Full-State Feedback of Delayed Systems using SOS: A New Theory of Duality

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Abstract: In this paper, we show that the controller synthesis of delayed systems can be formulated and solved in a convex manner through the use of a duality transformation, a structured class of operators, and the Sum-of-Squares (SOS) methodology. The contributions of this paper are as follows. We show that a dual stability condition can be formulated in terms of Lyapunov operators which are positive, self-adjoint and preserve the structure of the state-space. Second, we provide a class of such operators which can be parameterized using Sum-of-Squares. Next, we show how any operator in this class can be inverted using simple operations on the SOS variables which can be performed in Matlab. Next we use SOS and semidefinite programming to formulate a dual stability test for time-delay systems. Next, we use the dual stability results to formulate a convex test for stabilizability and show how SOS can be used to solve this test and recover the controller. Finally, we give a numerical example. The results of this paper are significant in that they open the way for dynamic output H_∞ optimal control of infinite-dimensional systems by giving the first truly convex, numerically realizable full-state feedback controller synthesis criterion.

Keywords: Sum-of-Squares; Delayed Systems; Infinite-Dimensional Systems; Duality; Controller Synthesis.

1. INTRODUCTION

Systems with delay have been well-studied for some time Niculescu [2001], Gu et al. [2003], Richard [2003]. Recently, there have been many results on the use of optimization and semidefinite programming for the stability analysis of these systems. Although the computational question of stability of a delayed system is believed to be NP-hard, several techniques have been developed to construct sequences of polynomial-time algorithms which provide sufficient stability conditions and appear to converge to necessity as the complexity of the algorithms increase. Examples of such sequential algorithms include the piecewise-linear approach Gu et al. [2003], the delay-partitioning approach Gouaisbaut and Peaucelle [2009] and the SOS approach Peet et al. [2009]. In addition, there are also frequency-domain approaches such as Michiels and Vyhlidal [2005]. These algorithms are sufficiently reliable so that for the purposes of this paper, we may consider the problem of robust stability analysis of linear fixed-delay systems to be solved.

The purpose of this paper is to explore methods by which the success in stability analysis of time-delay systems may be used to attack the relatively underdeveloped field of optimal controller synthesis. Although there have been a number of results on controller synthesis for time-delay systems, none of these results has been able to resolve the fundamental bilinearity of the synthesis problem. That is, controller synthesis is not convex in the combined Lyapunov operator P and controller operator K. Some papers use iterative methods to alternately optimize the Lyapunov operator and controller as in Moon et al. [2001] or Fridman and Shaked [2002] (via a "tuning parameter"). However, this iterative approach is not guaranteed to converge due to the non-convexity of the problem. In abstract space, there have been a number of results on dual and adjoint systems Bensoussan et al. [1992]. Unfortunately, however, these dual systems are not delay-type systems and there is no clear relationship between stability of these adjoint and dual systems and stability of the original delayed system. In this paper, we propose a broad set of conditions on the Lyapunov operator under which the controller synthesis problem may be convexified. Specifically, the operator must be invertible on the state-space, self-adjoint in L_2, and must preserve the structure of the state-space. Furthermore, we use polynomials to parameterize a set of operators which meet these three basic conditions. Although we do not expect that this set of operators is complete in any sense, the resulting synthesis conditions can be computed using SOS and can be used to solve a broad set of control problems for infinite-dimensional systems using LMI techniques developed for finite-dimensional systems.

This paper is organized as follows. Initially, we review previous work on the parametrization of positive operators using SOS and recall how this can be applied to stability analysis of time-delay systems. Furthermore, we recall previous work on inversion of SOS-derived positive operators and expand these results to a new class of operator. Next,

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we enumerate the conditions under which a dual stability condition may be derived. Then, we show how this dual condition may be tested using SOS. In Section 10, we briefly outline the convex operator conditions for full-state-feedback stabilization using a variable substitution trick. Next, we show how these synthesis conditions may be tested using SOS. Finally, we discuss numerical implementation, ongoing challenges and conclude the paper.

2. NOTATION

Standard notation includes the Hilbert spaces L_2 of square integrable operators and $W_2 := \{x : x, \dot{x} \in L_2\}$ with domains which will be clear from context. $\mathcal{C}[X]$ denotes the continuous functions on X. S^n denotes the symmetric matrices of dimension $n \times n$. $I_n \in \mathbb{S}^n$ denotes the identity matrix.

3. SUM-OF-SQUARES

Sum-of-Squares (SOS) refers to the optimization of positive polynomial variables by recasting the problem as a semidefinite programme. A polynomial p is SOS if it can be represented as the finite sum of squared polynomials $p(x) = \sum_{i=1}^{N} g_i(x)^2$. Clearly, any SOS polynomial is positive semidefinite and although there exist many positive polynomials which are not SOS, the set of SOS polynomials has been shown to approximate the set of positive polynomials to arbitrary accuracy. Most significantly, although it is NP-hard to determine whether a given polynomial is positive, it is relatively simple to parameterize the set of SOS polynomials for any given degree bound. Specifically, a polynomial p of degree $2d$ is SOS if and only if there exists some positive semidefinite matrix $Q \geq 0$ such that $p(x) = Z(x)^T Q Z(x)$, where Z is the vector of monomials in variables x of degree d or less. The resulting constraint that a polynomial p be SOS is then a semidefinite programming constraint on the coefficients of the polynomial p.

Throughout this paper, we will use the **Notation:** $p \in \Sigma_s$ to denote the constraint that p be SOS. This constraint implies that p is positive semidefinite and may be implemented in a straightforward manner through the use of Matlab toolboxes such as SOSTOOLS Prajna et al. [2002], Gloptipoly Henrion et al. [2009] or SOSOPT Seiler [2010].

4. LYAPUNOV STABILITY OF TIME-DELAY SYSTEMS

In this paper, we consider stability of linear discrete-delay systems of the form

$$\begin{aligned}\dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{K} A_i x(t-\tau_i) && \text{for all } \ t \geq 0,\\ x(t) &= \phi(t) && \text{for all } \ t \in [-\tau_K, 0]\end{aligned} \tag{1}$$

where $A_i \in \mathbb{R}^{n\times n}$, $\phi \in \mathcal{C}[-\tau_K, 0]$, $K \in \mathbb{N}$ and for convenience $\tau_1 < \tau_2 < \cdots < \tau_K$. We associate with any solution x and any time $t \geq 0$, the 'state' of System (1), $x_t \in \mathcal{C}[-\tau_K, 0]$, where $x_t(s) = x(t+s)$. Although we only consider discrete-delay systems, the results of this paper may easily be extended to systems with distributed delay. For linear discrete-delay systems of the form (1), the system has a unique solution for any $\phi \in \mathcal{C}[-\tau_k, 0]$ and global, local, asymptotic and exponential stability are all equivalent.

Stability of Equations (1) may be certified through the use of Lyapunov-Krasovskii functionals - an extension of Lyapunov theory to systems with infinite-dimensional state-space. In particular, it is known that stability of linear time-delay systems is equivalent to the existence of a quadratic Lyapunov-Krasovskii functional of the form

$$\begin{aligned}V(\phi) = &\int_{-\tau_K}^{0} \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix}^T M(s) \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix} ds\\ &+ \int_{-\tau_K}^{0}\int_{-\tau_K}^{0} \phi(s)^T N(s,\theta)\phi(\theta)\, ds\, d\theta,\end{aligned} \tag{2}$$

where the Lie (upper-Dini) derivative of the functional is negative along any solution x of (1). That is,

$$\dot{V}(x_t) = \lim_{h\to 0} \frac{V(x_{t+h}) - V(x_t)}{h} \leq 0$$

for all $t \geq 0$. Furthermore, the unknown functions M and N may be assumed to be continuous in their respective arguments everywhere except possibly at points $H := \{x_1, \cdots, x_K\}$.

5. POSITIVE OPERATORS

The use of Lyapunov-Krasovskii functionals can be simplified by considering stability in the semigroup framework - a generalization of the concept of differential equations. Although the results of this paper do not require the semigroup architecture, we adopt this notation in order to simplify the concepts and avoid unnecessary notation. Sometimes known as the 'flow map', a 'strongly continuous semigroup' is an operator, $S(t) : Z \to Z$, defined by the Hilbert space Z, which represents the evolution of the state of the system so that for any solution x, $x_{t+s} = S(s)x_t$. Note that for a given Z, the semigroup may not exist even if the solution exists for any initial conditions in Z. Associated with a semigroup on Z is an operator $\mathcal{A}$, called the 'infinitesimal generator' which satisfies

$$\frac{d}{dt} S(t)\phi = \mathcal{A} S(t)\phi$$

for any $\phi \in X$. The space X is often referred to as the domain of the generator $\mathcal{A}$, and is the space on which the generator is defined and need not be a closed subspace of Z. In this paper we will refer to X as the 'state-space'. For System (1), following the approach in Curtain and Zwart [1995], we define $Z := \{\mathbb{R}^n \times L_2\}$ and

$$\mathcal{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix}(s) := \begin{bmatrix} A_0 x_1 + \sum_{i=1}^{K} A_i x_2(-\tau_i)\\ \dot{x}_2(s)\end{bmatrix}.$$

The state-space is $X := \{\begin{bmatrix}x_1\\ x_2\end{bmatrix} \in Z : x_2 \in W_2 \text{ and } x_2(0) = x_1\}$. Using these definitions of $\mathcal{A}$, Z and X, the "complete-quadratic" Lyapunov functional (2) can be compactly represented as

$$V(\phi) = \langle \begin{bmatrix}\phi(0)\\ \phi\end{bmatrix}, \mathcal{P}\begin{bmatrix}\phi(0)\\ \phi\end{bmatrix}\rangle_Z$$

where

$$\mathcal{P}\begin{bmatrix}x_1\\ x_2\end{bmatrix} := M(s)\begin{bmatrix}x_1\\ x_2(s)\end{bmatrix} + \int_{-\tau}^{0} N(s,\theta)\begin{bmatrix}x_1\\ x_2(\theta)\end{bmatrix}. \tag{3}$$

That is, the Lyapunov functional is defined by a multiplier and integral operator whose multiplier and kernel are unknown. Likewise, the derivative of the functional can be represented as

$$\dot{V}(\phi) = \langle \begin{bmatrix} \phi(0) \\ \phi \end{bmatrix}, \mathcal{P}\mathcal{A} \begin{bmatrix} \phi(0) \\ \phi \end{bmatrix} \rangle_Z + \langle \mathcal{A} \begin{bmatrix} \phi(0) \\ \phi \end{bmatrix}, \mathcal{P} \begin{bmatrix} \phi(0) \\ \phi \end{bmatrix} \rangle_Z.$$

In fact, it is known Curtain and Zwart [1995] that a strongly continuous semigroup defined by a linear operator $\dot{x} = \mathcal{A}x$ on Hilbert space X is exponentially stable if and only if there exists a positive operator P such that

$$\langle \mathcal{A}x, \mathcal{P}x \rangle_Z + \langle \mathcal{A}x, \mathcal{P}\mathcal{A}x \rangle_Z \leq -\epsilon \|x\|$$

for all $x \in X$.

6. THE SOS POSITIVITY CONDITIONS

In previous work, we noted that for $\mathcal{P}$ to be positive, the multiplier and kernel functions in (5) must satisfy certain pointwise conditions. Specifically, we have the following two theorems.

Theorem 1. Suppose $M : [-\tau, 0] \to \mathbb{S}^{n+m}$ is continuous except possibly at points τ_i and is bounded. Then the following are equivalent.

(i) There exists an $\epsilon > 0$ so that for all $c \in \mathbb{R}^n$ and continuous $y : [-\tau, 0] \to \mathbb{R}^m$,

$$\int_{-\tau}^{0} \begin{bmatrix} c \\ y(t) \end{bmatrix}^T M(t) \begin{bmatrix} c \\ y(t) \end{bmatrix} dt \geq \epsilon \|y\|_{L_2} \qquad (4)$$

(ii) There exist an $\eta > 0$ and a function $T : [-\tau, 0] \to \mathbb{S}^n$, continuous except possibly at points τ_i, which is bounded and satisfies

$$M(t) + \begin{bmatrix} T(t) & 0 \\ 0 & -\eta I \end{bmatrix} \geq 0 \quad \text{for all } t \in [-\tau, 0]$$

$$\text{and} \quad \int_{-\tau}^{0} T(t)\, dt = 0.$$

Theorem 2. Suppose $N : [-\tau, 0] \times [-\tau, 0] \to \mathbb{R}^{n \times n}$ is a polynomial of degree $2d$. Then the following are equivalent:

- $$\int_{-\tau}^{0} \int_{-h}^{0} x(s)^T N(s,t) x(t) ds dt \geq 0 \qquad \text{for all } x \in \mathcal{C}$$
- There exists a $Q \geq 0$ such that
$$N(s,t) + N(t,s)^T = Z_d(s)^T Q Z_d(t)$$

where $Z_d(s) = I_n \otimes Z(t)$ where $Z(t)$ is the length $d+1$ vector of monomials in variable t of degree d or less.

Note that the conditions associated with Theorem 2 do not imply pointwise positivity of the function $N(s,t)$ and hence is not actually a SOS constraint. However, the conditions associated with Theorem 2 are semidefinite programming constraints on the coefficients of N and hence can be implemented alongside SOS constraints in such Matlab toolboxes as SOSTOOLS. To differentiate positivity of the integral operator from positivity of the function N, we will use the **Notation:** $N \in \Sigma_k$ to denote that N satisfies the conditions of Theorem 2.

6.1 Inverting Positive Operators

Positive operators are always invertible. However, in Peet and Papachristodoulou [2009], we demonstrated that if M and N are polynomial and $\mathcal{P}$ is positive in the sense of the Theorems 1 and 2, then the inverse of $\mathcal{P}$ may be calculated directly as per the following theorem.

Theorem 3. Consider the linear operator P defined by

$$Px(s) = M(s)x(s) + \int_I N(s,\theta)x(\theta)d\theta,$$

where $M(s) > 0$ for all $s \in I$ and N has a representation $N(s,\theta) = Z(s)^T R Z(\theta)$ where Z is a vector of basis functions and $R > 0$. Define the linear operator $\hat{P}$ by

$$\hat{P}x(s) = M(s)^{-1}x(s) + \int_I \hat{N}(s,\theta)x(\theta)d\theta$$

Where

$$\hat{N}(s,\theta) = M(s)^{-1} Z(s)^T Q Z(\theta) M(\theta)^{-1}$$
$$Q = -R(S^{-1} + R)^{-1} S^{-1}$$
$$S = \int_I Z(s) \left(M(s)^{-1}\right)_{22} Z(s)^T ds.$$

Then $\hat{P}Px = P\hat{P}x = x$ for any integrable function x.

In this paper, we expand this inversion formula to cover a broader class of operator. Specifically, we have the following.

Theorem 4. Define $L = L_1 + L_2$, where

$$(L_1 x)(s) := K(s)x(0)$$
$$(L_2 x)(s) := M(s)x(s) + \int_I N(s,\theta)x(\theta)\, d\theta$$

Suppose that L_2 is invertible as per Theorem 3 with

$$(L_2^{-1}x)(s) = Q(s)x(s) + \int_I R(s,\theta)x(\theta)\, d\theta$$

and that $\rho(J) < 1$, where

$$J := Q(0)K(0) + \int_I R(0,s)K(s)\, ds.$$

Then

$$((L_1 + L_2)^{-1}x)(s) :=$$
$$Y_0(s)x(0) + Y_1(s)x(s) + \int_I Y_2(s,\theta)x(\theta)\, d\theta.$$

where

$$Y_0(s) = -H(s)(I+J)^{-1}Q(0)$$
$$Y_1(s) = Q(s)$$
$$Y_2(s,\theta) = R(s,\theta) - H(s)(I+J)^{-1}R(0,\theta)$$
$$H(s) = Q(s)K(s) + \int_I R(s,\theta)K(\theta)\, d\theta$$

Proof. All proofs omitted to meet 6-page conference restriction. Proofs will be published in the journal version of this paper.

7. A STRUCTURED OPERATOR

In order to create a dual stability condition, we must restrict ourselves to a class of operators which are self-adjoint with respect to the given inner-product and which preserve the structure of the state-space. Recall that the state-space is $X := \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in Z \ : \ x_2 \in W_2 \text{ and } x_2(0) = x_1 \right\}$. To preserve this structure, we consider operators of the form

$$(Px)(s) := \begin{bmatrix} y_1 \\ y_2(s) \end{bmatrix}$$
$$= \begin{bmatrix} (\tau Q_2(0,0) + Q_1(0))x_1 + \int_{-\tau}^{0} Q_2(0,s)x_2(s)ds \\ \tau Q_2(s,0)x_2(0) + Q_1(s)x_2(s) + \int_{-\tau}^{0} Q_2(s,\theta)x_2(\theta)d\theta \end{bmatrix} \tag{5}$$

Clearly, we have that P is a bounded linear operator and maps $P : X \to X$. Furthermore, P is self-adjoint with respect to the L_2 inner product, as indicated in the following lemma.

Lemma 5. Suppose that $Q_2(s,\theta) = Q_2(\theta,s)^T$ and $Q_1(s) \in \mathbb{S}^n$. Then the operator P, as defined in Equation (5), is self-adjoint with respect to the L_2 inner product.

Proof. All proofs omitted to meet 6-page conference restriction. Proofs will be published in the journal version of this paper.

Now that we have show that P is self-adjoint, we briefly discuss constructing the inverse of P. Let us represent P as

$$(Px)(s) = \begin{bmatrix} (\bar{P}x_2)(0) \\ (\bar{P}x_2)(s) \end{bmatrix}$$

Assuming that $\bar{P}$ is invertible, we may construct the operator

$$(P^{-1}y)(s) = \begin{bmatrix} (\bar{P}^{-1}y_2)(0) \\ (\bar{P}^{-1}y_2)(s) \end{bmatrix}$$

P^{-1} is a left inverse since left composition yields

$$(P^{-1}Px)(s) = \begin{bmatrix} (\bar{P}^{-1}\bar{P}x_2)(0) \\ (\bar{P}^{-1}\bar{P}x_2)(s) \end{bmatrix} = \begin{bmatrix} x_2(0) \\ x_2(s) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2(s) \end{bmatrix}.$$

Likewise P^{-1} is also a right inverse.

8. A DUAL STABILITY CONDITION

Now that we have parameterized a class of positive, invertible, structure-preserving self-adjoint operators, we may easily construct a dual stability condition for time-delay systems.

Theorem 6. Suppose that $\mathcal{A}$ generates a strongly continuous semigroup on L_2 with domain X. Further suppose there exists a positive operator $P : X \to X$ which is self-adjoint with respect to the L_2 inner product and

$$\langle \mathcal{A}P, x\rangle + \langle x, \mathcal{A}Px\rangle \le -\langle x, x\rangle$$

for all $x \in X$. Then the dynamical system $\dot{x}(t) = \mathcal{A}x$ generates an exponentially stable semigroup.

Proof. Because P is positive, self-adjoint, any inverse must also be positive and self-adjoint. Define the Lyapunov function

$$V(y) = \langle y, P^{-1}y\rangle$$

where $y \in X$ and with derivative

$$\begin{aligned} \dot{V}(y) &= \langle \dot{y}, P^{-1}y\rangle + \langle y, P^{-1}\dot{y}\rangle \\ &= \langle \mathcal{A}y, P^{-1}y\rangle + \langle y, P^{-1}\mathcal{A}y\rangle \\ &= \langle \mathcal{A}y, P^{-1}y\rangle + \langle P^{-1}y, \mathcal{A}y\rangle. \end{aligned}$$

Now define $x = P^{-1}y \in X$. Then $y = Px$ and

$$\begin{aligned} \dot{V}(y) &= \langle \mathcal{A}y, P^{-1}y\rangle + \langle P^{-1}y, \mathcal{A}y\rangle \\ &= \langle \mathcal{A}Px, x\rangle + \langle x, \mathcal{A}Px\rangle \\ &\le -\langle x, x\rangle = -\langle y, P^{-1}P^{-1}y\rangle \le -\alpha\langle y, y\rangle \end{aligned}$$

where the last inequality holds for some $\alpha > 0$ by positivity of P^{-1}. Negativity of the derivative of the Lyapunov function implies exponential stability in the square norm of the state by, e.g. Curtain and Zwart [1995] or by the invariance principle.

9. AN SOS TEST OF THE DUAL STABILITY CONDITION

To use Sum-of-Squares to test the dual stability condition, we quantify the relevant operators. Recall that the generator, $\mathcal{A}$ is defined as

$$(\mathcal{A}x)(s) = \begin{bmatrix} A_0x_1 + \sum_{i=1}^{K} A_ix_2(t-\tau_i) \\ \frac{d}{ds}x_2(s) \end{bmatrix}. \tag{6}$$

Note that although we do not include a distributed delay term, such a term may easily be included. In Equation 5, we have restricted P to have the form

$$(Px)(s) :=$$
$$\begin{bmatrix} (\tau Q_2(0,0) + Q_1(0))x_1 + \int_{-\tau}^{0} Q_2(0,s)x_2(s)ds \\ \tau Q_2(s,0)x_2(0) + Q_1(s)x_2(s) + \int_{-\tau}^{0} Q_2(s,\theta)x_2(\theta)d\theta \end{bmatrix} \tag{7}$$

for polynomial functions Q_1 and Q_2.

Theorem 7. Suppose there exist polynomials Q_1, Q_2, T such that the following hold

$$\begin{bmatrix} \tau Q_2(0,0) + Q_1(0) + T(s) & \tau Q_2(0,s) \\ \tau Q_2(s,0) & Q_1(s) \end{bmatrix} - \epsilon I \in \Sigma_s, \tag{8}$$

$$\int_{-\tau}^{0} T(s)ds = 0, \qquad Q_2(s,\theta) \in \Sigma_k \tag{9}$$

$$\begin{bmatrix} S_{11} + S_{11}^T + U_{11}(s) & *^T & *^T \\ S_{12}^T + U_{21}(s) & S_{22} + U_{22}(s) & *^T \\ S_{13}(s)^T & 0 & \dot{Q}_1(s) \end{bmatrix} - \epsilon I \in \Sigma_s,$$

$$S_{11} = A_0(\tau Q_2(0,0) + Q_1(0)) + \tau A_1 Q_2(-\tau, 0) + \frac{1}{2\tau}Q_1(0),$$
$$S_{12} = A_1 Q_1(-\tau),$$
$$S_{22} = -\frac{1}{\tau}Q_1(-\tau),$$
$$S_{13}(s) = \tau A_0 Q_2(0,s) + \tau A_1 Q_2(-\tau, s) + \tau \dot{Q}_2(s,0)^T,$$
$$\int_{-\tau}^{0} \begin{bmatrix} U_{11}(s) & *^T \\ U_{21}(s) & U_{22}(s) \end{bmatrix} ds = 0, \tag{10}$$
$$\frac{d}{ds}Q_2(s,\theta) + \frac{d}{d\theta}Q_2(s,\theta) \in \Sigma_k. \tag{11}$$

Then the system defined by Equation (1) is exponentially stable.

Proof. All proofs omitted to meet 6-page conference restriction. Proofs will be published in the journal version of this paper.

10. FULL-STATE FEEDBACK

Given a dual stability condition, it is easy to construct a synthesis condition for full-state feedback.

Corollary 8. Suppose that $\mathcal{A}$ generates a strongly continuous semigroup on L_2 with domain X and $B : U \to X$. Further suppose there exists a positive operator $P : X \to X$

which is self-adjoint with respect to the L_2 inner product and an operator $Z : X \to U$ such that

$$\langle (\mathcal{A}P + BZ)\, x, x\rangle + \langle x, (\mathcal{A}P + BZ)\, x\rangle \le -\langle x, x\rangle$$

for all $x \in X$. Let $K = ZP^{-1}$. Then the dynamical system $\dot{x}(t) = (\mathcal{A} + BK)\, x$ generates an exponentially stable semigroup.

Proof. The proof follows immediately from Theorem 6 with $Z = KP$.

11. A SUM-OF-SQUARES IMPLEMENTATION

To use Sum-of-Squares to synthesize stabilizing controllers, we first recall and quantify the relevant operators. We use as a baseline, the single-delay system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + B_0 u(t). \qquad (12)$$

In this case the generator, $\mathcal{A}$ is defined as

$$(\mathcal{A}x)(s) = \begin{bmatrix} A_0 x_1 + A_1 x_2(t-\tau) \\ \frac{d}{ds} x_2(s) \end{bmatrix}. \qquad (13)$$

Note that although we do not include a distributed delay term, such a term may easily be included. In Equation 5, we have restricted P to have the form

$$(Px)(s) := \begin{bmatrix} (\tau Q_2(0,0) + Q_1(0))x_1 + \int_{-\tau}^{0} Q_2(0,s) x_2(s) ds \\ \tau Q_2(s,0) x_2(0) + Q_1(s) x_2(s) + \int_{-\tau}^{0} Q_2(s,\theta) x_2(\theta) d\theta \end{bmatrix} \qquad (14)$$

for polynomial functions Q_1 and Q_2. Next, we note that $B : \mathbb{R}^m \to X$ has the simple form

$$(Bu)(s) := \begin{bmatrix} B_0 u \\ 0 \end{bmatrix}.$$

Finally, we must assume some structure for the variable operator $Z : X \to \mathbb{R}^m$, which we will assume has the form

$$(Zx)(s) = Z_0 x_1 + Z_1 x_2(-\tau) + \int_{-\tau}^{0} Z_2(s) x_2(s) ds$$

We are now ready to state our controller synthesis condition. For simplicity, we will only state the condition for the case of a single delay.

Theorem 9. Suppose there exist matrices Z_0, Z_1 and polynomials Q_1, Q_2, Z_2, U, T such that the following hold

$$\begin{bmatrix} \tau Q_2(0,0) + Q_1(0) + T(s) & \tau Q_2(0,s) \\ \tau Q_2(s,0) & Q_1(s) \end{bmatrix} + \epsilon I \in \Sigma_s, \qquad (15)$$

$$\int_{-\tau}^{0} T(s) ds = 0, \qquad Q_2(s,\theta) \in \Sigma_k \qquad (16)$$

$$\begin{bmatrix} S_{11} + S_{11}^T + L_{11} + L_{11}^T & *^T & *^T \\ S_{21} + L_{12}^T & S_{22} & *^T \\ S_{31}(s) + L_{13}(s)^T & 0 & \dot{Q}_1(s) \end{bmatrix} \qquad (17)$$

$$+ \begin{bmatrix} U_{11}(s) & U_{21}(s)^T & 0 \\ U_{21}(s) & U_{22}(s) & 0 \\ 0 & 0 & 0 \end{bmatrix} + \epsilon I \in \Sigma_s,$$

$$S_{11} = A_0(\tau Q_2(0,0) + Q_1(0)) + \tau A_1 Q_2(-\tau, 0) + \frac{1}{2\tau} Q_1(0),$$
$$S_{21} = Q_1(-\tau)^T A_1^T,$$
$$S_{22} = -\frac{1}{\tau} Q_1(-\tau),$$
$$S_{31}(s) = \tau Q_2(0,s)^T A_0^T + \tau Q_2(-\tau,s)^T A_1^T + \tau \dot{Q}_2(s,0),$$
$$L_{11} = B_0 Z_0$$
$$L_{12} = B_0 Z_1$$
$$L_{13} = \tau B_0 Z_2(s)$$
$$\int_{-\tau}^{0} \begin{bmatrix} U_{11}(s) & *^T \\ U_{21}(s) & U_{22}(s) \end{bmatrix} ds = 0,$$
$$\frac{d}{ds} Q_2(s,\theta) + \frac{d}{d\theta} Q_2(s,\theta) \in \Sigma_k. \qquad (18)$$

Then the delayed system (1) is full-state feedback stabilizable. Furthermore, let

$$(P_1^{-1}x)(s) = Y_0(s) x_1 + Y_1(s) x_2(s) + \int_{-\tau}^{0} Y_2(s,\theta) x(\theta) d\theta$$

be the inverse of

$$(P_1 x)(s) = \tau Q_2(s,0) x_2(0) + Q_1(s) x_2(s) + \int_{-\tau}^{0} Q_2(s,\theta) x_2(\theta) d\theta$$

as defined in Theorem 3. Then a stabilizing controller is

$$u(t) = K_0 x(t) + K_1 x(t-\tau) + \int_{-\tau}^{0} K_2(s) x(t+s) ds$$

where

$$K_0 = Z_0 Y_0(0) + Z_1 Y_0(-\tau) + \int_{-\tau}^{0} Z_2(s) Y_0(s) ds + Z_0 Y_1(0)$$
$$K_1 = Z_1 Y_1(-\tau)$$
$$K_2(s) = Z_0 Y_2(0,s) + Z_1 Y_2(-\tau,s) + Z_2(s) Y_1(s) + \int_{-\tau}^{0} Z_2(\theta) Y_2(\theta, s) d\theta.$$

Proof. All proofs omitted to meet 6-page conference restriction. Proofs will be published in the journal version of this paper.

12. NUMERICAL RESULTS

SOS Dual Stability Condition: Although not the primary focus of this paper, it is worth considering the merit of the dual stability condition on its own. For testing stability, the dual stability condition by itself does not seem to perform as well as the primal version, we described in Peet et al. [2009]. This is most likely due to the restrictive structure placed on the Lyapunov operator. As an example, the dual test is only able to prove stability of $\dot{x}(t) = -x(t-\tau)$ for $\tau \in [0, .7]$ for a polynomial degree of 8. However, any conservatism in the analysis condition seems to be lost in the synthesis conditions. This is likely due to the controller being optimized for the structure imposed in the dual conditions.

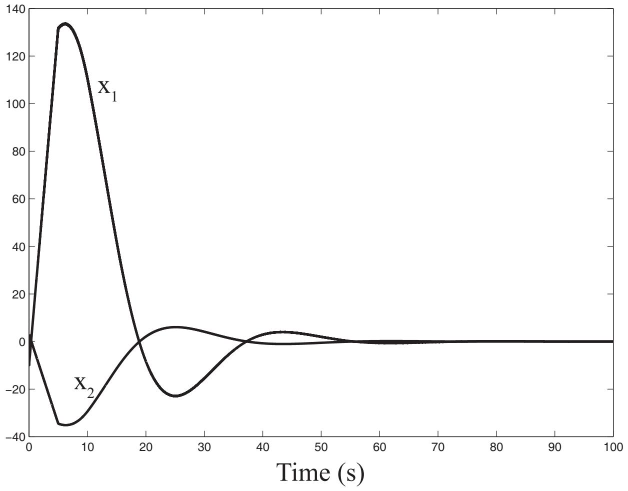

Fig. 1. A Matlab DDE23 simulation of System (19) with Controller (20) and delay $\tau = 5s$.

Synthesis Condition After a non-exhaustive search, we have yet to find a result which cannot be replicated using the method described here. However, this comparison is somewhat unfair, as most "state-feedback" results in the literature typically only use $x(t)$ or $x(t-\tau)$ and hence are working with more limited information. Often such results are appropriately justified by a presumed lack of knowledge of the delay. However, in such a case, the approach is not truly state feedback and should rather be considered output feedback, a topic we leave for future work. To illustrate our approach, we consider the commonly referenced dynamical system

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -2 & -.5 \\ 0 & -1 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t). \quad (19)$$

This system was stabilized using non-convex iterative/"tuning parameter" methods in e.g. Moon et al. [2001] and Fridman and Shaked [2002] for $\tau < 1$ (using only $x(t)$). We applied the methods of this paper for $\tau = 5$ using simple degree 2 polynomials and obtained the following exponentially stabilizing controller.

$$u(t) = \begin{bmatrix} -3601 \\ -944 \end{bmatrix}^T x(t) + \begin{bmatrix} -.00891 \\ .872 \end{bmatrix}^T x(t-\tau)$$
$$+ \int_{-5}^{0} \begin{bmatrix} 52.1 + 6.98s + .00839s^2 - .0710s^3 \\ 12.7 + 1.50s - .0407s^2 - .0190s^3 \end{bmatrix}^T x(t+s)ds \quad (20)$$

These results were obtained using a combination of Matlab, MuPad and SOSTOOLS to perform the optimization and controller reconstruction. The polynomial inversion was performed in MuPad and approximated using polynomial functions to simplify presentation. Simulations for fixed initial conditions were performed and can be seen in Figure 1.

13. CONCLUSION

In conclusion, we have proposed a new form of duality which allows us to convexify the controller synthesis problem for infinite-dimensional systems. This dual principle requires a Lyapunov operator which is positive, invertible, self-adjoint and preserves the structure of the state-space. We have used Sum-of-Squares to parameterize a class of such operators. We applied these results to generate full-state feedback controllers for time-delay systems. Numerical tests indicate the algorithm compares favorably with results in the literature, although this comparison is somewhat specious as we were unable to find any literature which uses true full-state feedback for control. The contribution of the present paper is not in the accuracy of the results, however, as these are likely conservative when compared to previous SOS results due to the highly structured nature of the operators used. Rather the contribution is in the convexification of the synthesis problem which opens the door for dynamic output-feedback H_∞ synthesis for infinite-dimensional systems. Future work will entail this extension as well as an expansion of the class of Lyapunov operators over which it is possible to optimize.

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