A Converse Sum of Squares Lyapunov Result With a Degree Bound

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Abstract—**Although sum of squares programming has been used extensively over the past decade for the stability analysis of nonlinear systems, several fundamental questions remain unanswered. In this paper, we show that exponential stability of a polynomial vector field on a bounded set implies the existence of a Lyapunov function which is a sum of squares of polynomials. In particular, the main result states that if a system is exponentially stable on a bounded nonempty set, then there exists a sum of squares Lyapunov function which is exponentially decreasing on that bounded set. Furthermore, we derive a bound on the degree of this converse Lyapunov function as a function of the continuity and stability properties of the vector field. The proof is constructive and uses the Picard iteration. Our result implies that semidefinite programming can be used to answer the question of stability of a polynomial vector field with a bound on complexity.**

Index Terms—**Computational complexity, linear matrix inequalities (LMIs), Lyapunov functions, nonlinear systems, ordinary differential equations, stability, sum-of-squares.**

I. Introduction

CONVEX optimization algorithms are used extensively in control. An important example is the use of semidefinite programming for solving linear control problems which have been reformulated as feasibility of a set of linear matrix inequalities (LMIs). Once a control problem has been reformulated as a convex optimization problem, computation of a solution is relatively straightforward. Indeed, the prevalence of LMIs in control has increased since the 1990s [1] to the point that once the solution of a control problem has been reformulated as the solution to an LMI, it is considered solved.

When it comes to nonlinear and infinite-dimensional systems, analysis and synthesis problems can sometimes be reformulated as convex optimization of polynomial variables subject to polynomial nonnegativity constraints using a Lyapunov framework. However, these optimization problems are not, at first glance, as easy to solve as LMIs. Verifying polynomial nonnegativity is, in fact, an NP hard problem. It is for this reason that researchers have looked at alternative, polynomial-time tests which are sufficient for proving nonnegativity of a polynomial, but which may not be necessary. One such relaxation is the existence of a sum of squares decomposition—a constraint which can be expressed as an LMI. Indeed, the ability to optimize over the cone of positive polynomials using the sum of squares relaxation has opened up new methods for nonlinear analysis and control in much the same way linear matrix inequalities were used to solve analysis and control problems for linear finite-dimensional systems. For references on early work on optimization of polynomials, see [2], [3], and [4]. For more recent work see [5] and [6]. For a recent review paper, see [7]. Today, there exist a number of software packages for optimization over positive polynomials, e.g., SOSTOOLS [8] and GloptiPoly [9].

While the sum of squares method is being increasingly used to solve nonlinear stability problems, there are still a number of basic unanswered questions about the limits of this approach. Unanswered questions include, for example, whether duality can be used to convexify either the nonlinear synthesis problem or the problem of estimating regions of attraction of equilibria. On the optimization side, it is unclear whether multi-core computing or sparsity can be used efficiently to increase the size and complexity of the models we consider. Finally, the question we consider in this paper is whether the use of sum of squares Lyapunov functions is conservative for the stability analysis of polynomial vector fields and if not, what degree of polynomial Lyapunov function is required to be necessary and sufficient for stability of a given vector field.

Note that this paper does not propose any fundamentally new methods for optimizing over sum of squares Lyapunov functions. Such work can be found in, e.g., [4], [10]–[12]. Instead, we consider existing methods and attempt to determine the accuracy and complexity of these methods. Specifically, we consider systems of the form

$$\dot{x}(t) = f(x(t))$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is polynomial. In particular, we address the question of whether a nonlinear system of this form which is exponentially stable on a bounded set will have a bounded-degree sum of squares Lyapunov function which establishes this property. This result adds to our previous work [13], wherein we were able to show that exponential stability on a bounded set implies the existence of an exponentially decreasing polynomial Lyapunov function on that set.

Research on the properties of converse Lyapunov functions relevant to the results of this paper can be found in, e.g., [14]–[17] and the overview in [18]. Infinitely-differentiable

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functions were explored in [19] and [20]. Other innovative results are found in [21] and [22]. The books [23] and [24] treat further converse theorems of Lyapunov. Continuity of Lyapunov functions is inherited from continuity of the vector field. An excellent treatment of this problem can be found in the text of Arnol'd [25].

Unlike the work in [13], the methods of this paper are closely tied to systems theory as opposed to approximation theory. Our method is to take a well-known form of converse Lyapunov function based on the solution map and use the Picard iteration to approximate the solution map. The advantage of this approach is that if the vector field is polynomial, the Picard iteration will also be polynomial. Furthermore, the Picard iteration inductively retains almost all the properties of the solution map. The result is a new form of iterative converse Lyapunov function, V_k. This function is discussed in Section VI.

The first practical contribution of this paper is to give a bound on the number of decision variables involved in the question of exponential stability of polynomial vector fields on bounded sets. Roughly speaking, this bound scales as $2q^{c(L/\lambda)}$, where c is a constant, q is the degree of the vector field, L is the Lipschitz constant for the vector field and λ is the decay rate of the system. This yields a bound on the number of decision variables because SOS functions of fixed degree can be parameterized by the set of positive matrices of fixed dimension. Furthermore, we note that the question of existence of a Lyapunov function with negative derivative is convex. Therefore, if the question of polynomial positivity on a bounded set is decidable, we can conclude that the problem of exponential stability of polynomial vector fields on that set is decidable. The further complexity benefit of using SOS Lyapunov functions is discussed in Section VIII.

The main result of the paper is stated and proven in Section VI. Preceding the main result is a series of lemmas that are used in the proof of the main theorem. In Section V, we show that the Picard iteration is contractive on a certain metric space; and in Section V-A we propose a new way of extending the Picard iteration. In Section V-B, we show that the Picard iteration approximately retains the differentiability properties of the solution map. The implications of the main result are then explored in Section VIII and Section VII. A detailed example is given in Section IX. The paper is concluded in Section X.

II. Main Result

Before we begin the technical part of the paper, we give a simplified version of the main result.

Theorem 1: Suppose that f is polynomial of degree q and that solutions of $\dot{x} = f(x)$ satisfy

$$\|x(t)\| \le K \|x(0)\| e^{-\lambda t}$$

for some $\lambda > 0$, $K \ge 1$ and for any $x(0) \in M$, where M is a bounded nonempty region of radius r. Then there exist $\alpha, \beta, \gamma > 0$ and a sum of squares polynomial $V(x)$ such that for any $x \in M$,

$$\alpha\|x\|^2 \le V(x) \le \beta\|x\|^2$$
$$\nabla V(x)^T f(x) \le -\gamma\|x\|^2.$$

Further, the degree of V will be less than $2q^{(Nk-1)}$, where $k(L,\lambda,K)$ is any integer such that $c(k) := \sum_{i=0}^{N-1} \left(e^{TL} + K(TL)^k\right)^i K^2(TL)^k < K$, and

$$c(k)^2 + \frac{\log 2K^2}{2\lambda} K \frac{(TL)^k}{T} (1 + c(k)) (K + c(k)) < \frac{1}{2},$$
$$c(k)^2 < \frac{\lambda}{KL \log 2K^2} \left(1 - (2K^2)^{-\frac{L}{\lambda}}\right)$$

and $N(L,\lambda,K)$ is any integer such that $NT > (\log 2K^2/2\lambda)$ and $T < (1/2L)$ for some T and where L is a Lipschitz bound on f on the ball of radius $4Kr$ centered at 0.

For fixed K, n, r, this theorem implies the existence of constants c_i, such that the degree can be upper bounded by $c_1 2q^{c_2(L/\lambda)}$.

III. Sum of Squares

Sum of squares (SOS) methods allow for the numerical solution of problems which can be formulated as the optimization of polynomial variables subject to global polynomial nonnegativity constraints. Consider, for example, the problem of ensuring that a polynomial $p(x) \in \mathbb{R}[x]$ satisfies $p(x) \ge 0$ for all $x \in \mathbb{R}^n$. This problem, while NP-hard [26], arises naturally when trying to construct Lyapunov functions for the stability analysis of dynamical systems. In [27], an algorithm was proposed to test for the existence of a sum of squares decomposition—i.e., a set of new polynomials $p_i(x)$ such that

$$p(x) = \sum_{i=1}^{k} p_i(x)^2. \tag{1}$$

New algorithms for finding an SOS decomposition appeared in the 1990s [28] but it was not until the turn of the century that the SOS question was reformulated as a linear matrix inequality [29]. In particular, (1) can be shown equivalent to the existence of a $Q \succeq 0$ and a vector of monomials $Z(x)$ of degree less than or equal half the degree of $p(x)$, such that

$$p(x) = Z(x)^T Q Z(x)$$

In the above representation, the matrix Q is not unique, in fact it can be represented as

$$Q = Q_0 + \sum_i \lambda_i Q_i \tag{2}$$

where Q_i satisfy $Z(x)^T Q_i Z(x) = 0$. The search for λ_i such that Q in (2) satisfies $Q \succeq 0$ is a linear matrix inequality, which can be solved using semidefinite programming. Moreover, if $p(x)$ has unknown coefficients that enter affinely in the representation (1), Semidefinite programming can be used to find values for these coefficients such that the resulting polynomial is SOS.

This latter observation allows us to *search* for polynomials that satisfy SOS conditions: the most important example is in the construction of Lyapunov functions, which is the topic of this paper. For more details, please see [10], [29].

IV. Notation and Background

The core concept we use in this paper is the Picard iteration. We use this to construct an approximation to the solution map and then use the approximate solution map to construct the Lyapunov function. Construction of the Lyapunov function will be discussed in more depth later on.

Denote the Euclidean ball centered at 0 of radius r by B_r. Consider an ordinary differential equation of the form

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad f(0) = 0 \tag{3}$$

where $x \in \mathbb{R}^n$ and f satisfies appropriate smoothness properties for local existence and uniqueness of solutions. The solution map is a function ϕ which satisfies

$$\frac{\partial}{\partial t}\phi(t,x) = f(\phi(t,x)) \quad \text{and} \quad \phi(0,x) = x.$$

A. Lyapunov Stability

The use of Lyapunov functions to prove stability of ordinary differential equations is well-established. The following theorem illustrates the use of Lyapunov functions.

Definition 2: We say that the system defined by the equations in (3) is exponentially stable on X if there exist $\gamma, K > 0$ such that for any $x_0 \in X$

$$\|\phi(t,x_0)\| \le K\|x_0\|e^{-\gamma t}$$

for all $t \ge 0$.

Theorem 3 (Lyapunov): Suppose there exist constants $\alpha, \beta, \gamma > 0$ and a continuously differentiable function V such that the following conditions are satisfied for all $x \in U \subset \mathbb{R}^n$:

$$\alpha\|x\|^2 \le V(x) \le \beta\|x\|^2$$
$$\nabla V(x)^T f(x) \le -\gamma\|x\|^2$$

Then we have exponential stability of System (3) on $\{x : \{y : V(y) \le V(x)\} \subset U\}$.

B. Fixed-Point Theorems

Definition 4: Let X be a metric space. A mapping $F : X \to X$ is *contractive* with coefficient $d \in [0,1)$ if

$$\|Fx - Fy\| \le d\|x - y\| \quad x, y \in X.$$

The following is a *Fixed-Point* Theorem.

Theorem 5 (Contraction Mapping Principle [30]): Let X be a complete metric space and let $F : X \to X$ be a contraction with coefficient d. Then there exists a unique $y \in X$ such that

$$Fy = y.$$

Furthermore, for any $x_0 \in X$

$$\|F^k x_0 - y\| \le d^k\|x_0 - y\|.$$

To apply these results to the existence of the solution map, we use the Picard iteration.

V. Picard Iteration

We begin by reviewing the Picard iteration. This is the basic mathematical tool we will use to define our approximation to the solution map and can be found in many texts, e.g., [31].

Definition 6: For given T and r, define the complete metric space

$$X_{T,r} := \left\{ q(t) : \sup_{t\in[0,T]} \|q(t)\| \le r, \ q \text{ is continuous.} \right\} \tag{4}$$

with norm

$$\|q\|_X = \sup_{t\in[0,T]} \|q(t)\|.$$

For a fixed $x \in B_r$ and $q \in X_{T,r}$, the *Picard Iteration* [32], is defined as

$$(Pq)(t) \triangleq x + \int_0^t f(q(s))\, ds.$$

In this paper, we also define the Picard iteration on functions $z(t,x)$ as

$$(Pz)(x,t) \triangleq x + \int_0^t f(z(x,s))\, ds.$$

We begin by showing that for any radius r, there exists a T such that the Picard iteration is contractive on $X_{T,2r}$ for any $x \in B_r$.

Lemma 7: Given $r > 0$, let $T < \min\{r/Q, 1/L\}$ where f has Lipschitz factor L on B_{2r} and $Q = \sup_{x\in B_{2r}} f(x)$. Then $P : X_{T,2r} \to X_{T,2r}$ and there exists some $\phi \in X_{T,2r}$ such that for $t \in [0,T]$ and $x \in B_r$

$$\frac{d}{dt}\phi(t) = f(\phi(t)), \quad \phi(0) = x$$

and for any $z \in X_{T,2r}$

$$\|\phi - P^k z\| \le (TL)^k\|\phi - z\|.$$

Proof: We first show that for $x \in B_r$, $P : X_{T,2r} \to X_{T,2r}$. If $q \in X_{T,2r}$, then $\sup_{t\in[0,T]}\|q(t)\| \le 2r$ and so

$$\begin{aligned}
\|Pq\|_X &= \sup_{t\in[0,T]} \left\| x + \int_0^t f(q(s)) \right\| ds \\
&\le \|x\| + \int_0^T \|f(q(s))\|\, ds \\
&\le r + \int_0^T \sup_{y\in B_{2r}} \|f(y)\|\, ds \\
&\le r + TQ < 2r.
\end{aligned}$$

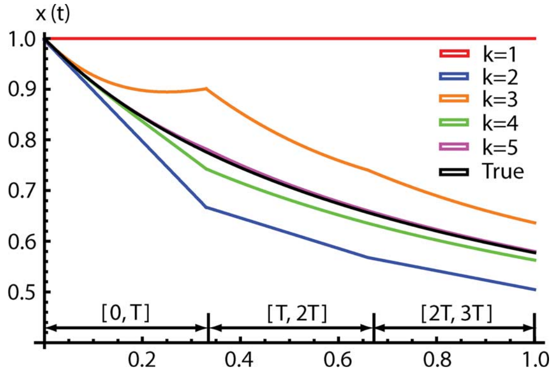

Fig. 1. The Solution map ϕ and the functions G_i^k for $k = 1, 2, 3, 4, 5$ and $i = 1, 2, 3$ for the system $\dot{x}(t) = -x(t)^3$. The interval of convergence of the Picard Iteration is $T = 1/3$.

Thus, we conclude that $Pq \in X_{T,2r}$. Furthermore, for $q_1, q_2 \in X_{T,2r}$

$$\begin{aligned}\|Pq_1 - Pq_2\|_X &= \sup_{t\in[0,T]} \left\| \int_0^t \left(f\left(q_1(s)\right) - f\left(q_2(s)\right)\right) ds \right\| \\ &\le \int_0^T \sup_{t\in[0,T]} \left\| f\left(q_1(s)\right) - f\left(q_2(s)\right)\right\| ds \\ &\le TL \sup_{t\in[0,T]} \|q_1(s) - q_2(s)\| \\ &= TL \|q_1 - q_2\|_X .\end{aligned}$$

Therefore, by the contraction mapping theorem, the Picard iteration converges on [0, T] with convergence rate $(TL)^k$. ■

A. Picard Extension Convergence Lemma

In this section, we propose an extension to the Picard iteration to intervals of arbitrary length. To begin, we divide the interval into subintervals sufficiently short so that the Picard iteration is guaranteed to converge on each subinterval. On each subinterval, we apply the Picard iteration using the final value of the solution estimate from the previous subinterval as the initial condition, x. For a polynomial vector field, the result is a piece-wise polynomial approximation which is guaranteed to converge on an arbitrary interval—see Fig. 1 for an illustration.

Definition 8: Suppose that the solution map ϕ exists on $t \in [0, \infty)$ and $\|\phi(t, x)\| \le K\|x\|$ for any $x \in B_r$. Suppose that f has Lipschitz factor L on B_{4Kr} and is bounded on B_{4Kr} with bound Q. Given $T < \min\{2Kr/Q, 1/L\}$, let $z = 0$ and define

$$G_0^k(t, x) := (P^k z)(t, x)$$

and for $i > 0$, define the functions G_i recursively as

$$G_{i+1}^k(t, x) := (P^k z)\left(t, G_i^k(T, x)\right).$$

The G_i^k are Picard iterations $P^k z(t, x)$ defined on each subinterval where we substitute the initial condition $x \mapsto G_{i-1}^k(t, x)$. Define the concatenation of these G_i^k as

$$G^k(t, x) := G_i^k(t - iT, x) \quad \forall \quad t \in [iT, iT + T] \quad \text{and} \quad i = 1, \cdots, \infty.$$

If f is polynomial, then the G_i^k are polynomial for any i, k and G^k is continuously differentiable in x for any k. The following lemma provides several properties for the functions G^k.

Lemma 9: Given $\delta > 0$, suppose that the solution map $\phi(t, x)$ exists on $t \in [0, \delta]$ and on $x \in B_r$. Further suppose that $\|\phi(t, x)\| \le K\|x\|$ for any $x \in B_r$. Suppose that f is Lipschitz on B_{4Kr} with factor L and bounded with bound Q. Choose $T < \min\{2Kr/Q, 1/L\}$ and integer $N > \delta/T$. Then let G^k and G_i^k be defined as above.

Define the function

$$c(k) = \sum_{i=0}^{N-1} \left(e^{TL} + K(TL)^k\right)^i K^2 (TL)^k.$$

Given any k sufficiently large so that $c(k) < K$, then for any $x \in B_r$,

$$\sup_{s\in[0,\delta]} \left\|G^k(s, x) - \phi(s, x)\right\| \le c(k)\|x\|. \tag{5}$$

Proof: Suppose $x \in B_r$. By assumption, the conditions of Lemma 7 are satisfied using $r' = 2Kr$. Let $z(t, x) = 0$. Define the convergence rate $d = TL < 1$. By Lemma 7,

$$\begin{aligned}\sup_{s\in[0,T]} \left\|G_0^k(s, x) - \phi(s, x)\right\| &= \sup_{s\in[0,T]} \left\|(P^k z)(s, x) - \phi(s, x)\right\| \\ &\le d^k \sup_{s\in[0,T]} \|\phi(s, x)\| \le K d^k \|x\|.\end{aligned}$$

Thus, (5) is satisfied on the interval [0, T]. We proceed by induction. Define

$$c_i(k) = \sum_{j=1}^{i} \left(e^d + K d^k\right)^j K^2 d^k.$$

and suppose that $\|G^k - \phi\|_\infty \le c_{i-1}(k)\|x\|$ on interval $[iT - T, iT]$. Then

$$\begin{aligned}&\sup_{s\in[iT,iT+T]} \left\|G^k(s, x) - \phi(s, x)\right\| \\ &= \sup_{s\in[iT,iT+T]} \left\|G_i^k(s - iT, x) - \phi(s, x)\right\| \\ &= \sup_{s\in[iT,iT+T]} \Big\|P^k z\left(s - iT, G_{i-1}^k(T, x)\right) \\ &\qquad - \phi\left(s - iT, \phi(iT, x)\right)\Big\| \\ &\le \sup_{s\in[iT,iT+T]} \Big\|P^k z\left(s - iT, G_{i-1}^k(T, x)\right) \\ &\qquad - \phi\left(s - iT, G_{i-1}^k(T, x)\right)\Big\| \\ &\quad + \sup_{s\in[iT,iT+T]} \Big\|\phi\left(s - iT, G_{i-1}^k(T, x)\right) \\ &\qquad - \phi\left(s - iT, \phi(iT, x)\right)\Big\| .\end{aligned}$$

We treat these final two terms separately. First note that

$$\begin{aligned}\left\|G_{i-1}^k(T, x)\right\| &\le \|\phi(iT, x)\| + \left\|\phi(iT, x) - G_{i-1}^k(T, x)\right\| \\ &\le K\|x\| + c_{i-1}(k)\|x\| \\ &\le \left(K + c_{i-1}(k)\right)\|x\|.\end{aligned}$$

Since $c_{i-1}(k) \leq c(k) < K$ and $x \in B_r$, we have $\|G_{i-1}^k(T,x)\| \leq (K + Kc_{i-1}(k))\|x\| \leq 2Kr$. Hence,

$$\begin{aligned}
\sup_{s\in[iT,iT+T]} &\left\|P^k z\left(s-iT, G_{i-1}^k(T,x)\right)\right. \\
&\qquad \left. -\phi\left(s-iT, G_{i-1}^k(T,x)\right)\right\| \\
&\leq \sup_{s\in[iT,iT+T]} d^k \left\|\phi\left(s-iT, G_{i-1}^k(T,x)\right)\right\| \\
&\leq Kd^k \left\|G_{i-1}^k(T,x)\right\| \\
&\leq Kd^k\left(K + c_{i-1}(k)\right)\|x\|.
\end{aligned}$$

Now, if $x \in B_r$, $\|\phi(s,x)\| \leq Kr$ and since $\|G_{i-1}^k(T,x)\| \leq 2Kr$ and f is Lipschitz on B_{4Kr}, it is well-known that

$$\begin{aligned}
&\sup_{s\in[iT,iT+T]} \left\|\phi\left(s-iT, G_{i-1}^k(T,x)\right) - \phi\left(s-iT, \phi(iT,x)\right)\right\| \\
&\leq \sup_{s\in[iT,iT+T]} e^{L(s-iT)} \left\|G_{i-1}^k(T,x) - \phi(iT,x)\right\| \\
&\leq e^{TL} c_{i-1}(k)\|x\|.
\end{aligned}$$

Combining, we conclude that

$$\begin{aligned}
&\sup_{s\in[iT,iT+T]} \left\|G_i^k(s-iT,x) - \phi(s,x)\right\| \\
&\quad\leq e^{TL} c_{i-1}(k)\|x\| + Kd^k\left(K + c_{i-1}(k)\right)\|x\| \\
&\quad= \left((e^d + Kd^k)c_{i-1}(k) + K^2 d^k\right)\|x\| = c_i(k)\|x\|.
\end{aligned}$$

Since $c_i(k) \leq c(k)$, and $\delta < NT$, by induction, we conclude that

$$\sup_{s\in[0,\delta]} \left\|G^k(s,x) - \phi(s,x)\right\| \leq c(k)\|x\|.$$

■

B. Derivative Inequality Lemma

In this critical lemma, we show that the Picard iteration approximately retains the differentiability properties of the solution map. The proof is based on induction, with a key step based on an approach in [33, Proof of Th. 4.14]. This lemma is then adapted to the extended Picard iteration introduced in the previous section.

Lemma 10: Suppose that the conditions of Lemma 7 are satisfied. Then for any $x \in B_r$ and any $k \geq 0$

$$\sup_{t\in[0,T]} \left\|\frac{\partial}{\partial x}(P^k z)(t,x)^T f(x) - \frac{\partial}{\partial t}(P^k z)(t,x)\right\| \leq \frac{(TL)^k}{T}\|x\|.$$

Proof: Begin with the identity for $k \geq 1$

$$\begin{aligned}
(P^k z)(t,x) &= x + \int_0^t f\left((P^{k-1}z)(s,x)\right) ds \\
&= x + \int_{-t}^0 f\left((P^{k-1}z)(s+t,x)\right) ds.
\end{aligned}$$

Then, by differentiating the right-hand side, we get

$$\begin{aligned}
&\frac{\partial}{\partial t}(P^k z)(t,x) \\
&= f\left((P^{k-1}z)(0,x)\right) \\
&\quad + \int_{-t}^0 \nabla f\left((P^{k-1}z)(s+t,x)\right)^T \frac{\partial}{\partial 1}(P^{k-1}z)(s+t,x)ds \\
&= f\left((P^{k-1}z)(0,x)\right) \\
&\quad + \int_0^t \nabla f\left((P^{k-1}z)(s,x)\right)^T \frac{\partial}{\partial s}(P^{k-1}z)(s,x)ds \\
&= f(x) + \int_0^t \nabla f\left((P^{k-1}z)(s,x)\right)^T \frac{\partial}{\partial s}(P^{k-1}z)(s,x)ds
\end{aligned}$$

where $(\partial/\partial i)f$ denotes partial differentiation of f with respect to its ith variable and

$$\begin{aligned}
&\frac{\partial}{\partial x}(P^k z)(t,x) \\
&\qquad = I + \int_0^t \nabla f\left((P^{k-1}z)(s,x)\right)^T \frac{\partial}{\partial x}(P^{k-1}z)(s,x)ds.
\end{aligned}$$

Now define for $k \geq 1$

$$y_k(t,x) := \frac{\partial}{\partial x}(P^k z)(t,x)^T f(x) - \frac{\partial}{\partial t}(P^k z)(t,x).$$

For $k \geq 2$, we have

$$\begin{aligned}
y_k(t,x) &:= \frac{\partial}{\partial x}(P^k z)(t,x)^T f(x) - \frac{\partial}{\partial t}(P^k z)(t,x) \\
&= -\int_0^t \nabla f\left((P^{k-1}z)(s,x)\right)^T \frac{\partial}{\partial t}(P^{k-1}z)(s,x)ds \\
&\quad + \int_0^t \nabla f\left((P^{k-1}z)(s,x)\right)^T \frac{\partial}{\partial x}(P^{k-1}z)(s,x) \\
&\qquad \times f(x)ds \\
&= \int_0^t \nabla f\left((P^{k-1}z)(s,x)\right)^T \\
&\qquad \cdot\left[\frac{\partial}{\partial x}(P^{k-1}z)(s,x)f(x) - \frac{\partial}{\partial s}(P^{k-1}z)(s,x)\right] ds \\
&= \int_0^t \nabla f((P^{k-1}z)(s,x))^T y_{k-1}(s,x)ds.
\end{aligned}$$

This means that since $(P^{k-1}z)(t,x) \in B_{2r}$, by induction

$$\begin{aligned}\sup_{[0,T]} \|y_k(t)\| \le & T \sup_{t\in[0,T]} \left\|\nabla f\left((P^{k-1}z)(t,x)\right)\right\| \\ & \times \sup_{t\in[0,T]} \|y_{k-1}(t,x)\| \\ \le & TL \sup_{t\in[0,T]} \|y_{k-1}(t,x)\| \le (TL)^{(k-1)} \\ & \times \sup_{t\in[0,T]} \|y_1(t,x)\| .\end{aligned}$$

For $k = 1$, $(Pz)(t,x) = x$, so $y_1(t) = f(x)$ and $\sup_{[0,T]}\|y_1(t)\| \le L\|x\|$. Thus,

$$\sup_{t\in[0,T]} \|y_k(t)\| \le \frac{(TL)^k}{T}\|x\|.$$

■

We now adapt this lemma to the extended Picard iteration.

Lemma 11: Suppose that the conditions of Lemma 9 are satisfied. Then for any $x \in B_r$

$$\sup_{t\in[0,T]} \left\| \frac{\partial}{\partial x}G^k(t,x)^T f(x) - \frac{\partial}{\partial t}G^k(t,x)\right\| \le \frac{(TL)^k}{T}\left(K + c(k)\right)\|x\|.$$

Proof: Recall that

$G^k(t,x) := G_i(t - iT, x)\, \forall\, t \in [iT, iT+T]$ and $i = 1,\cdots,\infty$.

and $G^k_{i+1}(t,x) = P^k z(t, G^k_i(T,x))$ where $z = 0$. Then for $t \in [iT, iT+T]$

$$\begin{aligned}&\left\| \frac{\partial}{\partial x}G^k(t,x)^T f(x) - \frac{\partial}{\partial t}G^k(t,x)\right\| \\ &= \left\| \frac{\partial}{\partial x}G^k(t-iT,x)^T f(x) - \frac{\partial}{\partial t}G^k_i(t-iT,x)\right\| \\ &= \left\| -\frac{\partial}{\partial t}P^k z\left(t-iT, G^k_i(T,x)\right) \right. \\ &\qquad \left. + \frac{\partial}{\partial x}P^k(t-iT, G^k_i(T,x))^T f(x)\right\| \\ &\le \frac{(TL)^k}{T}\left\|G^k_i(T,x)\right\|.\end{aligned}$$

As was shown in the proof of Lemma 9, $\|G^k_i(T,x)\| \le (K + c_i(k))\|x\|$. Thus, for $t \in [iT, iT+T]$

$$\left\| \frac{\partial}{\partial x}G^k(t,x)^T f(x) - \frac{\partial}{\partial t}G^k(t,x)\right\| \le \frac{(TL)^k}{T}\left(K + c_i(k)\right)\|x\|.$$

Since the c_i are non-decreasing

$$\sup_{t\in[0,\delta]} \left\| \frac{\partial}{\partial x}G^k(t,x)^T f(x) - \frac{\partial}{\partial t}G^k(t,x)\right\| \le \frac{(TL)^k}{T}\left(K + c(k)\right)\|x\|.$$

■

VI. Main Result—A Converse SOS Lyapunov Function

In this section, we combine the previous results to obtain a converse Lyapunov function which is also a sum of squares polynomial. Specifically, we use a standard form of converse Lyapunov function and substitute our extended Picard iteration for the solution map. Consider the system

$$\dot{x}(t) = f\left(x(t)\right), \quad x(0) = x_0. \tag{6}$$

Theorem 12: Suppose that f is polynomial of degree q and that system (6) is exponentially stable on M with

$$\|x(t)\| \le K\left\|x(0)\right\| e^{-\lambda t}$$

where M is a bounded nonempty region of radius r. Then there exist $\alpha, \beta, \gamma > 0$ and a sum of squares polynomial $V(x)$ such that for any $x \in M$,

$$\alpha\|x\|^2 \le V(x) \le \beta\|x\|^2 \tag{7}$$

$$\nabla V(x)^T f(x) \le -\gamma\|x\|^2. \tag{8}$$

Further, the degree of V will be less than $2q^{(Nk-1)}$, where $k(L,\lambda,K)$ is any integer such that $c(k) < K$ and

$$c(k)^2 + \frac{\log 2K^2}{2\lambda}K\frac{(TL)^k}{T}\left(1 + c(k)\right)\left(K + c(k)\right) < \frac{1}{2}, \tag{9}$$

$$c(k)^2 < \frac{\lambda}{KL\log 2K^2}\left(1 - (2K^2)^{-\frac{L}{\lambda}}\right) \tag{10}$$

where $c(k)$ is defined as

$$c(k) = \sum_{i=0}^{N-1}\left(e^{TL} + K(TL)^k\right)^i K^2 (TL)^k \tag{11}$$

and $N(L,\lambda,K)$ is any integer such that $NT > (\log 2K^2/2\lambda)$ and $T < (1/2L)$ for some T and where L is a Lipschitz bound on f on B_{4Kr}.

Proof: Define $\delta = \log 2K^2/2\lambda$ and $d = TL$. By assumption $N > (\delta/T)$. Next, we note that since stability implies $f(0) = 0$, f is bounded on any B_r with bound $Q = Lr$. Thus, for B_{4Kr}, we have the bound $Q = 4KrL$. By assumption, $T < (1/2L) = 2Kr/4KrL = 2Kr/Q$. Therefore, if k is defined as above, the conditions of Lemma 9 are satisfied. Define G^k as in Lemma 9. By Lemma 9, if k is defined as above, $\|G^k(s,x) - \phi(s,x)\| \le c(k)\|\phi(s,x)\|$ on $s \in [0,\delta]$ and $x \in B_r$.

We propose the following Lyapunov functions, indexed by k:

$$V_k(x) := \int_0^\delta G^k(s,x)^T G^k(s,x)ds$$

We will show that for any k which satisfies Inequalities (9), (10) and (11), then if we define $V(x) = V_k(x)$, we have that V satisfies the Lyapunov Inequalities (7) and (8) and has degree less than $2q^{(Nk-1)}$. The proof is divided into four parts:

Upper and Lower Bounded: To prove that V_k is a valid Lyapunov function, first consider upper boundedness. If $x \in B_r$ and $s \in [0, \delta]$. Then

$$\left\|G^k(s,x)\right\|^2 = \left\|\phi(s,x) + \left[G^k(s,x)^T - \phi(s,x)\right]\right\|^2 \leq \|\phi(s,x)\|^2 + \left\|\left[G^k(s,x)^T - \phi(s,x)\right]\right\|^2.$$

As per Lemma 9, $\|G^k(s,x) - \phi(s,x)\| \leq c(k)\|\phi(s,x)\| \leq Kc(k)\|x\|$. From stability we have $\|\phi(s,x)\| \leq K\|x\|$. Hence,

$$V_k(x) = \int_0^\delta \left\|G^k(s,x)\right\|^2 ds \leq \delta K^2\left(1 + c(k)^2\right)\|x\|^2.$$

Therefore, the upper boundedness condition is satisfied for any $k \geq 0$ with $\beta = \delta K^2(1 + c(k)^2) > 0$.

Next we consider the strict positivity condition. First we note

$$\|\phi(s,x)\|^2 = \left\|G^k(s,x) + \left[\phi(s,x) - G^k(s,x)\right]\right\|^2 \leq \left\|G^k(s,x)\right\|^2 + \left\|\phi(s,x) - G^k(s,x)\right\|^2$$

which implies

$$\left\|G^k(s,x)\right\|^2 \geq \|\phi(s,x)\|^2 - \left\|\phi(s,x) - G^k(s,x)\right\|^2.$$

By Lipschitz continuity of f, $\|\phi(s,x)\|^2 \geq e^{-2Ls}\|x\|^2$ and $\|G^k(s,x) - \phi(s,x)\| \leq Kc(k)\|x\|$. Thus,

$$V_k(x) = \int_0^\delta \left\|G^k(s,x)\right\|^2 ds \geq \left(\frac{1}{2L}(1 - e^{-2L\delta}) - \delta Kc(k)^2\right)\|x\|^2.$$

Therefore, for k as defined previously, $(1/2L)(1 - e^{-2L\delta}) - \delta Kc(k)^2 > 0$ and so the positivity condition holds for some $\alpha > 0$.

Negativity of the Derivative: Next, we prove the derivative condition. Recall

$$V_k(x) := \int_0^\delta G^k(s,x)^T G^k(s,x) ds = \int_t^{t+\delta} G^k(s-t,x)^T G^k(s-t,x) ds.$$

Then, since $\nabla V(x(t))^T f(x(t)) = (d/dt)V(x(t))$, we have by the Leibniz rule for differentiation of integrals

$$\begin{aligned}
&\frac{d}{dt}V_k(x(t)) \\
&= \left[G^k(\delta, x(t))^T G^k(\delta, x(t))\right] \\
&- \left[G^k(0, x(t))^T G^k(0, x(t))\right] \\
&- \int_t^{t+\delta} 2G^k(s-t, x(t))^T \frac{\partial}{\partial 1}G^k(s-t, x(t))\, ds \\
&+ \int_t^{t+\delta} 2G^k(s-t, x(t))^T \frac{\partial}{\partial 2}G^k(s-t, x(t)) \\
&\quad \times f(x(t))\, ds \\
&= \left\|G^k(\delta, x(t))\right\|^2 - \|x(t)\|^2 \\
&+ \int_0^\delta 2G^k(s, x(t))^T \left[\frac{\partial}{\partial 2}G^k(s, x(t))\, f(x(t)) \right. \\
&\qquad \left. - \frac{\partial}{\partial s}G^k(s, x(t))\right] ds
\end{aligned}$$

where recall $(\partial/\partial i)f$ denotes partial differentiation of f with respect to its ith variable. As per Lemma 11, we have

$$\left\|\frac{\partial}{\partial 2}G^k(t, x(t))^T f(x(t)) - \frac{\partial}{\partial 1}G^k(t, x(t))\right\| \leq \frac{d^k}{T}(K + c(k))\|x(t)\|$$

and as previously noted $\|G^k(\delta, x(t))\|^2 \leq (K^2 e^{-2\lambda(s-t)} + c(k)^2)\|x(t)\|^2$. Also, $\|G^k(s, x(t))\| \leq K(1 + c(k))\|x(t)\|$. We conclude that

$$\begin{aligned}
\frac{d}{dt}V_k(x(t)) &\leq \left(K^2 e^{-2\lambda\delta} + c(k)^2\right)\|x(t)\|^2 - \|x(t)\|^2 \\
&\quad + 2\delta\frac{d^k}{T}K(1 + c(k))(K + c(k))\|x(t)\|^2 \\
&= \left(K^2 e^{-2\lambda\delta} + c(k)^2 - 1 \right. \\
&\quad \left. + 2\delta K\frac{d^k}{T}(1 + c(k))(K + c(k))\right)\|x(t)\|^2.
\end{aligned}$$

Therefore, we have strict negativity of the derivative since

$$\begin{aligned}
&K^2 e^{-2\lambda\delta} + c(k)^2 + 2\delta\frac{d^k}{T}(1 + c(k))(K + c(k)) \\
&= T\frac{1}{2} + c(k)^2 + 2\frac{K\log 2K^2}{2\lambda}\frac{d^k}{T}(1 + c(k))(K + c(k)) < 1.
\end{aligned}$$

Thus, $(d/dt)V_k(x(t)) \leq -\gamma\|x(t)\|^2$ for some $\gamma > 0$.

Sum of Squares: Since f is polynomial and z is trivially polynomial, $(P^k z)(s,x)$ is a polynomial in x and s. Therefore,

$V_k(x)$ is a polynomial for any $k > 0$. To show that V_k is sum of squares, we first rewrite the function

$$V_k(x) = \sum_{i=1}^{N} \int_{iT-T}^{iT} \left[G_i^k(s - iT, x)^T G_i^k(s - iT, x)\right] ds.$$

Since $G_i^k z$ is a polynomial in all of its arguments, $G_i^k(s - iT, x)^T G_i^k(s - iT, x)$ is sum of squares. It can therefore be represented as $R_i(x)^T Z_i(s)^T Z_i(s) R_i(x)$ for some polynomial vector R_i and matrix of monomial bases Z_i. Then

$$\begin{aligned} V_k(x) &= \sum_{i=1}^{N} R_i(x)^T \int_{iT-T}^{iT} Z_i(s)^T Z_i(s) ds \, R_i(x) \\ &= \sum_{i=1}^{N} R_i(x)^T M_i R_i(x) \end{aligned}$$

where $M_i = \int_{iT-T}^{iT} Z_i(s)^T Z_i(s) ds \geq 0$ is a constant matrix. This proves that V_k is sum of squares since it is a sum of sums of squares.

We conclude that $V = V_k$ satisfies the conditions of the theorem for any k which satisfies (9) and (10).

Degree Bound: Given a k which satisfies the inequality conditions on $c(k)$, we consider the resulting degree of G^k, and hence, of V_k. If f is a polynomial of degree q, and y is a polynomial of degree d in x, then Py will be a polynomial of degree $\max\{1, dq\}$ in x. Thus, since $z = 0$, the degree of $P^k z$ will be q^{k-1}. If $N > 1$, then the degree of G_i^k will be q^{Nk-1}. Thus, the maximum degree of the Lyapunov function is $2q^{(Nk-1)}$. ∎

We conclude this section with several comments on Theorem 12 and its proof. In the proof of Theorem 12, the integration interval, δ was chosen such that the conditions will always be feasible for some $k > 0$. However, this choice may not be optimal. Numerical experimentation has shown that a better degree bound may be obtained by varying this parameter in the proof. Note that while calculation of an accurate degree bound is best done numerically, we can still give a conservative upper bound on this value in order to gain insight into the complexity of the stability question for nonlinear systems. Specifically, for fixed K, n, and r, Theorem 12 implies the existence of constants c_i, such that the degree can be upper bounded by $c_1 2q^{c_2(L/\lambda)}$. This implies that the key factor in the complexity of the decision problem is the ratio, L/λ of roughness of the vector field versus magnitude of the decay rate.

Before we conclude this section, let us comment on the form of the converse Lyapunov function:

$$V_k(x) := \int_0^{\delta} G^k(s, x)^T G^k(s, x) ds.$$

Our Lyapunov function is defined using an approximation of the solution map. A dual approach to the solution of the Hamilton–Jacobi–Bellman Equation was taken in [34] using occupation measures instead of Picard iteration. Indeed, the dual space of the sum of squares Lyapunov functions can be understood in terms of moments of such occupation measures [35].

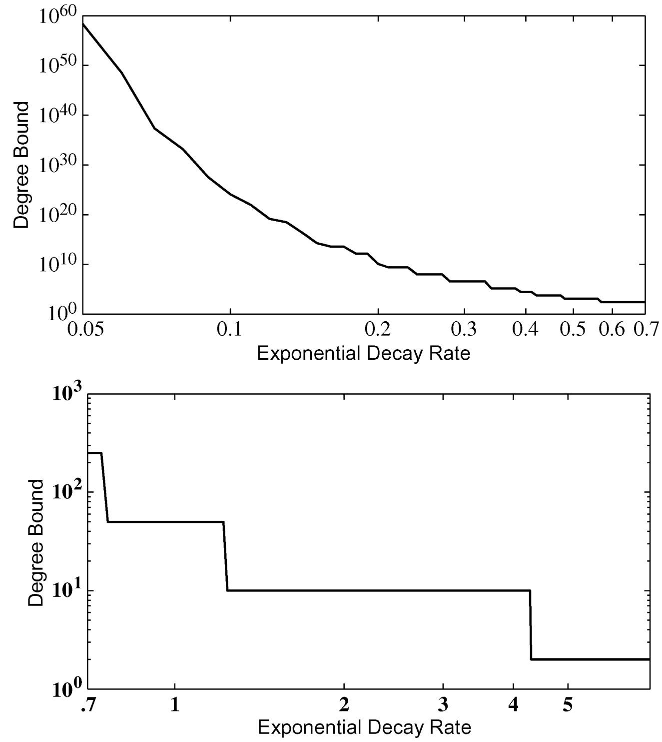

Fig. 2. Degree bound versus exponential convergence rate for $K = 1.2$, $r = L = 1$, $q = 5$. Domains $\lambda < .7$ and $\lambda > .7$ are plotted separately for clarity.

As a final note, the proof of Theorem 12 also holds for time-varying systems. Indeed the original proof was for this case. However, because sum of squares is rarely used for time-varying systems, the result has been simplified to improve clarity of presentation.

A. Numerical Illustration

To illustrate the degree bound and hence the complexity of analyzing a nonlinear system, we plot in Fig. 2 the degree bound versus the exponential convergence rate of the system. For given parameters, this bound is obtained by numerically searching for the smallest k which satisfies the conditions of Theorem 12. The convergence rate parameter can be viewed as a metric for the accuracy of the sum of squares approach: suppose we have a degree bound as a function of convergence rate, $d(\lambda)$. If it is not possible to find a sum of squares Lyapunov function of degree $d(\lambda)$ proving stability, then we know that the convergence rate of the system must be less than λ.

As can be seen, as the convergence rate increases, the degree bound decreases super-exponentially, so that at $\lambda = 2.4$, only a quadratic Lyapunov function is required to prove stability. For cases where high accuracy is required, the degree bound increases quickly; scaling approximately as $q^{L/\lambda}$. To reduce the complexity of the problem, in come cases less conservative bounds on the degree can be found by considering the monomial terms in the vector field. If the complexity is still unacceptably high, then one can consider the use of parallel computing: unlike single-core processing, parallel computing power continues to increase exponentially. For a discussion on using parallel computing to solve polynomial optimization problems, we refer to [36].

VII. Quadratic Lyapunov Functions

In this section, we briefly explore the implications of our result for the existence of quadratic Lyapunov functions proving exponential stability of nonlinear systems. Specifically, we look at when the theorem predicts the existence of a degree bound of 2. We first note that when the vector field is linear, then $q = 1$, which implies that $2q^{(Nk-1)} = 2$ independent of N and k. Recall N is the number of Picard iterations, k is the number of extensions and q is the degree of the polynomial vector field, f. Hence, an exponentially stable linear system has a quadratic Lyapunov function—which is not surprising.

Instead we consider the case when $q \neq 1$. In this case, for a quadratic Lyapunov function, we require $N = k = 1$—a single Picard iteration and no extensions. By examining the proof of Theorem 12, we see that if the conditions of the theorem are satisfied with $N = k = 1$ then $V(x) = x^T x$ is a Lyapunov function which establishes exponential stability of the system. Since this is perhaps the most commonly used form of Lyapunov function, it is worth considering how conservative it is when applied to nonlinear systems of the form

$$\dot{x}(t) = f(x(t)).$$

In the following corollary, we give sufficient conditions on the vector field and decay rate for the Lyapunov function $x^T x$ to prove exponential stability.

Corollary 1: Suppose that system (6) is exponentially stable with

$$\|x(t)\| \leq K \|x(0)\| e^{-\lambda t}$$

for some $\lambda > 0$, $K \geq 1$ and for any $x(0) \in M$, where M is a bounded nonempty region of radius r. Let L be a Lipschitz bound for f on B_{4Kr}. Suppose that there exists some $(1/2)L > \delta > 0$ such that

$$K^2 e^{-2\lambda\delta} + c_1^2 + 2K\delta L(1 + c_1)(K + c_1) < 1$$

and $K\delta L < 1$, where $c_1 = K^2 \delta L$. Let $V(x) = x^T x$. Then for any $x \in M$,

$$\dot{V}(x) = \nabla x^T f(x) \leq -\beta \|x\|^2.$$

for some $\beta > 0$.

Proof: We reconsider the proof of Theorem 12. This time, we set $N = k = 1$ and $T = \delta$ and determine if there exists a $\delta = T < (1/2L)$ which satisfies the upper-boundedness, lower-boundedness and derivative conditions. Because $V(x) = x^T x$, the upper and lower boundedness conditions are immediately satisfied. The derivative negativity condition is

$$K^2 e^{-2\lambda\delta} + c(1)^2 + 2K\delta L (1 + c(1)) (K + c(1)) < 1$$

where $c(1) = c_1 = K^2 \delta L$. This is satisfied by the statement of the theorem. ■

Note that neither the size of the region we consider nor the degree of the vector field plays any role in determining the degree bound. To illustrate the conditions for existence of a quadratic Lyapunov function, we plot the required decay rate versus the Lipschitz continuity factor in Fig. 3 for $K = 1.2$. This plot shows that as the Lipschitz continuity of the vector field increases (and the field becomes less smooth), the conservatism of using the quadratic Lyapunov function $x^T x$ increases.

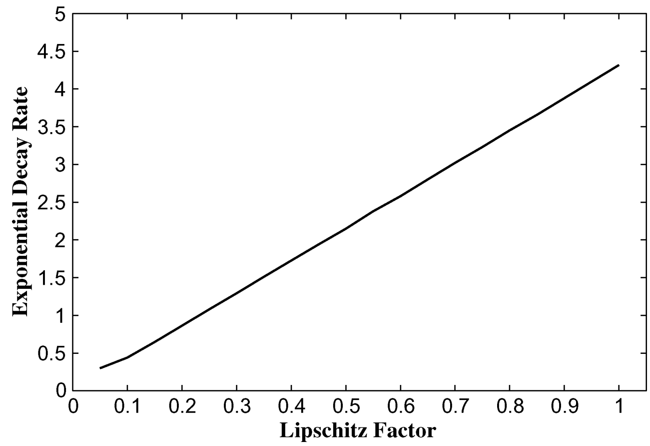

Fig. 3. Required decay rate for a quadratic Lyapunov function versus Lipschitz bound for $K = 1.2$.

VIII. Implications for Sum of Squares Programming

In this section, we consider the implications that the above results have on sum of squares programming.

A. *Bounding the Number of Decision Variables*

Because the set of continuously differentiable functions is an infinite-dimensional vector space, the general problem of finding a Lyapunov function is an infinite-dimensional feasibility problem. However, the set of sum of squares Lyapunov functions with bounded degree is finite-dimensional. The most significant implication of our theorem is a bound on the number of variables in the problem of determining stability of a nonlinear vector field. The nonlinear stability problem can now be expressed as a feasibility problem of the following form.

Theorem 13: For a given λ, let $2d$ be the degree bound associated with Theorem 12 and define $N = (n + d)!/n!d!$. If System (6) is exponentially stable on M with decay rate λ or greater, the following is feasible for some $\alpha, \beta, \gamma > 0$:

$$\begin{aligned} &\textbf{Find} : P \in \mathbb{S}^N : \\ &P \geq 0 \\ &\alpha\|x\|^2 \leq Z(x)^T P Z(x) \leq \beta\|x\|^2 \quad \text{for all } x \in M \\ &\nabla \left(Z(x)^T P Z(x)\right)^T f(x) \leq -\gamma\|x\|^2 \quad \text{for all } x \in M \end{aligned}$$

where $Z(x)$ be the vector of monomials in x of degree d or less.

Proof: The proof follows immediately from the fact that a polynomial V of degree $2d$ is SOS if and only if there exists a $P \geq 0$ such that $V(x) = Z(x)^T P Z(x)$. ■

Our condition bounds the number of variables in the feasibility problem associated with Theorem 13. If M is semialgebraic, then the conditions in Theorem 13 can be enforced using sum of squares and the Positivstellensatz [37]. The complexity of solving the optimization problem will depend on the complexity of the Positivstellensatz test. If positivity on a semialgebraic set is decidable, as indicated in [38], this implies the question of exponential stability on a bounded set is decidable.

B. Local Positivity

Another implication of our result is that it reduces the complexity of enforcing the positivity constraint. As discussed in Section III, semidefinite programming is used to optimize over the cone of sums of squares of polynomials. There are several different ways the stability conditions can be enforced. For example, we have the following theorem.

Theorem 14: Suppose there exist polynomial V and sum of squares polynomials s_1, s_2, s_3, and s_4 such that the following conditions are satisfied for $\alpha, \gamma > 0$:

$$\begin{aligned} V(x) - \alpha\|x\|^2 &= s_1(x) + g(x)s_2(x) \\ -\nabla V(s)^T f(x) - \gamma\|x\|^2 &= s_3(x) + g(x)s_4(x). \end{aligned}$$

Then we have exponential stability of (6) on $\{x : \{y : V(y) \le V(x)\} \subset U\}$.

The complexity of the conditions associated with Theorem 14 is determined by the four sum of squares variables, s_i. Theorem 14 uses the Positivstellensatz multipliers s_2 and s_4 to ensure that the Lyapunov function need only be positive and decreasing on the region $X = \{x : g(x) \ge 0\}$. However, as we now know that the Lyapunov function can be assumed SOS, we can eliminate the multiplier s_2, reducing complexity of the problem.

Theorem 15: Suppose there exist polynomial V and sum of squares polynomials s_1, s_2, and s_3 such that the following conditions are satisfied for $\alpha, \gamma > 0$:

$$\begin{aligned} V(x) - \alpha\|x\|^2 &= s_1(x) \\ -\nabla\left(V(x) + \alpha\|x\|^2\right)^T f(x) - \gamma\|x\|^2 &= s_2(x) + g(x)s_3(x). \end{aligned}$$

Then we have exponential stability of (6) for any $x(0)$ such that $\{y : V(y) \le V(x(0))\} \subset X$ where $X := \{x : g(x) \ge 0\}$.

This simplification reduces the size of the SOS variables by 25% (from 4 to 3). If the semialgebraic set X is defined using several polynomials (e.g., a hypercube), then the reduction in the number of variables can approach 50%. SDP solvers are typically of complexity $O(n^6)$, where n is the dimension of the symmetric matrix variable. In the above example, we reduced $n = 4N$ to $n = 3N$. Thus, this simplification can potentially decrease computation by a factor of 82%.

IX. Numerical Example

In this section, we use the Van-der-Pol oscillator to illustrate how the degree bound influences the accuracy of the stability test. The zero equilibrium point of the Van-der-Pol oscillator is unstable. In reverse-time, however, this equilibrium is stable with a domain of attraction bounded by the well-known forward-time limit-cycle. The reverse-time dynamics are as follows:

$$\begin{aligned} \dot{x}_1(t) &= -x_2(t) \\ \dot{x}_2(t) &= -\mu\left(1 - x_1(t)^2\right)x_2(t) + x_1(t). \end{aligned}$$

For simplicity, we choose $\mu = 1$. On a ball of radius r, the Lipschitz constant can be found from $L = \sup_{x \in B_r} \|Df(x)\|$, where $\|\cdot\|$ is the maximum singular value norm. We find a

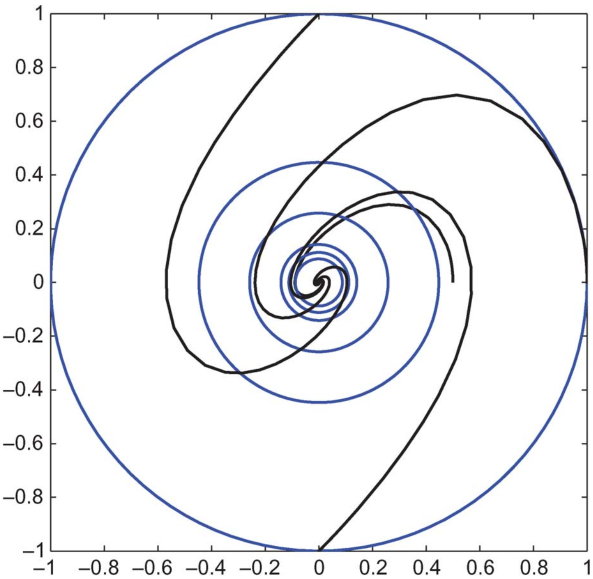

Fig. 4. Plot of trajectories of the Van-der-Pol oscillator. We estimate the overshoot parameter as $K \cong 1$.

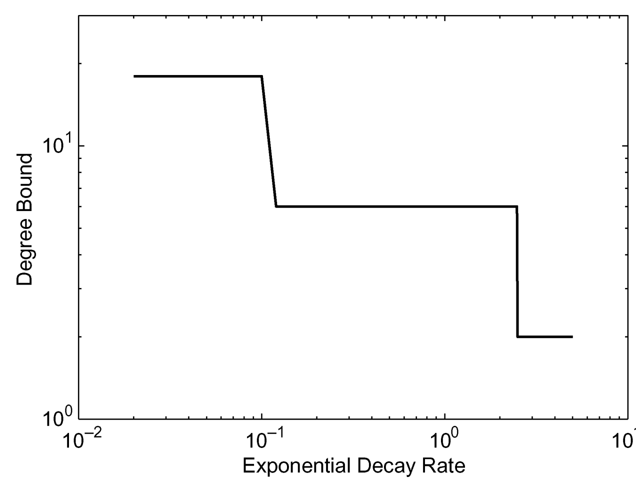

Fig. 5. Degree bound for the Van-der-Pol oscillator as a function of decay rate.

Lipschitz constant for the Van-der-Pol oscillator on radius $r = 1$ to be 2.1. Numerical simulations indicate $K \cong 1$, as illustrated in Fig. 4. Given these parameters, the degree bound plot is illustrated in Fig. 5. Note that the choice of $K = 1$ dramatically improves the degree bound. Numerical simulation shows the decay rate to be a relatively constant $\lambda = .542$ throughout the unit ball. This is illustrated in Fig. 6. This gives us an estimate of the degree bound as $d = 6$.

To find the converse Lyapunov function associated with this degree bound we construct the Picard iteration:

$$(Pz)(t,x) = x + \int_0^t f(0)ds = x.$$

$$\begin{aligned} (P^2 z)(t,x) &= x + \int_0^t f\left(Pz(s,x)\right)ds \\ &= x + \int_0^t f(x)ds = x + f(x)t. \end{aligned}$$

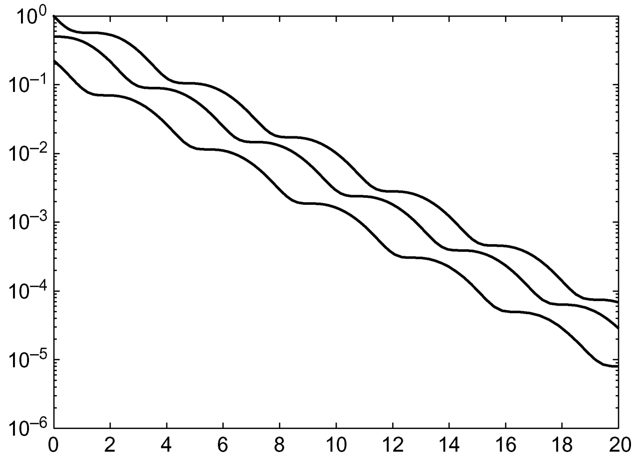

Fig. 6. A semi-log plot of $\|x\|$ for three trajectories. We estimate $\lambda = .542$ for the Van-der-Pol oscillator.

The converse Lyapunov function is

$$
\begin{aligned}
V(x) &= \int_0^\delta \left(P^2 z(s,x)\right)^T \left(P^2 z(s,x)\right) ds \\
&= \int_0^\delta (x + f(x)s)^T (x + f(x)s)\, ds \\
&= \int_0^\delta \begin{bmatrix} x \\ f(x) \end{bmatrix}^T \begin{bmatrix} I \\ sI \end{bmatrix} [I\ sI] \begin{bmatrix} x \\ f(x) \end{bmatrix} ds \\
&= \begin{bmatrix} x \\ f(x) \end{bmatrix}^T \int_0^\delta \begin{bmatrix} I & sI \\ sI & s^2 I \end{bmatrix} ds \begin{bmatrix} x \\ f(x) \end{bmatrix} \\
&= \begin{bmatrix} x \\ f(x) \end{bmatrix}^T \begin{bmatrix} \delta I & \delta^2/2I \\ \delta^2/2I & \delta^3/3I \end{bmatrix} \begin{bmatrix} x \\ f(x) \end{bmatrix}.
\end{aligned}
$$

If $\delta = T = 1/2L = 1/4$, for the Van-der-Pol oscillator, we get the SOS Lyapunov function

$$
\begin{aligned}
192 \cdot V(x) &= \begin{bmatrix} x \\ f(x) \end{bmatrix}^T \begin{bmatrix} 48I & 6I \\ 6I & I \end{bmatrix} \begin{bmatrix} x \\ f(x) \end{bmatrix} \\
&= \begin{bmatrix} x \\ f(x) \end{bmatrix}^T \begin{bmatrix} 6.93I & 2.45I \\ 2.45I & I \end{bmatrix}^2 \begin{bmatrix} x \\ f(x) \end{bmatrix} \\
&= \begin{bmatrix} 6.93x + 2.45f(x) \\ 2.45x + f(x) \end{bmatrix}^T \begin{bmatrix} 6.93x + 2.45f(x) \\ 2.45x + f(x) \end{bmatrix} \\
&= (6.93x_1 - 2.45x_2)^2 \\
&\quad + \left(2.45(x_1 + x_1^2 x_2) + 4.48x_2\right)^2 \\
&\quad + (2.45x - x_2)^2 + (x_1 + x_1^2 x_2 + 1.45x_2)^2.
\end{aligned}
$$

As per the previous discussion, we use SOSTOOLS to verify that this Lyapunov function proves stability. Note that we must show the function is decreasing on the ball of radius $r = .25$, as the Lipschitz bound used in the theorem is for the ball of radius B_{4r}. We are able to verify that the Lyapunov function is decreasing on the ball of radius $r = .25$. Some level sets of this Lyapunov function are illustrated in Fig. 7. Through experimentation, we find that when we increase the ball to radius $r = 1$, the Lyapunov function is no longer decreasing. We also found that the quadratic Lyapunov function $V(x) = x^T x$ is not decreasing on the ball of radius $r = .25$. Although we believe that our degree bound is somewhat conservative, these results indicate the conservatism is not excessive.

To explore the limits of the SOS approach, for degree bound 2, 4, 6, 8, and 10, we find the maximum unit ball on which we are able to find a sum of squares Lyapunov function. We then use the largest sublevel set of this Lyapunov function on which the trajectories decrease as an estimate for the domain of attraction of the system. These level sets are illustrated in Fig. 8. We see that as the degree bound increases, our estimate of the domain of attraction improves.

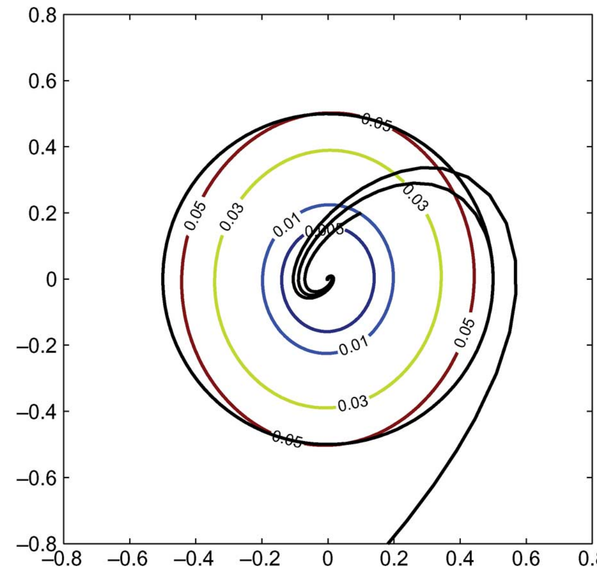

Fig. 7. Level sets of the converse Lyapunov function, with ball of radius $r = .25$.

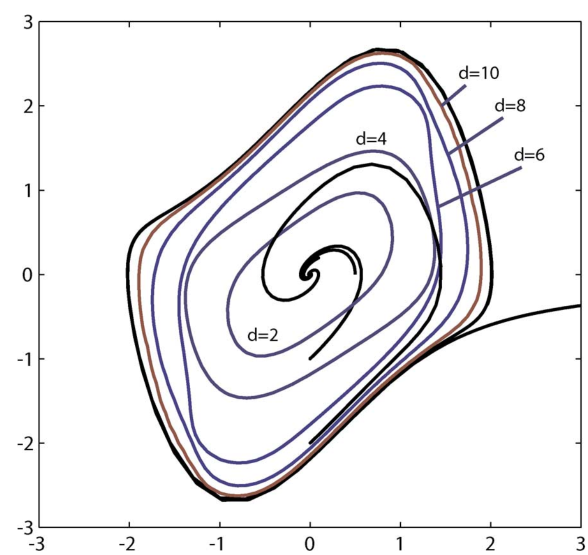

Fig. 8. Best invariant region versus degree bound with limit cycle.

X. Conclusion

In this paper, we have used the Picard iteration to construct an approximation to the solution map on arbitrarily long intervals. We have used this approximation to prove that exponential stability of a polynomial vector field on a bounded set implies the existence of a Lyapunov function which is a sum of squares of polynomials with a bound on the degree. This implies that the question of exponential stability on a bounded set may be decidable with a complexity determined by the ratio, L/λ, of roughness of the vector field to magnitude of the decay rate. Furthermore, the converse Lyapunov function we have used in this paper is relatively easy to construct given the vector field and may find applications in other areas of control. The main result also holds for time-varying systems.

Recently, there has been interest in using semidefinite programming for the analysis on nonlinear systems using sum of squares. This paper clarifies several questions on the application of this method. We now know that exponential stability on a bounded set implies the existence of an SOS Lyapunov function and we know how complex this function may be. It has been recently shown that *globally* asymptotically stable vector fields do not always admit sum of squares Lyapunov functions [39]. Still unresolved is the question of the existence of polynomial Lyapunov functions for stability of globally *exponentially* stable vector fields.

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