Lyapunov-Krasovskii functional and it's derivative. If there exists such quadratic Lyapunov-Krasovskii functional that satisfies the Lyapunov-Krasovskii theorem, in other words, the Lypaunov-Krasovskii functional is positive and its derivative is negative, then the system will be stable. So our next mission is to find an algorithm to test the existence of such Lypaunov-Krasovskii functional.

III.C. Positive Conditions of Lyapunov-Krasovskii Functional

The Lyapunov functional is expressed as

$$V(x_t) = V_1(x_t) + V_2(x_t),$$

if both $V_1(x_t)$ and $V_2(x_t)$ are positive, then obviously $V(x_t)$ will be positive.

First, recall that

$$V_2(x_t) = \int_{-\tau(x(t))}^{0} \int_{-\tau(x(t))}^{0} v_2\big(x(t+\theta), x(t+\xi), \theta, \xi\big) d\theta d\xi = \int_{-\tau(x(t))}^{0} \int_{-\tau(x(t))}^{0} z_d^T(x_2, \theta) M z_d(x_3, \xi) d\theta d\xi. \tag{22}$$

Since M is a semidefinite symmetric matrix with appropriate dimension, it can be decomposed as $M = S^T S$. It is easy to show that $V_2(x_t)$ will always be positive by following equation[12]

$$\begin{aligned} V_2(x_t) = &\int_{-\tau(x_0)}^{0} \int_{-\tau(x_0)}^{0} z_d^T(x_2, \theta) M z_d(x_3, \xi) d\theta d\xi = \int_{-\tau(x_0)}^{0} \int_{-\tau(x_0)}^{0} z_d^T(x_2, \theta) S^T S z_d(x_3, \xi) d\theta d\xi \\ &= (\int_{-\tau(x_0)}^{0} S z_d^T(x_2, \theta) d\theta)^T (\int_{-\tau(x_0)}^{0} S z_d^T(x_3, \xi) d\xi) \geq 0. \end{aligned} \tag{23}$$

Next, we introduce following theorem to test the positivity of $V_1(x_t)$.

Theorem 5: Let

$$g_1 = (\theta - \tau(x_0))(\theta + \tau(x_0)),$$

$$g_2 = (x_0 - \tau_0/(2b))(x_0 + \tau_0/(2b)),$$

$$g_3 = (x_2 - \tau_0/(2b))(x_2 + \tau_0/(2b)),$$

if there exist $U_i : \mathbb{R}^3 \to \mathbb{R}$ and $U_i \in \Sigma_{2d}$, for $i = 1, 2, 3$, a polynomial $R_1(x_0, \theta)$ and some $\epsilon_1 > 0$, such that

$$v_1(x_0, x_2, \theta) + R_1(x_0, \theta) - \epsilon_1 x_0^2 + \sum_{i=1}^{3} U_i(x_0, x_2, \theta) g_i \geq 0, \tag{24}$$

$$\int_{-\tau(x_0)}^{0} R_1(x_0, \theta) d\theta = 0, \tag{25}$$

then $V_1(x_t)$ is positive for all $x_t \in \Omega$.

Proof: By s-procedure, it is easy to see that

$$v_1(x_0, x_2, \theta) + R_1(x_0, \theta) \geq \epsilon_1 x_0^2$$

for all $\theta \in [-\tau(x_0), 0], \|x_0\| \leq \tau_0/(2b), \|x_2\| \leq \tau_0/(2b)$.

Integral both sides from $-\tau(x_0)$ to 0 on θ, we have

$$V_1(x_t) + \int_{-\tau(x_0)}^{0} R_1(x_0, \theta) d\theta \geq \tau(x_0)\epsilon_1 x_0^2.$$

Since $x_0 \in X$, we have $\tau(x_0) \geq \epsilon$, ϵ is a small positive real number and together with condition (25) we have

$$V_1(x_t) \geq \tau(x_0)\epsilon_1 x_0^2 \geq \epsilon\epsilon_1 x_0^2$$

for all $x_t \in \Omega$.

III.D. Negative Conditions of the Derivative

By using exactly the same procedure of testing the positivity of $V(x_t)$, we have following theorem to test the negativity of the derivative.

Theorem 6: If there exist $L_i : \mathbb{R}^4 \to \mathbb{R}$ and $L_i \in \Sigma_{2d}$, for $i = 1, 2, 3, 4$, a polynomial $R_2(x_0, x_1, \theta)$, semidefinite symmetric matrix N and some $\epsilon_2 > 0$, such that

$$-v_3(x_0, x_1, x_2, \theta) + R_2(x_0, x_1, \theta) - \epsilon_2 x_0^2 + \sum_{i=1}^{4} L_i(x_0, x_1, x_2, \theta) g_i \in \Sigma, \tag{26}$$

$$v_4(x_2, x_3, \theta, \xi) = z_d^T(x_2, \theta) N z_d(x_3, \xi), \tag{27}$$

$$\int_{-\tau(x_0)}^{0} R_2(x_0, x_1, \theta) d\theta = 0, \tag{28}$$

where g_4 is given by

$$g_4 = (x_1 - \tau_0/(2b))(x_1 + \tau_0/(2b)),$$

then $\dot{V}(x_t)$ is negative for all $x_t \in \Omega$.

Proof: By using s-procedure and following exactly the same procedure as the proof of theorem 5, we have

$$V_3(x_t) \leq -\tau(x_0)\epsilon_2 x_0^2 \leq -\epsilon\epsilon_2 x_0^2.$$

for all $\theta \in [-\tau(x), 0], \|x_0\| \leq \tau_0/(2b), \|x_1\| \leq r, \|x_2\| \leq \tau_0/(2b)$.

From equation (23), we get

$$V_4(x_t) \geq 0,$$

so we have

$$\dot{V}(x_t) = V_3(x_t) - V_4(x_t) \leq -\epsilon\epsilon_2 x_0^2$$

for all $x_t \in \Omega$.

III.E. Stability Criteria

Based on previous analysis, we have following proposition to test the stability of the state-dependent delay system:

Theorem 7: Recall the State-Dependent Delay System

$$\dot{x}(t) = ax(t - \tau(x(t))), \tag{1}$$

suppose there exist $U_i : \mathbb{R}^3 \to \mathbb{R}$ and $U_i \in \Sigma^{2d}$ for $i = 1, 2, 3$, $L_i : \mathbb{R}^4 \to \mathbb{R}$ and $L_i \in \Sigma^{2d}$ for $i = 1, 2, 3, 4$, some $\epsilon_j > 0$ for $j = 1, 2$, polynomial $R_1(x_0, \theta)$, $R_2(x_0, x_1, \theta)$ and semidefinite symmetric matrix M and N, such that the conditions

1) $v_1(x_0, x_2, \theta) + R_1(x_0, \theta) - \epsilon_1 x_0^2 + \sum_{i=1}^{3} U_i(x_0, x_2, \theta) g_i \in \Sigma$,

2) $-v_3(x_0, x_1, x_2, \theta) + R_2(x_0, x_1, \theta) - \epsilon_2 x_0^2 + \sum_{i=1}^{4} L_i(x_0, x_1, x_2, \theta) g_i \in \Sigma$,

3) $v_2(x_2, x_3, \theta, \xi) = z_d^T(x_2, \theta) M z_d(x_3, \xi)$,

4) $v_4(x_2, x_3, \theta, \xi) = z_d^T(x_2, \theta) N z_d(x_3, \xi)$,

5) $\int_{-\tau(x_0)}^{0} R_1(x_0, \theta) d\theta = 0$,

6) $\int_{-\tau(x_0)}^{0} R_2(x_0, x_1, \theta) d\theta = 0$,

hold, then the system will be asymptotically stable for all $x_t \in \Omega$, where Ω is defined as

$$\Omega := \{x_t \in \mathbb{C} : \|x_t\| \leq \tau_0/(2b)\}.$$

Proof: Based on the theorem 5 and equation (23), for all $x_t \in \Omega$, we have

$$V(x_t) = V_1(x_t) + V_2(x_t) \geq \epsilon \epsilon_1 x_0^2,$$

and based on theorem 6, we have

$$\dot{V}(x_t) = V_3(x_t) - V_4(x_t) \leq -\epsilon \epsilon_2 x_0^2,$$

these two equations satisfy the Lyapunov theorem, thus the system will be stable for all $x_t \in \Omega$.

III.F. Numerical and Simulation Results

The validity of our proposed algorithm is demonstrated by following numerical results. Firstly, we set some typical values of a and τ_0, then we use the proposed algorithm to determine the maximum and minimum b which renders the system stable. Similarly, we can determine the maximum and minimum τ_0 which renders the system stable when a and b are given through the proposed algorithm. The detailed numerical results are shown in Table 1.

Table 1. Parameters that Make the System Stable

	a=−0.1	a=−0.5	a=−1
τ_0=0.1	b=4e-4 ∼ 1	b=1e-4 ∼ 1	b=2e-4 ∼ 1
τ_0=0.5	b=6e-4 ∼ 2	b=3e-4 ∼ 2	b=3e-3 ∼ 0.02
τ_0=1	b=7e-4 ∼ 3	b=8e-4 ∼ 2	b=3e-3 ∼ 0.02

The minimum and maximum values of b which make the system stable when a and τ_0 are given.

	a=−0.1	a=−0.5	a=−1
b=0.1	τ_0=3e-2 ∼ 5	τ_0=1e-2 ∼ 1	τ_0=7e-3 ∼ 0.7
b=0.5	τ_0=3e-2 ∼ 6	τ_0=4e-2 ∼ 1	τ_0=4e-2 ∼ 0.7
b=1	τ_0=3e-2 ∼ 6	τ_0=6e-2 ∼ 1	τ_0=7e-2 ∼ 0.7

The minimum and maximum values of τ_0 which make the system stable when a and b are given.

Fig. 2 shows the simulation results of a stable situation. We choose $a = -0.1, b = 1, \tau_0 = 6$, initial condition $\phi(\theta) = 0.5sin(\theta)$ for $\theta \in [-9, 0]$. Simulation results indicate that the system will be stable in this case, which is consistent with our numerical results.

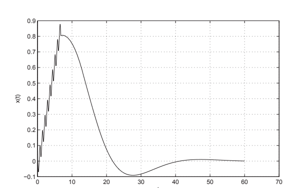

Figure 2. Simulation Results

IV. CONCLUSIONS AND FUTURE WORKS

IV.A. Conclusions

This paper provides sufficient conditions for the stability of the state-dependent delay system which are easy to apply. In summary, we can use these conditions to test the stability of a given system, and to determine the value of the parameters for system design as well.

IV.B. Future Work

There are several works to be done in the future. Firstly, we may try some other kind of Lyapunov-Krasovskii functional to get some more accurate results. One way worth to attempt is using Semi-Separable kernels to build the Lyapunov-Krasovskii functional. In this case we assume

$$v_2(x_2, x_3, \theta, \xi) = \begin{cases} z_d^T(x_2, \theta) M_1 z_d(x_3, \xi), \theta \leq \xi \\ z_d^T(x_2, \theta) M_2 z_d(x_3, \xi), \theta > \xi \end{cases}$$

where M_1 and M_2 are both positive semidefinite matrix. Numerical results shows that the Semi-Separable kernels is more effective then the separable kernels.[18]

Secondly, we could apply our algorithm to some more practical systems and evaluate the effectiveness of our algorithm. For example, some biological systems[19] and economical systems[20] can be expressed as state-dependent delay differential equation. We can determine on what condition the biological systems or the economical systems will be stable by using our algorithm.

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