A Sum-of-Squares Approach to the Analysis of Zeno Stability in Polynomial Hybrid Systems

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***Abstract*— Hybrid dynamical systems can exhibit many unique phenomena, such as Zeno behavior. Zeno behavior is the occurrence of infinite discrete transitions in finite time. Zeno behavior has been likened to a form of finite-time asymptotic stability, and corresponding Lyapunov theorems have been developed. In this paper, we propose a method to construct Lyapunov functions to prove Zeno stability of compact sets in cyclic hybrid systems with parametric uncertainties in the vector fields, domains and guard sets, and reset maps utilizing sum-of-squares programming. This technique can easily be applied to cyclic hybrid systems without parametric uncertainties as well. Examples illustrating the use of the proposed technique are also provided.**

I. Introduction

Hybrid systems are dynamical systems with trajectories that exhibit both continuous flows and discrete transitions. As such, a variety of man-made systems can be modeled using the hybrid systems framework. Some exampleks are electrical systems with switching [1], communication networks [2], embedded systems [3], and air traffic control [4].

Recent research into hybrid systems has yielded results on stability of equilibria [5] and observability and controllability [6]. Several Lyapunov-based techniques for the analysis of hybrid systems, including the use of multiple Lyapunov functions [7], the construction of piecewise-quadratic Lyapunov functions [8], and the utilization of Lyapunov techniques for robust stability analysis [9] have also been presented. More recently, a means to assess stability of hybrid systems by constructing higher-order polynomial Lyapunov functions using sum-of-squares techniques was presented in [10], and a method to perform robust stability analysis using sum of squares techniques was provided in [11]. However, there are still behaviors of hybrid systems that require further study. Among these phenomena are chattering and zeno behavior.

Zeno behavior is the occurrence of infinite transitions between discrete states in a finite period of time. Trajectories exhibiting this behavior are called Zeno executions, and converge to a set of points known as a Zeno equilibrium. Hybrid systems exhibiting Zeno behavior are described in detail in, for example, [12]. Zeno behavior can cause simulations to halt or fail, since infinitely many transitions would need to be simulated, as noted in, e.g., [12]. This problem was addressed in [13] and [14], which describe methods to regularize hybrid systems to ensure that trajectories continue after the Zeno equilibrium. Sufficient conditions for Zeno behavior in first quadrant hybrid systems were given in [15], and further sufficient conditions for systems with nonlinear vector fields based on constant approximations were given in [16]. More recently, necessary and sufficient Lyapunov conditions for the existence of isolated Zeno equilibria were first given in [17]. These results were extended in [18], where the concept of Zeno stability was described as an extension of finite-time asymptotic stability. Moreover, [18] provided Lyapunov conditions for Zeno stability of compact sets. The results in [18] were also shown to be equivalent to the theorem presented in [17]. The results of [17] were also exended by Ames and Lamperski to non-isolated Zeno equilibria in [19]. In a similar vein, a Lyapunov characterization of Filippov solutions was provided in [20].

In this paper, we use sum-of-squares programming to construct Lyapunov functions which prove Zeno stability of compact sets, based on the results of [17] and [18]. Moreover, the method presented in this paper allows for the verification of Zeno stability for polynomial hybrid systems with nonlinear polynomial vector fields and transitions. We also present a method to verify Zeno stability for systems with parametric uncertainties.

The outline of the paper is as follows: in Section II, definitions of sum-of-squares polynomials, hybrid systems and their executions, and Zeno executions and equilibria are presented. Section II also details Lyapunov conditions for Zeno stability as described in [17]. In Section III, we present a method to construct Lyapunov functions to prove Zeno stability of hybrid systems with parametric uncertainties using Sum-of-Squares optimization, and in Section IV, illustrative examples are provided.

II. Preliminaries

In this section, we provide a brief introduction to Sum-of-Squares polynomials and definitions for hybrid systems, their executions, and Zeno behavior.

A. Sum of Squares Polynomials

We use $\mathbf{R}[x]$ to denote the ring of polynomials generated by variables $x = (x_1, .., x_n)$.

Definition 1. (Sum of Squares Polynomial) A polynomial $p(x) : \mathbb{R}^n \to \mathbb{R}$ is said to be Sum of Squares (SOS) if there exist polynomials $f_i(x) : \mathbb{R}^n \to \mathbb{R}$ such that

$$p(x) = \sum_i (f_i(x))^2$$

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We use $p \in \Sigma_x \subset \mathbf{R}[x]$ to denote that p is SOS.

The following result gives a polynomial-time complexity test to determine whether a polynomial is SOS.

Theorem 1. *For a polynomial, p of degree $2d$, $p \in \Sigma_x$ if and only if there exists a positive semidefinite matrix Q, such that*

$$p(x) = Z(x)^T Q Z(x)$$

where $Z(x)$ is the vector of monomials of degree d or less

Therefore, checking whether a polynomial is SOS is equivalent to checking the existence of a positive-semidefinite matrix Q under some affine constraints, which can be solved with semidefinite programming. Thus, while checking polynomial positivity is NP-hard, checking whether a polynomial is SOS is decidable in polynomial time.

In this paper, Positivstellensatz results from algebraic geometry are used extensively to create constraints that can be implemented using sum of squares programming. We use the Positivstellensatz to construct Lyapunov functions which are positive on bounded sets (see section IV).
For further details and proofs, we refer to [21] and [22].

B. Hybrid Systems

In this section, we define hybrid systems and their executions. We use similar notation to that given in [23] and, more recently, [17].

Definition 2. (Hybrid System) A hybrid system H is a tuple $H = (Q, E, D, F, G, R)$ where

- Q is a finite collection of discrete states or indices
- $E \subset Q \times Q$ is a collection of edges, where for any edge $e = (q, q')$ we use the functions s and t to denote the start and end, so that for $e = (q, q')$, $s(e) = q$ and $t(e) = q'$
- $D = \{D_q\}_{q \in Q}$ is a collection of Domains, where for each $q \in Q$, $D_q \subseteq \mathbb{R}^n$
- $F = \{f_q\}_{q \in Q}$ is a collection of vector fields, where for each $q \in Q$, $f_q : D_q \to \mathbb{R}^n$
- $G = \{G_e\}_{e \in E}$ is a collection of guard sets, where for each $e = (q, q') \in E$, $G_e \subset D_q$
- $R = \{\phi_e\}_{e \in E}$ is a collection of Reset Maps, where for each $e = (q, q') \in E$, $\phi_e : G_e \to D_{q'}$.

Definition 3. (Cyclic hybrid system) A cyclic hybrid system H_c is a hybrid system where for each domain $q \in Q$, we can associate a unique edge $e(q) = (q, q_i) \in E$ such that $s(e(q)) = q$ and such that for any $q \in Q$, $q = t(e(t(e(\cdots t(e(t(e(q))))))))$. That is, the set of edges forms a directed graph.

Definition 4. (Hybrid System Execution) Consider the tuple $\chi = (I, T, p, C)$ where

- $I \subseteq \mathbb{N}$ is index of intervals
- $T = \{T_i\}_{i \in I}$ are a set of open time intervals associated with points in time τ_i as $T_i = (\tau_i, \tau_{i+1}) \subset \mathbb{R}_+^n$ where $T_{i+1} = (\tau_{i+1}, \tau_{i+2})$
- $p : I \to Q$ maps each interval to a domain,
- $C = \{c_i(t)\}_{i \in I}$ is a set of continuously differentiable functions. We say χ is an *execution* of the hybrid system $H = (Q, E, D, F, G, R)$ with initial condition (q_0, x_0) if $c_1(0) = x_0$ and $p(1) = q_0$.
- $\dot{c}_i(t) = f_{p(i)}(c_i(t))$ for $t \in T_i$ and for all $i \in I$; $c_i(t) \in D_{p(i)}$ for $t \in T_i$ and for all $i \in I$; $c_i(\tau_{i+1}) \in G_{(p(i),p(i+1))}$ for all $i \in I$; $c_{i+1}(\tau_{t+1}) = \phi_{(p(i),p(i+1))}(c_i(\tau_i))$ for all $i \in I$.

C. Zeno Stability in Hybrid Dynamical Systems

We now present definitions of Zeno executions, equilibria, and stability, along with necessary and sufficient conditions for Zeno stability as presented in [17] and [18].

Definition 5. (Zeno Execution) We say an execution $\chi = (I, T, p, C)$ starting from (q_0, x_0) of a hybrid System $= (Q, E, D, F, G, R)$ is Zeno if

1) $I = \mathbb{N}$
2) $\lim_{i \to \infty} \tau_i < \infty$

Definition 6. (Zeno Equilibrium) A set $z = \{z_q\}_{q \in Q}$ is a Zeno equilibrium of a Hybrid System $H = (Q, E, D, F, G, R)$ if it satisfies

1) For each edge $e = (q, q') \in E$, $z_q \in G_e$ and $\phi_e(z_q) = z_{q'}$.
2) $f_q(z_q) \neq 0$ for all $q \in Q$.

Note that for any $z \in \{z_q\}_{q \in Q}$, where $\{z_q\}_{q \in Q}$ is a Zeno equilibrium of a cyclic hybrid system H_c,

$$(\phi_{i-1} \circ \cdots \circ \phi_0 \cdots \phi_i)(z) = z$$

Next, we define Zeno stability:

Definition 7. (Zeno Stability) Let $H = (Q, E, D, F, G, R)$ be a hybrid system, and let $z = \{z_q\}_{q \in Q}$ be a compact set. The set z is Zeno stable if, for each $q \in Q$, there exist neighborhoods Z_q, where $z_q \in Z_q$, such that for any initial condition $x_0 \in \bigcup_{q \in Q} Z_q$, the execution $\chi = (I, T, p, C)$, with $c_o(t_0) = x_0$ is Zeno, and converges to z.
Note that this definition of Zeno stability is consistent with the stability definitions provided in [18]. We now reiterate the Lyapunov conditions for the stability of Zeno equilibria in cyclic hybrid systems presented in [17], which are as follows:

Theorem 2. (Lamperski and Ames) *Consider a cyclic hybrid system $H = (Q, E, D, F, G, R)$, with an isolated Zeno equilibrium $\{z_q\}_{q \in Q}$. Let $\{W_q\}_{q \in Q}$ be a collection of open neighborhoods of $\{z_q\}_{q \in Q}$. Suppose there exist continuously differentiable functions $V_q : \mathbb{R}^n \to \mathbb{R}$ and $B_q : \mathbb{R}^n \to \mathbb{R}$, and non-negative constants $\{r_q\}_{q \in Q}$, γ_a, and γ_b, where $r_q \in [0, 1]$, and $r_q < 1$ for some q and such that*

$$\begin{aligned}
V_q(x) &> 0 \quad \textit{for all } x \in W_q \backslash z_q, q \in Q && (1)\\
V_q(z_q) &= 0, \quad \textit{for all } q \in Q && (2)\\
\nabla V_q^T(x) f_q(x) &\leq 0 \quad \textit{for all } x \in W_q, q \in Q && (3)\\
B_q(x) &\geq 0 \quad \textit{for all } x \in W_q, q \in Q && (4)\\
\nabla B_q^T(x) f_q(x) &< 0 \quad \textit{for all } x \in W_q, q \in Q && (5)\\
V_{q'}(R_{(q,q')}(x)) &\leq r_q V_q(x), && (6)\\
&\quad \textit{for all } e = (q, q') \in E \textit{ and } x \in G_e \cap W_q\\
B_q(R_{(q',q)}(x)) &\leq \gamma_b \left(V_q(R_{(q,q')}(x))\right)^{\gamma_a} && (7)\\
&\quad \textit{for all } e = (q, q') \in E \textit{ and } x \in G_e \cap W_q.
\end{aligned}$$

Then $\{z_q\}_{q\in Q}$ is Zeno stable.

As noted in [18], the conditions above are equivalent to those given in [18, Proposition 5.2]. Thus, satisfying EC1-C2 is also sufficient to prove asymptotic Zeno stability of a compact set. This in turn allows us to relax the restriction $f_q(z_q) \neq 0$.

To simplify notation, we will use the sufficient conditions of Theorem 3 as follows. Note that our subsequent analysis can be easily applied directly to the conditions of Theorem 2 and in our numerical examples we have tested both sets of conditions and they yield similar results.

Theorem 3. *Let $H = (Q,E,D,F,G,R)$ be a cyclic hybrid system, and let $z = \{z_q\}_{q\in Q}$ be a compact set. Let $\{W_q \subset D_q\}_{q\in Q}$, be a collection of neighborhoods of the $\{z_q\}_{q\in Q}$. Suppose that there exist continuously differentiable functions $V_q : W_q \to \mathbb{R}$, and positive constants $\{r_q\}_{q\in Q}$ and γ, where $r_q \in (0,1]$, and $r_q < 1$ for some q and such that*

$$V_q(x) > 0 \quad \text{for all } x \in W_q \backslash z_q, q \in Q \tag{8}$$

$$V_q(z_q) = 0, \quad \text{for all } q \in Q \tag{9}$$

$$\nabla V_q^T(x) f_q(x) \leq -\gamma \quad \text{for all } x \in W_q, q \in Q \tag{10}$$

$$r_q V_q(x) \geq V_{q'}(\phi_e(x)) \tag{11}$$

$$\text{for all } e = (q,q') \in E \text{ and } x \in G_e \cap W_q.$$

then z is Zeno stable.

Proof:
We show that if for each $q \in Q$, we can find a V_q such that (8)-(11) are satisfied, then the same V_q also satisfies (1)-(7). From inspection, it is clear that if V_q satisfies (8)-(11), then (1)-(3) and (6) are satisfied. Second, choose $B_q = V_q$ for each $q \in Q$. From inspection, it is clear that V_q also satisfies (4) and (5). Last, if $\gamma_a = \gamma_b = 1$, we get $V_q \leq V_q$, where the equality holds. From this, we see that for each $q \in Q$, V_q also satisfies (7). Thus, the theorem is proved. □

III. Using Sum-of-Squares Programming to prove Zeno Stability

Theorem 3 provides sufficient conditions for Zeno stability in cyclic hybrid systems. We now show that these conditions can be enforced using SOS, even for systems with parametric uncertainties. First, define the vector of parametric uncertainties P to lie within a semialgebraic set

$$P := \{p \in \mathbb{R} : \tilde{p}_k(p) \geq 0, k = 1,2,...,K_1\}. \tag{12}$$

We then present the following assumption:

Assumption 1. For the purposes of this paper, we consider hybrid systems with polynomial vector fields and resets, and semialgebraic domains and guard sets, with parametric uncertainties in the each of the above. Let P be defined as in (12). We implicitly assume that associated with every hybrid system is a set of polynomials $g_{qi}(x,p)$, $h_{e,k}(x,p)$ for $q \in Q$, $e \in E$, $i = k = 1,\cdots,K_q$ and $k = 1,\cdots,N_q$ for some $K_q, N_q > 0$, and $p \in P$.

In this framework, the domains of the hybrid system H are defined as

$$D_q = \{x \in \mathbb{R}^n \,:\, g_{qk}(x,p) \geq 0,\ k = 1,2,\cdots,K_q\} \tag{13}$$

where $g_{qk} \in \mathbf{R}[x,p]$, $K_q \in \mathbb{N}$, and $p \in P$. The guard sets are defined as

$$G_e = \{x \in \mathbb{R}^n \,:\, h_{e,0}(x,p) = 0,\ h_{e,k}(x,p) \geq 0,\ k = 1,2,\cdots,N_q\} \tag{14}$$

where each $h_{ek} \in \mathbf{R}[x,p]$, $N_q \in \mathbb{N}$, and $p \in P$. Lastly, for each $e = (q,q') \in E$, the reset map ϕ_e is given by the vector-valued polynomial function

$$\phi_e = [\phi_{e,1}(x,p),\cdots,\phi_{e,n}(x,p)]^T \tag{15}$$

where $\phi_{e,j} \in \mathbf{R}[x,p]$ for $j = 1,\cdots,n$, and $p \in P$.

Let $H = (Q,E,D,F,G,R)$ be a cyclic hybrid system, and let $z = \{z_q\}_{q\in Q}$ be a compact set. Let $\{W_q\}_{q\in Q}$ be a collection of neighborhoods of $\{z_q\}_{q\in Q}$. We consider W_q of the form

$$W_q := \{x \in \mathbb{R}^n : w_{qk}(x) > 0, k = 1,2,...,K_q\}$$

where each $w_{qk}(x) \in \mathbf{R}[x]$.
Consider Feasibility Problem 1:
Feasibility Problem 1:
For hybrid system $H = (Q,E,D,F,G,R)$, find

- a_{qk}, c_{qk}, i_{qk}, $\in \Sigma_{x,p}$, for $k = 1,2,...,K_{qw}$, $p \in P$ and $q \in Q$;
- b_{qk}, d_{qk}, $j_{qk} \in \Sigma_{x,p}$, for $k = 1,2,...,K_q$, $p \in P$, and $q \in Q$.
- η_{qk}, β_{qk}, $\zeta_{qk} \in \Sigma_{x,p}$, for $k = 1,2,...,K_1$, $p \in P$, and $q \in Q$.
- $m_{e,l} \in \Sigma_{x,p}$ for $e \in E$, $p \in P$, and $l = 1,2,...,N_q$
- V_q, $m_{e,0} \in \mathbf{R}[x,p]$ for $e \in E$, $p \in P$, and $q \in Q$.
- Constants $\alpha,\gamma > 0$, $\{r_q\}_{q\in Q} \in (0,1]$ such that $r_q < 1$ for some $q \in Q$.

such that

$$V_q - \alpha x^T x - \sum_{k=1}^{K_{qw}} a_{qk} w_{qk} - \sum_{k=1}^{K_q} b_{qk} g_{qk} - \sum_{k_1=1}^{K_1} \eta_{qk_1} \tilde{p}_{qk} \in \Sigma_{x,p} \quad \text{for all } q \in Q \tag{16}$$

$$V_q(z_q,p) = 0 \quad \text{for all } q \in Q \tag{17}$$

$$-\nabla V_q^T f_q - \gamma - \sum_{k=1}^{K_{qw}} c_{qk} w_{qk} - \sum_{k=1}^{K_q} d_{qk} g_{qk} - \sum_{k_1=1}^{K_1} \beta_{qk_1} \tilde{p}_{qk} \in \Sigma_{x,p} \quad \text{for all } q \in Q \tag{18}$$

$$r_q V_q - V_{q'}(\phi_e) - m_{e,0} h_{e,0} - \sum_{l=1}^{N_q} m_{e,l} h_{e,l} - \sum_{k=1}^{K_{qw}} i_{qk} w_{qk} - \sum_{k=1}^{K_q} j_{qk} g_{qk} - \sum_{k=1}^{K_1} \zeta_{qk} \tilde{p}_{qk} \in \Sigma_{x,p} \quad \text{for all } e = (q,q') \in E. \tag{19}$$

Theorem 4. *Consider a cyclic hybrid system $H = (Q,E,D,F,G,R)$, and let $z = \{z_q\}_{q\in Q}$ be a compact set. If Feasibility Problem 2 has a solution, then z is Zeno stable for all $p \in P$.*

Proof:

To prove the theorem we show that if V_q, $q \in Q$ are elements of a solution of Feasibility Problem 1, then for each $q \in Q$, the same V_q also satisfy (8)-(11) of Theorem 3 for all $p \in P$. That is, we show that if the V_q satisfy (16)-(19), then the same V_q also satisfies (8)-(11) for all $p \in P$.
First, we observe that (17) directly implies (9) for $p \in P$. Next, from (16), we know that

$$V_q(x,p) \geq \sum_{k=1}^{K_{qw}} a_{qk}(x,p) w_{qk}(x) + \sum_{k=1}^{K_q} b_{qk}(x,p) g_{qk}(x,p) + \alpha x^T x + \\ + \sum_{k_1=1}^{K_1} \eta_{qk_1}(x,p) \tilde{p}_{qk}(p)$$

Since $a_{qk}(x,p)$, $b_{qk}(x,p)$, and $\eta_{qk}(x,p)$ are SOS, and thus, always nonnegative, by the Positivstellensatz and the definitions of W_q, P, and D_q, we have that $V_q(x) \geq \alpha x^T x$ for all $x \in W_q \subset D_q$ and all $p \in P$. Thus, (16) implies (8) is satisfied. Similarly, from (19),

$$-\nabla V_q^T(x,p) f_q(x,p) - \gamma \geq \sum_{k=1}^{K_{qw}} c_{qk}(x,p) w_{qk}(x) + \\ \sum_{k=1}^{K_q} d_{qk}(x,p) g_{qk}(x,p) - \sum_{k_1=1}^{K_1} \beta_{qk_1} \tilde{p}_{qk}.$$

Since $c_{qk}(x,p)$ and $d_{qk}(x,p)$ are always nonnegative, by the definition of P, D_q and W_q, $\nabla V_q(x)^T f_q(x,p) \leq -\gamma$ for $x \in \{x \in \mathbb{R}^n : g_{qk}(x,p) \geq 0,\ w_{qk}(x,p) \geq 0\} = D_q \cap W_q$ and $p \in P$, which implies (10) is satisfied. Next, from (19) we have that for all $e = (q,q') \in Q$,

$$r_q V_q(x,p) - V_{q'}(\phi_e(x,p),p) \geq m_{e,0}(x,p) h_{e,0}(x,p) \\ + \sum_{l=1}^{N_q} m_{e,l}(x,p) h_{e,l}(x,p) + \sum_{k=1}^{K_q} i_{qk}(x,p) w_{qk}(x) \\ + \sum_{k=1}^{K_q} j_{qk}(x,p) g_{qk}(x,p) + \sum_{k=1}^{K_1} \zeta_{qk}(x,p) \tilde{p}_{qk}(p).$$

First note that $h_{e,0}(x) = 0$ and hence $m_{e,0}(x,p) h_{e,0}(x,p) = 0$ on G_e. Since $m_{e,l} \in \Sigma_x$, we have $m_{e,l}(x) h_{e,l}(x) \geq 0$ on G_e. Similarly $j_{qk}(x,p) g_{qk}(x,p) \geq 0$ on D_q and $i_{qk}(x,p) w_{qk}(x) \geq 0$ on W_q. It follows that $r_q V_q(x) - V_{q'}(\phi_e(x)) \geq 0$ when $x \in G_e \cap W_q \cap D_q$ for all $p \in P$ $e = (q,q') \in E$. Thus, we have shown that (19) implies (11).
Thus we conclude that the solution elements V_q of Feasibility Problem 1 satisfy the conditions (8)-(11) of Theorem 4. Thus by Theorem 3 we conclude Zeno stability of z for all $p \in P$. $\square$

Remark: For systems without parametric uncertainty, we simply take the set of uncertain parameters to be empty. Thus, all elements of Feasibility Problem 1 become dependent only on x, and all $\tilde{p}_{qk}$, η_{qk}, β_{qk}, and ζ_{qk} are 0. A similar theorem for Zeno stability of hybrid systems without parametric uncertainty is stated explicitly in [24].

IV. EXAMPLES

In this section, we provide some examples that illustrate the application of the given technique. We demonstrate Zeno stability in hybrid systems with polynomial vector fields and semialgebraic domains and guard sets, and parametric uncertainties.

Example 1.

In this first example, we analyze Zeno stability of a hybrid system without time-invariant parametric uncertainties.
Consider the hybrid system $H = (Q,E,D,F,G,R)$, where

- $Q = \{1,2,3\}$
- $E = \{(1,2),(2,3),(3,1)\}$
- $D := \{D_1, D_2, D_3\}$ where

$$D_1 = \{x \in \mathbb{R}^2 : x_1 > 0, x_2 + \frac{1}{2}x_1 \geq 0\}$$
$$D_2 = \{x \in \mathbb{R}^2 : x_2 - \frac{1}{2}x_1 \geq 0, x_2 + \frac{1}{2}x_1 < 0\}$$
$$D_3 = \{x \in \mathbb{R}^2 : x_1 < 0, x_2 + \frac{1}{2}x_1 \geq 0\};$$

- $F = \{f_1, f_2, f_3\}$, where

$$\dot{x} = f_1(x) = (x_2,\ -5x_1 - x)^T$$
$$\dot{x} = f_2(x) = \left(-x_1^2 - 3,\ 2x_2^2 - \frac{1}{2}x_1^2\right)$$
$$\dot{x} = f_3(x) = \left(x_2^2 + x_1.\ -3x_1\right);$$

- $G := \{G_{12}, G_{23}, G_{31}\}$ where

$$G_{12} := \left\{x \in \mathbb{R}^2 : x_2 \leq 0, \frac{1}{2}x_1 + x_2 = 0\right\}$$
$$G_{23} := \left\{x \in \mathbb{R}^2 : x_2 \leq 0, \frac{1}{2}x_1 - x_2 = 0\right\}$$
$$G_{31} := \left\{x \in \mathbb{R}^2 : x_2 > 0, x_1 = 0\right\};$$

- $R = \{\phi_{12}(x), \phi_{23}(x), \phi_{31}(x)\}$ where each $\phi_{ij}(x) = x$.

We note that this hybrid system is cyclic, as the pair (Q,E) forms a directed cycle, with vertices Q and edges E. A phase portrait of the system is given below in Figure 1.
Results:
We wish to analyze Zeno stability for $z = \{z_1, z_2, z_3\}$, where $z_1 = z_2 = z_3 = [0,0]^T$. To solve Feasibility Problem 1, we consider each

$$W_q := \mathbb{B}^2 \cap D_q$$

where $\mathbb{B}^2 := \{x \in \mathbb{R}^2 : |x| \leq 1\}$.

We then search for 3 degree 8 polynomials to solve Feasibility Problem 1. Since we are able to solve Feasibility Problem 1 with such polynomials, we show using Theorem 4 that $z = [0,0]^T$ is Zeno stable for H.

Example 2.

Consider the hybrid system $H = (Q,E,D,F,G,R)$ with uncertain parameter $p \in (C, \infty)$ where

- $Q = \{1,2\}$
- $E = \{(1,2),(2,1)$

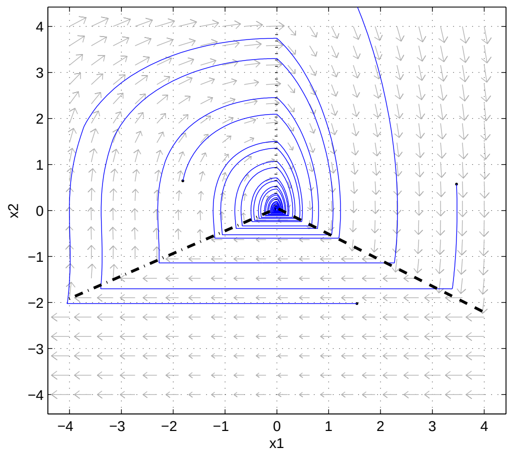

Fig. 1. Hybrid System in Example 1. Dashed line indicates G_{12}, dash-dotted line indicates G_{23} and dotted line indicates G_{31}

- $D = \{D_1, D_2\}$ where

$$D_1 := \{x \in \mathbb{R}^2 : x_1 + x_2 \geq 0, px_1 - x_2 \geq 0\}$$
$$D_2 := \mathbb{R}^2 \backslash D_1$$

- $F = \{f_1, f_2\}$ where

$$f_1 = \begin{pmatrix} -0.1 \\ 2 \end{pmatrix}$$
$$f_2 = \begin{pmatrix} -x_2 - x_1^3 \\ x_1 \end{pmatrix}$$

- $G = \{G_{12}, G_{21}\}$ where

$$G_{12} := \{x \in \mathbb{R}^2 : x_2 - px_1 = 0\}$$
$$G_{21} := \{x \in \mathbb{R}^2 : x_1 + x_2 = 0\}$$

- $R = \{\phi_{12}(x), \phi_{21}(x)\}$ where each $\phi_{ij}(x) = x$.

In this example, the uncertain parameter affects the switching rule. Provided below are simulations with 3 different fixed values of p. First, we consider the case when $p = 1$, in Figure 2.

We see from inspection that the origin is Zeno stable. Furthermore, if we consider Figure 3, we see that even if we increase p (thereby increasing the slope of G_{21}), we notice that the system remains Zeno stable.

However, when we reduce the value of p, we notice that the system exhibits different asymptotic behavior. First, we see that if $p \leq -0.1$, the system will no longer exhibit Zeno stability. Indeed, in that circumstance, the system would not display any form of stable behavior. This is because the trajectories in D_1 would never reach the guard set (since the direction of the vector field would be parallel to the guard set). But even if $p \in (-0.1, 1)$, we notice that the system asymptotically converges to limit cycles, as seen in Figure 4.

Results:

We wish to analyze Zeno stability of $z = \{z_1, z_2\} = (0, 0)$.

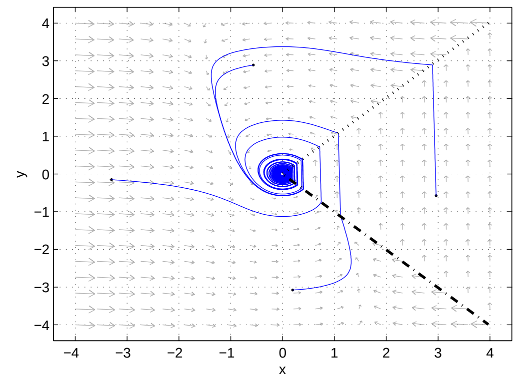

Fig. 2. Trajectories of Hybrid System in Example 2 with p=1. Dotted line indicates G_{12} and dash-dotted line indicates G_{21}

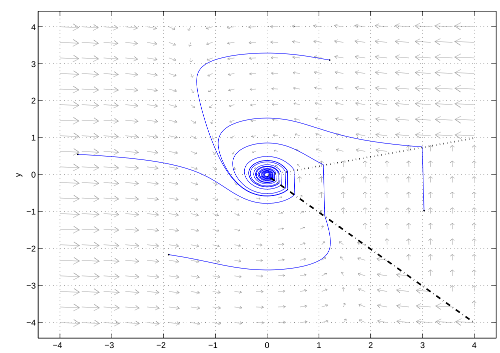

Fig. 3. Trajectories of Hybrid System in Example 2 with p=4. Dotted line indicates G_{12} and dash-dotted line indicates G_{21}

For our computational analysis, we first divide D_2 into D_{21} and D_{22},where

$$D_{21} := \{x \in \mathbb{R}^2 : x_1 + x_2 \geq 0, -px_1 + x_2 \geq 0\}$$
$$D_{22} := \{x \in \mathbb{R}^2 : x_1 + x_2 \leq 0\}.$$

We then search for a common Lyapunov function for both D_{21} and D_{22}. The set of uncertain parameters is given by the inequality $P := \{p \in \mathbb{R} : \tilde{p} := p - C > 0\}$, where C is determined a priori. The goal is to find a lower bound on C such that z is Zeno stable. We use $W = W_1 \cup W_2 = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < 5\}$. We then search for Lyapunov functions of varying degrees for different values of C. We note that as we increase the degree of V_1 and V_2, we are able to obtain a tighter lower bound on C. These results are given below in table I.

Degree of V_1, V_2	Lower bound on C
8	2.11
10	1.87
12	1.73

TABLE I

LOWER BOUND ON C FOR WHICH z IS ZENO STABLE OBTAINED FOR DIFFERENT DEGREES OF V_1 AND V_2

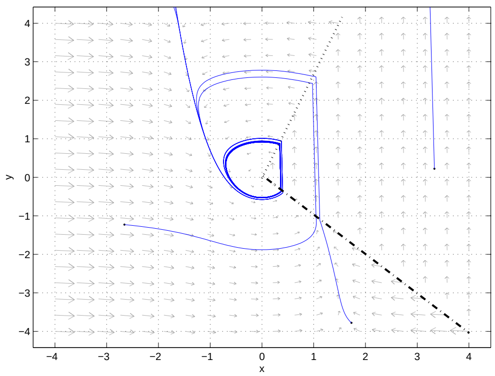

Fig. 4. Trajectories of Hybrid System in Example 2 with p=0.4. Dotted line indicates G_{12} and dash-dotted line indicates G_{21}

We were unable to find a feasible V_1 and V_2 of degree less than 8. Unfortunately, we were unable to search for polynomials of degree greater than 12 owing to computational limitations.

V. Conclusions

In this paper, we present a Lyapunov based method for determining Zeno stability of compact sets in hybrid dynamical systems. The method presented makes use of the sum-of-squares decomposition, thus enabling the construction of higher-order Lyapunov functions. As such, the theorem presented can be used to certify Zeno stability in systems with nonlinear vector fields and reset maps, and time-invariant parametric uncertainties. The result can easily be simplified to certify Zeno stability for hybrid systems without parametric uncertainties as well. Examples of hybrid systems with polynomial domains, guard sets, vector fields and reset maps illustrating the use of the proposed method are also provided. We also provide an example of a nonlinear polynomial hybrid system with an uncertain parameter in the guard set.

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