Constructing Piecewise Polynomial Lyapunov Functions Over Arbitrary Convex Polytopes Using Handelman Basis

Abstract— **We introduce a new algorithm to check the local stability and compute the region of attraction of isolated equilibria of nonlinear systems with polynomial vector fields. First, we consider an arbitrary convex polytope that contains the equilibrium in its interior. Then, we decompose the polytope into several convex sub-polytopes with a common vertex at the equilibrium. Then, by using Handelman's theorem, we derive a new set of affine equality and inequality feasibility conditions -solvable by linear programming- on each sub-polytope. The solution to this feasibility problem yields a piecewise polynomial Lyapunov function on the entire polytope. To the best of the authors' knowledge, this is the first effort that utilizes Handelman's theorem to construct piecewise polynomial Lyapunov functions on sub-divided arbitrary polytopes. In a computational complexity analysis, we show that for large number of states and large degrees of the Lyapunov function, the complexity of the proposed feasibility problem is less than the complexity of the semi-definite programs associated with Sum-of-Squares and Polya's algorithms. Using different types of convex polytopes, we assess the accuracy of the algorithm in estimating the region of attraction of the equilibrium point of reverse-time Van Der Pol oscillator.**

I. INTRODUCTION

Analyzing the stability of dynamical systems by applying Lyapunov-based methods often requires checking the feasibility of a set of inequalities. Solving these feasibility problems can also be thought of as deciding the positivity of a set of functions. In the case of dynamical systems with polynomial vector fields, choosing polynomial Lyapunov functions results in the problem of deciding the positivity of a set of polynomials. However, it has been shown that the general problem of deciding the positivity of polynomials is NP-hard [1]. One approach to solve this decision problem is to find a representation which certifies the positivity of the polynomial and can be calculated in polynomial time.

Thus far, several positivity certificates for polynomials have been proposed [2]. One of the most well-known positivity certificates is the Sum of Squares (SOS) representation. Perhaps the most fundamental result on the existence of SOS representations for positive polynomials is Artin's theorem [3]. It states that for every positive-semi-definite polynomial f, there exists a polynomial g such that $g^2 f$ is SOS. However, it has been shown that there exists no single polynomial g (a uniform denominator) that satisfies Artin's theorem for every positive-semi-definite polynomial f [4]. The problem of searching for the SOS representation of a positive polynomial can be formulated as a Semi-Definite Program (SDP) [5] using the SOS algorithm [6]. Since solving SDPs is computationally tractable, SOS algorithm has been used extensively in stability analysis and control of a variety of complex systems. These include stability analysis of nonlinear systems [7], [8], robust stability analysis of switched and hybrid systems [9], and stability analysis of time-delay systems [10], [11]. Moreover, it was shown in [12] that there exists an SOS Lyapunov function with a degree bound for every exponentially stable nonlinear system on a bounded region.

Another well-known positivity certificate is given by Polya's Theorem [13]. It states that every positive homogeneous polynomial defined on the positive orthant can be represented as a sum of even-powered monomials with positive coefficients. Different variants and extensions of Polya's theorem have been utilized in analysis and control, e.g., Polya's certificate on a simplex and multi-simplex for robust stability analysis [14], [15], non-negativity certificate for polynomials with zeros [16] and a certificate for positivity on the entire real domain [17].

Using SOS or Polya's algorithm for stability analysis of large-scale systems requires setting up and solving large SDPs. For example, using the SOS algorithm to construct a degree 6 Lyapunov function on a hypercube for a system with 5 states requires solving an SDP with $\sim 10^6$ variables with an LMI of size $\sim 10^3$. However, unlike the SOS algorithm, the SDPs associated with Polya's algorithm possess a block-diagonal structure. This enables efficient decentralization of the computation required for setting up and solving the SDP. Decentralized implementations of Polya's algorithm applied to robust and nonlinear stability problems can be found in [18], [19]. However, Polya's algorithm can only be used to obtain positivity certificates on simple geometries such as simplices and hypercubes. To find certificates of positivity on a broader class of convex sets, one can use Handelman's theorem [20]. The theorem gives a complete parameterization of the cone of positive polynomials defined on arbitrary convex polytopes by giving a representation in the basis of affine functions (Handelman basis) with positive coordinates. A degree bound for Handelman's certificate is given in [21]. A non-negativity certificate expressed in Handelman basis and with arbitrary positive polynomial coordinates is given in [22].

Thus far, Handelman's theorem has been used to construct polynomial Lyapunov functions on hypercubes [23]. In this paper, we employ a decomposition strategy which allows us to use Handelman's theorem to construct piecewise polynomial Lyapunov functions on arbitrary convex polytopes. First, we decompose a given convex polytope into a set of convex sub-polytopes that share a common vertex at the origin. Then, on each sub-polytope we derive a set of constraints that are affine in the unknown coefficients of the Lyapunov function expressed in the Handelman basis. Finally, we derive an additional set of affine constraints which ensure the zeroness of the Lyapunov function at the origin and its continuity on the entire polytope. The feasibility of the derived affine constraints can be checked using linear programming. Our complexity analysis shows that for certain decompositions, the computational cost of solving the linear program associated with the derived constraints scales polynomially with the state-space dimension. We also show that for large state-space dimensions, high degrees of the Lyapunov functions and certain decompositions, the complexity of the proposed LP is lower than the complexities of the SDPs associated with SOS and Polya's algorithms. Finally, we perform a numerical experiment to evaluate the accuracy of the proposed method in computing the invariant set of the reverse-time Van Der Pol oscillator.

II. Definitions and Notation

In this section, we define convex polytopes, facets of polytopes, decompositions, interfaces, Handelman bases and representations with respect to polytopes.

Definition 1: (Convex Polytope) Given the set of vertices $P := \{p_i \in \mathbb{R}^n, i = 1, \cdots, K\}$, define the convex polytope Γ_P as

$$\Gamma_P := \{p \in \mathbb{R}^n : p = \sum_{i=1}^{K} \mu_i p_i : \mu_i \in [0,1] \text{ and } \sum_{i=1}^{K} \mu_i = 1\}.$$

Every convex polytope can be represented as

$$\Gamma := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\},$$

for some $w_i \in \mathbb{R}^n, u_i \in \mathbb{R}, i = 1, \cdots, K$. Throughout the paper, every polytope that we use contains origin.

Definition 2: (Facets of a polytope) Given a polytope of the form

$$\Gamma := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\},$$

define the $i-$th facet of the polytope Γ as

$$\zeta^i := \{x \in \mathbb{R}^n : w_i^T x + u_i = 0 \text{ and } w_j^T x + u_j \geq 0 \text{ for } j \in \{1, \cdots, K\}, j \neq i\}.$$

Definition 3: (D-decomposition) Given a polytope of the form

$$\Gamma := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\},$$

we say $D_\Gamma := \{D_i\}_{i=1,\cdots,L}$ is a D-decomposition of Γ, if $D_i \subset \mathbb{R}^n$ are polytopes of the form

$$D_i := \{x \in \mathbb{R}^n : h_{i,j}^T x + g_{i,j} \geq 0, j = 1, \cdots, m_i\}$$

such that $\cup_{i=1}^{L} D_i = \Gamma$, $\cap_{i=1}^{L} D_i = \{0\}$ and $\text{int}(D_i) \cap \text{int}(D_j) = \emptyset$ for $i, j \in \{1, \cdots, L\}, i \neq j$.

Definition 4: (Collection of adjacent sub-polytopes) Given a D-decomposition $\{D_i\}_{i=1,\cdots,L}$ of polytope Γ, where $D_i := \{x \in \mathbb{R}^n : h_{i,j}^T x + g_{i,j} \geq 0,\ j = 1, \cdots m_i\}$, for any i, we denote the polytopes adjacent to D_i as

$$\{H_j^i\} := \{H_j^i \in D_\Gamma : h_{j,k_1}^T x + g_{j,k_1} = h_{i,k_2}^T x + g_{i,k_2} \text{ for some } k_1, k_2 \in \mathbb{N}\}$$

Definition 5: (Handelman basis associated with a polytope) Given a polytope of the form

$$\Gamma := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0,\ i = 1, \cdots, K\},$$

we define the set of Handelman bases, indexed by

$$\alpha \in E_d := \{\alpha \in \mathbb{N}^K : |\alpha|_1 \leq d\}, \text{ as}$$

$$B(\Gamma) := \{\lambda_\alpha(x) : \lambda_\alpha(x) = \prod_{i=1}^{K} (w_i^T x + u_i)^{\alpha_i},\ \alpha \in E_d\}.$$

Definition 6: (Restriction of a polynomial to a facet) Given a polytope of the form $\Gamma := \{x \in \mathbb{R}^n : w_i^T x + u_i,\ i = 1 \cdots K\}$, and a polynomial $f(x)$ of the form

$$f(x) = \sum_{\alpha \in E_d} b_\alpha \prod_{i=1}^{K} (w_i^T x + u_i)^{\alpha_i},$$

define the restriction of $f(x)$ to the k-th facet of Γ as

$$f(x)|_{\zeta^k} := \sum_{\alpha \in E_d : \alpha_k = 0} b_\alpha \prod_{i=1}^{K} (w_i^T x + u_i)^{\alpha_i}.$$

Definition 7: (Upper Dini Derivative) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map. Then, define the upper Dini derivative of a function $V : \mathbb{R}^n \to \mathbb{R}$ in the direction of $f(x)$ as

$$D^+(V, f(x)) = \limsup_{h \to 0^+} \frac{V(x + hf(x)) - V(x)}{h}.$$

It can be shown that for a continuously differentiable $V(x)$,

$$D^+(V, f(x)) = \langle \nabla V(x), f(x) \rangle.$$

Finally, given a polynomial of the form

$$f(x) = \sum_{\alpha \in E_d} b_\alpha \prod_{i=1}^{K} (w_i^T x + u_i)^{\alpha_i},\ x \in \mathbb{R}^n$$

with $N = \sum_{j=0}^{d} \frac{(j+n-1)!}{j!(n-1)!}$ monomials, we will use the map $\mathcal{C} : \mathbb{R}[x] \to \mathbb{R}^N$, $\mathcal{C}(f) := [c_1, \cdots, c_N]$, where c_j is the coefficient of the j-th monomial in $f(x)$ in the lexicographical ordering. We have derived c_j as a function of $b_\alpha, d, u_i, w_i, i = 1, \cdots, K$ in the Appendix.

III. Background and Problem Statement

We address the problem of local stability of nonlinear systems of the form

$$\dot{x} = f(x), \tag{1}$$

around the origin, where $f : \mathbb{R}^n \to \mathbb{R}^n$. Consider the classic Lyapunov result for nonlinear systems.

Theorem 1: For any $\Omega \subset \mathbb{R}^n$ such that $0 \in \Omega$, if there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ and continuous positive definite functions W_1, W_2, W_3,

$$W_1(x) \leq V(x) \leq W_2(x) \text{ for } x \in \Omega \text{ and}$$
$$D^+(V, f(x)) \leq -W_3(x) \text{ for } x \in \Omega,$$

then system (1) is asymptotically stable on $\{x : \{y : V(y) \leq V(x)\} \subset \Omega\}$.

In this paper, we construct piecewise polynomial Lyapunov functions, which may not have classical derivatives. As such, we use Dini derivatives instead, which exist for all continuous functions, thus for piecewise polynomial functions as well.

In this paper, we solve the following problem.

Problem statement: Given the vertices $p_i \in \mathbb{R}^n, i = 1, \cdots, K$, we would like to find $\max_{s\in\mathbb{R}^+} s$ and a polynomial $V(x)$ such that $V(x)$ satisfies the conditions of Theorem 1 on the convex polytope $\{x \in \mathbb{R}^n : x = \sum_{i=1}^{K} \mu_i p_i : \mu_i \in [0,s]$ and $\sum_{i=1}^{K} \mu_i = s\}$.

Every convex polytope can be represented as a semi-algebraic set. It has been shown that deciding whether a real-valued polynomial is positive over a semi-algebraic set is NP-hard [1]. To avoid intractability, we use a result given by Handelman [20] which defines a tractable certificate for positivity of polynomials on convex polytopes.

Theorem 2: (Handelman's Theorem) Given $w_i \in \mathbb{R}^n, u_i \in \mathbb{R}, i = 1, \cdots, K$, define the polytope $\Gamma := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\}$. If polynomial $f(x) > 0$ for all $x \in \Gamma$, then there exist $b_\alpha \geq 0$, $\alpha \in \mathbb{N}^m$ such that for some $d \in \mathbb{N}$,

$$f(x) = \sum_{\alpha\in E_d} b_\alpha (w_1^T x + u_1)^{\alpha_1} \cdots (w_K^T x + u_K)^{\alpha_K}.$$

Recall that the set of exponents E_d is defined in Definition 5. For any arbitrary positive polynomial and some polytope Γ, Theorem 2 gives a representation of the polynomial using the Handelman basis $B(\Gamma)$ associated with the polytope Γ and with non-negative coordinates b_α. In other words, the theorem parameterizes every positive polynomial using the positive orthant. We now present the converse of Theorem 2, which gives a certificate of nonnegativity of a polynomial on a polytope using the Handelman basis.

Proposition 1: Given $w_i \in \mathbb{R}^n, u_i \in \mathbb{R}, i = 1, \cdots, K$, define the polytope $\Gamma := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\}$. Suppose the polynomial $f(x)$ of degree d is of the form

$$f(x) = \sum_{\alpha\in E_d} b_\alpha (w_1^T x + u_1)^{\alpha_1} \cdots (w_K^T x + u_K)^{\alpha_K},$$

where each $b_\alpha \geq 0$. Then, $f(x) \geq 0$ for all $x \in \Gamma$.

In the following section, we use Proposition 1 to construct a set of affine constraints for a given polynomial on a given polytope. The feasibility of these constraints can be tested using linear programming. Any feasible point yields a certificate for the positivity of the polynomial on the given polytope.

IV. Problem setup

We first present some lemmas necessary for the proof of our main result.

Lemma 1: (Zeroness at the origin) Let $D_\Gamma := \{D_i\}_{i=1,\cdots,L}$ be a D-decomposition of a polytope Γ, where for each i

$$D_i := \{x \in \mathbb{R}^n : h_{i,j}^T x + g_{i,j} \geq 0, j = 1, \cdots, m_i\},$$

and let

$$f_i(x) = \sum_{\alpha\in E_d} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_i}, \;\; i = 1, \cdots L.$$

Then, for all $i \in \{1, \cdots, L\}$ and all

$$\alpha \in \hat{E}_d := \{\alpha \in \mathbb{N}^{m_i} : |\alpha|_1 \leq d, \; \alpha_k = 0 \text{ for all } k : g_{i,k} = 0\},$$

if $b_{\alpha,i} = 0$, then each $f_i(0) = 0$.

Proof: Note that

$$f_i(x) = \sum_{\alpha\in E_d\setminus\hat{E}_d} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_i} + \sum_{\alpha\in\hat{E}_d} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_i}.$$

First, for each $\alpha \in E_d\setminus\hat{E}_d$, there exists at least one $k \in \{1, \cdots, m_i\}$ such that $g_{i,k} = 0$ and $\alpha_k > 0$. Thus, at $x = 0$,

$$\sum_{\alpha\in E_d\setminus\hat{E}_d} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_i} = 0. \qquad (*)$$

Second, for each $\alpha \in \hat{E}_d$, $b_{\alpha,i} = 0$. Thus, for any x

$$\sum_{\alpha\in\hat{E}_d} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_i} = 0. \qquad (**)$$

Thus, from (*) and (**), we show that if $b_{\alpha,i} = 0$ for all $\alpha \in \hat{E}_d$, then $f_i(0) = 0$. ■

Lemma 2: (Continuity of piecewise polynomial functions) Let $D_\Gamma := \{D_i\}_{i=1,\cdots,L}$ be a D-decomposition of a polytope Γ, where for each i

$$D_i := \{x \in \mathbb{R}^n : h_{i,j}^T x + g_{i,j} \geq 0, j = 1, \cdots, m_i\},$$

and let $\{H_s^i\}$ be the set of all polytopes adjacent to D_i. For $i = 1, \cdots, L$, define

$$f_i(x) = \sum_{\alpha\in E_d} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_i}.$$

If

$$C(f_i(x)|_{\zeta^k}) = C(f_q(x)|_{\zeta^l})$$

for every facet $\zeta^k \subset D_i, k = 1, \cdots, m_i$ and every facet $\zeta^l \subset D_q, l = 1, \cdots, m_i$ and every $q \in \{1, \cdots, L\}$ such that $D_q \in \{H_s^i\}$ and $\zeta^k = \zeta^l$, then $f_i(x) = f_q(x)$ for all $x \in D_i \cap D_q$.

Proof: For any $i \in \{1 \cdots, L\}$ and $k \in \{1, \cdots, m_i\}$, we can write

$$f_i(x) = \sum_{\alpha\in E_d:\alpha_k>0} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_i} + \sum_{\alpha\in\hat{E}_d:\alpha_k=0} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_i}.$$

Since for any $x \in \zeta^k \subset D_i$, $h_{i,k}^T x + g_{i,k} = 0$, it follows that for any $x \in \zeta^k$,

$$\sum_{\alpha\in E_d:\alpha_k>0} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_j} = 0.$$

Therefore, for any $x \in \zeta^k$

$$f_i(x) = \sum_{\alpha\in E_d:\alpha_k=0} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_j}.$$

Then, from Definition 6 we have

$$f_i(x) = f_i(x)|_{\zeta^k}. \;\; (*)$$

Furthermore, since for every facet $\zeta^k \subset D_i, K = 1,\cdots,m_i$ and every facet $\zeta^l \subset D_q, l = 1,\cdots,m_i$ and every $q \in \{1,\cdots,L\}$ such that $D_q \in \{H_s^i\}$ and $\zeta^k = \zeta^l$ we have

$$\mathcal{C}(f_i(x)|_{\zeta^k}) = \mathcal{C}(f_q(x)|_{\zeta^l}),$$

it follows that

$$f_i(x)|_{\zeta^k} = f_q(x)|_{\zeta^l}. \qquad (**)$$

Last, for any $x \in \zeta^l \subset D_q$,

$$f_q(x)|_{\zeta^l} = f_q(x). \qquad (***)$$

Thus, if for every facet $\zeta^k \subset D_i, K = 1,\cdots,m_i$ and every facet $\zeta^l \subset D_q, l = 1,\cdots,m_i$ and every $q \in \{1,\cdots,L\}$ such that $D_q \in \{H_s^i\}$ and $\zeta^k = \zeta^l$,

$$\mathcal{C}(f_i(x)|_{\zeta^k}) = \mathcal{C}(f_q(x)|_{\zeta^l})$$

then from (*), (**) and (***), $f_i(x) = f_q(x)$ for all $x \in D_i \cap D_q$.

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Theorem 3: (Main result) Consider system (1) with polynomial vector field $f(x)$ of degree d_f. Given w_i, $h_{i,j} \in \mathbb{R}^n$, u_i, $g_{i,j} \in \mathbb{R}$, define the polytope

$$\Gamma := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1,\cdots,K\},$$

with D-decomposition $D_\Gamma := \{D_i\}_{i=1,\cdots,L}$, where

$$D_i := \{x \in \mathbb{R}^n : h_{i,j}^T x + g_{i,j} \geq 0, j = 1,\cdots,m_i\}.$$

Suppose for each $i \in \{1,\cdots,L\}$, $\{H_s^i\}$ is the set of polytopes adjacent to each D_i. Then, for some $d \in \mathbb{N}$ and for $i = 1,\cdots,L$, if there exist

$$b_{\alpha,i} \geq 0 \quad \text{for} \quad \alpha \in E_d \text{ and} \qquad (2)$$

$$c_{\beta,i} \leq 0 \quad \text{for} \quad \beta \in E_{d+d_f-1}, \qquad (3)$$

such that

I. $b_{\alpha,i} > 0$ and $c_{\beta,i} < 0$ for at least one $\alpha \in E_d$ and $\beta \in E_{d+d_f-1}$,

II. $b_{\alpha,i} = 0$ for all $\alpha \in \hat{E}_d := \{\alpha \in \mathbb{N}^{m_i} : |\alpha|_1 \leq d,$ $\alpha_k = 0,$ for all $k : g_{i,k} = 0\}$,

III. $\mathcal{C}(V_i(x)|_{\zeta^k}) = \mathcal{C}(V_q(x)|_{\zeta^l})$ for every facet $\zeta^k \subset D_i, k = 1,\cdots,m_i$ and every facet $\zeta^l \subset D_q, l = 1,\cdots,m_i$ and every $q \in \{1,\cdots,L\}$ such that $D_q \in \{H_s^i\}$ and $\zeta^k = \zeta^l$,

IV. $\mathcal{C}(\langle \nabla V_i(x), f(x)\rangle) = \mathcal{C}(Z_i(x))$,

where

$$V_i(x) := \sum_{\alpha \in E_d} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_j} \text{ and} \qquad (4)$$

$$Z_i(x) := \sum_{\beta \in E_{d+d_f-1}} c_{\beta,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\beta_j},$$

then system 1 is asymptotically stable at the origin.

Proof: Let us choose $V(x)$ as

$$V(x) = V_i(x) = \sum_{\alpha \in E_d} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_j}$$

for $x \in D_i$, $i = 1,\cdots,L$.

In order to show that $V(x)$ is a Lyapunov function for system 1, we need to prove the following:

1) $V_i(x) > 0$ for all $x \in D_i \backslash \{0\}$, $i = 1,\cdots,L$,
2) $D^+(V_i(x), f(x)) < 0$ for all $x \in D_i \backslash \{0\}$, $i = 1,\cdots,L$,
3) $V(0) = 0$,
4) Continuity of $V(x)$.

Then, by Theorem 1, it follows that the system 1 is asymptotically stable at the origin. Now, let us prove items (1)-(4). First, since for each $i \in \{1,\cdots,L\}$ and $\alpha \in E_d$, $b_{\alpha,i} \geq 0$ and $c_{\beta,i} \leq 0$, from Proposition 1, it follows that for each $i \in \{1,\cdots,L\}$, $V_i(x) \geq 0$ and $Z_i(x) \leq 0$ for all $x \in D_i \backslash \{0\}$. Then, from condition I, since for each $i \in \{1,\cdots,L\}$, there exists at least one $b_{\alpha,i} > 0$, and at least one $c_{\beta,i} < 0$, thus, for each $i \in \{1,\cdots,L\}$, $V_i(x) > 0$ and $Z_i(x) < 0$ for all $x \in D_i \backslash \{0\}$. Second, since $Z_i(x) < 0$ for $x \in D_i \backslash \{0\}$, $i \in \{1,\cdots,L\}$, and $\mathcal{C}(\langle \nabla V_i(x), f(x)\rangle) = \mathcal{C}(Z_i(x))$, it follows that for each $i \in \{1\cdots,L\}$, $\langle \nabla V_i(x), f\rangle(x) < 0$ for all $x \in D_i \backslash \{0\}$. Then, since for each $i \in \{1,\cdots,L\}$, $D^+(V_i(x), f(x)) = \langle \nabla V_i(x), f(x)\rangle$ for all $x \in D_i$, $D^+(V_i(x), f(x)) < 0$ for all $x \in D_i \backslash \{0\}$ and $i \in \{1\cdots,L\}$. Third, by Lemma 1, condition II directly implies that $V_i(0) = 0$ for all $i \in \{1,\cdots,L\}$. Thus, $V(0) = 0$. Finally, since each $V_i(x)$ is continuous on $\text{int}(D_i)$, we only need to show that $V(x)$ is continuous on $D_i \cap D_q$, for all $i, q \in \{1,\cdots,L\} : D_q \in \{H_s^i\}$. By Lemma 2, condition III directly implies that $V_i(x) = V_q(x)$ for all $x \in D_q \cap D_i$.

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Remark 1: The inequality and equality constraints (2), (3), I and II in Theorem 3 are affine in $\{b_{\alpha,i}\}$ and $\{c_{\beta,i}\}$. Furthermore, in the Appendix, we have shown that the equality constraints in III and IV are affine in $\{b_{\alpha,i}\}$ and $\{c_{\beta,i}\}$. Thus, for some $d \in \mathbb{N}$, one can use linear programming to find $\{b_{\alpha,i}\}$ and $\{c_{\beta,i}\}$ that satisfy the constraints (2), (3) and I to IV.

Using Theorem 3, we define the following algorithm to find the set of coefficients $\{b_{\alpha,i}\}$ of Lyapunov functions of the form (4).

V. Complexity Analysis

In this section, we analyze and compare the complexity of the LP and SDP problems associated with Algorithm 1, Polya's algorithm in [19] and SOS algorithm. In this comparison, we consider the polytopic domain of analysis to be a hypercube, centered at the origin. We will use the formula

$$N_{vars} := \sum_{i=0}^{d} \frac{(i+K-1)!}{i!(K-1)!},$$

which gives the number of basis functions in $B(\Gamma)$ for a convex polytope Γ with K facets.

A. Complexity of the LP associated with Algorithm 1

We consider the following assumption in the analysis.

Assumption 1: We perform the analysis on an $n-$dimensional hypercube, centered at the origin. The hypercube is decomposed into $L = 2n$ sub-polytopes such that each sub-polytope has $m = 2n-1$ facets, $\bar{m} = 1$ of which do not contain the origin. Fig. 1 shows the $1-$, $2-$ and $3-$demensional decomposed hypercube.

Algorithm 1: Search for piecewise polynomial Lyapunov functions

Inputs:

- The polytope's data: w_i, u_i for $i = 1, \cdots, L$.
- D-decomposition data: $h_{i,j}$ and $g_{i,j}$ for $i = 1, \cdots, K$ and $j = 1, \cdots, m_i$.
- The vector field of the dynamical system: $f(x)$.
- Maximum degree of the Lyapunov function: d_{max}

for $i = 1, \cdots, L$ **do**
  Calculate the set of adjacent polytopes to D_i, i.e., $\{H_j^i\}$ using Definition 4
  Calculate the set of facets of D_i, i.e., $\{\zeta^j\}$ using Definition 2

while $d < d_{max}$ **do**
  Check the feasibility of the constraints (2), (3) and I to IV in Theorem 3 using an LP solver
  if *feasible* **then**
    Break the while loop
  else
    Set $d = d + 1$

Outputs:

- In case of feasibility: the coefficients $b_{\alpha,i}$ for $\alpha \in E_d$, $i = 1, \cdots, L$ of the piecewise polynomial Lyapunov function $V_i(x) = \sum_{\alpha \in E_d} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_j}$.

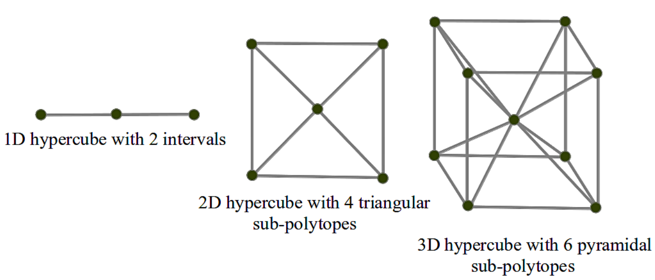

Fig. 1. Decomposition of the hypercube in $1-,2-$ and $3-$dimensions

Let n be the state-space dimension, d_V be the degree of the Lyapunov function and d_f be the degree of the vector field $f(x)$ in system (1). Then, the number of decision variables in the LP associated with Algorithm 1 is

$$N_{vars}^H = L\left(\sum_{d=0}^{d_V} \frac{(d+m-1)!}{d!(m-1)!} + \sum_{d=0}^{d_V+d_f-1} \frac{(d+m-1)!}{d!(m-1)!} - \sum_{d=0}^{d_V} \frac{(d+\bar{m}-1)!}{d!(\bar{m}-1)!}\right), \quad (5)$$

where the first term is the number of b_α coefficients, the second term is the number of c_β coefficients and the third term is the number of b_α coefficients that are set to zero in condition II of Theorem 3. By substituting for L, m and $\bar{m}$ in (5) from Assumption 1, we have

$$N_{vars}^H = 2n\left(\sum_{d=0}^{d_V} \frac{(d+2n-2)!}{d!(2n-2)!} + \sum_{d=0}^{d_V+d_f-1} \frac{(d+2n-2)!}{d!(2n-2)!} - d_V - 1\right).$$

Then, for large number of states, i.e., large n,

$$N_{vars}^H \sim 2n\left((2n-2)^{d_V} + (2n-2)^{d_V+d_f-1}\right) \sim n^{d_V+d_f}$$

The number of constraints in the LP is

$$N_{cons}^H = N_{vars}^H + L\left(\sum_{d=0}^{d_V} \frac{(d+n-1)!}{d!(n-1)!} + \sum_{d=0}^{d_V+d_f-1} \frac{(d+n-1)!}{d!(n-1)!}\right), \quad (6)$$

where the first term is total number of inequality constraints associated with the positivity of b_α and negativity of c_β, the second term is the number of equality constraints on the coefficients of the Lyapunov function to maintain the continuity (see condition III of Theorem 3) and the third term is the number of equality constraints on the coefficients of the Lie derivative to maintain its negativity (see condition IV of Theorem 3). By substituting for L in (6) from Assumption 1, for large n we get

$$N_{cons}^H \sim n^{d_V+d_f} + 2n(n^{d_V} + n^{d_V+d_f-1}) \sim n^{d_V+d_f}.$$

Since solving an LP with an interior-point algorithm requires $O(N_{vars}^2 N_{cons})$ operations [24], the computational cost of solving the LP associated with Algorithm 1 is

$$\sim n^{3(d_V+d_f)}.$$

B. Complexity of the SDP associated with Polya's algorithm

Polya's algorithm [25] checks the positivity of a polynomial over $n-$hupercubes as follows. First, the algorithm defines each variable of the polynomial on a distinct simplex. Then, it constructs a homogeneous representation of the polynomial on the multi-simplex generated by the cross product of the simplices. Finally, it checks the positivity of the coefficients of the homogeneous polynomial. Since the multi-simplex lies on the positive orthant, positivity of the coefficients of the homogeneous representation of the polynomial is a certificate for the positivity of the polynomial.

By using semi-definite programming, the Polya's algorithm in [19] searches for the coefficients of polynomial $P(x)$ over the cone of positive definite matrices, to construct Lyapunov functions of the form $V(x) = x^T P(x) x$. In [19], we have shown that the number of decision variables in the SDP associated with Polya's algorithm is

$$N_{vars}^P = \frac{n(n+1)}{2} \sum_{d=0}^{d_V-2} \frac{(d+n-1)!}{d!(n-1)!}.$$

The number of rows in the LMI constraint of the SDP is

$$N_{cons}^P = \frac{n(n+1)}{2}\left((d_V+e-1)^n + (d_V+d_f+e-2)^n\right),$$

where e is the Polya's exponent. Then, for large n,

$$N_{vars}^P \sim n^{d_V} \quad \text{and} \quad N_{cons}^P \sim (d_V+d_f+e-2)^n.$$

Since solving an SDP with an interior-point algorithm typically requires $O(N_{cons}^3 + N_{var}^3 N_{cons} + N_{var}^2 N_{cons}^2)$ operations [24], the computational cost of solving the SDP associated with Polya's algorithm is

$$\sim (d_V+d_f+e-2)^{3n}.$$

C. Complexity of the SDP associated with SOS algorithm

To find a Lyapunov function for (1) over the polytope of the form

$$\Gamma = \left\{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i \in \{1, \cdots, K\}\right\}$$

using SOS algorithm and the Positivstellensatz results [26], we find polynomial $V(x)$ and SOS polynomials $s_i(x)$ and $t_i(x)$ for $i = 1, \cdots, K$ such that for any $\varepsilon > 0$

$$V(x) - \varepsilon x^T x - \sum_{i=1}^{K} s_i(x)(w_i^T x + u_i) \text{ is SOS} \quad \text{and}$$
$$-\langle \nabla V(x), f(x) \rangle - \varepsilon x^T x - \sum_{i=1}^{K} t_i(x)(w_i^T x + u_i) \text{ is SOS.}$$

Let us choose the degree of $s_i(x)$ to be $d_V - 2$ and the degree of $t_i(x)$ to be $d_V + d_f - 2$. Then, it can be shown that the total number of decision variables in the SDP associated with the SOS algorithm is

$$N_{vars}^S = \frac{N_1(N_1+1)}{2} + K\frac{N_2(N_2+1)}{2} + K\frac{N_3(N_3+1)}{2}, \quad (7)$$

where N_1 is the number of monomials in a polynomial of degree $d_V/2$, N_2 is the number of monomials in a polynomial of degree $(d_V - 2)/2$ and N_3 is the number of monomials in a polynomial of degree $(d_V + d_f - 2)/2$ calculated as

$$N_1 = \sum_{d=1}^{d_V/2} \frac{(d+n-1)!}{(d)!(n-1)!},$$

$$N_2 = \sum_{d=0}^{(d_V-2)/2} \frac{(d+n-1)!}{(d)!(n-1)!} \quad \text{and} \quad N_3 = \sum_{d=0}^{(d_V+d_f-2)/2} \frac{(d+n-1)!}{(d)!(n-1)!}.$$

The first, second and third terms in (7) are the number of decision variables associated with the SOS format $(z(x)^T M z(x)$, $z(x)$ is the vector of monomial basis) of the polynomials $V(x), s_i(x)$ and $t_i(x)$, respectively. It can be shown that the number of rows in the LMI constraint of the SDP is

$$N_{cons}^S = N_1 + 2K N_2 + N_3,$$

where the first term is the number of rows associated with the positivity of $V(x)$, the second term is the number of rows associated with the positivity of $s_i(x)$ and $t_i(x), i = 1, \cdots, K$ and the third term is the number of rows associated with the negativity of the Lie derivative. By substituting $K = 2n$ (For the case of a hypercube), for large n we have

$$N_{vars}^S \sim n^{d_V + d_f - 1} \quad \text{and} \quad N_{cons}^S \sim n^{(d_V + d_f - 2)/2}.$$

Finally, the computational cost of solving the SDP associated the SOS algorithm, using an interior-point algorithm is

$$\sim n^{3.5(d_V + d_f) - 4}$$

D. Comparison of the Complexities

We draw the following conclusions from our complexity analysis.

1) For large number of states and with Assumption 1, the complexities of the LP associated with Algorithm 1 and the SDP associated with the SOS algorithm grows **polynomially** with n, whereas the complexity of the SDP associated with Polya's algorithm grows **exponentially** with n. Furthermore, for large state-space dimensions and degrees of the Lyapunov polynomial, the LP has the least computational complexity.
2) The complexity of the LP associated with Algorithm 1 scales linearly with the number of sub-polytopes L. In case simplicial complex decomposition, where $L = 2^n$, the complexity of the LP grows exponentially with the state-space dimension.
3) In Fig. 2, we show the number of decision variables and constraints of the LP and SDPs, for different degrees of the Lyapunov function and different degrees of the vector field. The figure shows that in general, SDP associated with Polya's algorithm has the least number of variables and the greatest number of constraints, whereas the SDP associated with SOS algorithm has the greatest number of variables and the least number of constraints.

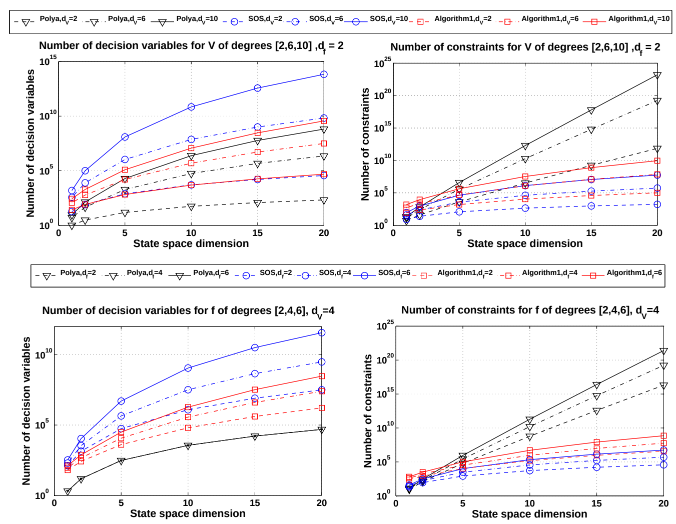

Fig. 2. Number of decision variables and constraints of the optimization problems associated with Algorithm 1 and Polya's and SOS algorithms, for different degrees of Lyapunov function and vector field $f(x)$

VI. Numerical Results

In this section, we test the accuracy of our algorithm in approximating the region of attraction of locally-stable nonlinear systems through numerical examples. We perform the stability analysis on the reverse-time Van Der Pol oscillator.

$$\dot{x}_1 = -x_2, \; \dot{x}_2 = x_1 + x_2(x_1^2 - 1), \quad (8)$$

using the following polytopes:

1) Parallelogram Γ_P: the convex hull of the vertices

$$p_1 = \begin{bmatrix} -1.31 \\ 0.18 \end{bmatrix}, p_2 = \begin{bmatrix} 0.56 \\ 1.92 \end{bmatrix}, p_3 = \begin{bmatrix} -0.56 \\ -1.92 \end{bmatrix} p_4 = \begin{bmatrix} 1.31 \\ -0.18 \end{bmatrix}$$

2) Square Φ_Q: the convex hull of the vertices

$$q_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, q_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, q_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} q_4 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

3) Diamond Θ_R: the convex hull of the vertices

$$r_1 = \begin{bmatrix} -1.41 \\ 0 \end{bmatrix}, r_2 = \begin{bmatrix} 0 \\ 1.41 \end{bmatrix}, r_3 = \begin{bmatrix} 1.41 \\ 0 \end{bmatrix} r_4 = \begin{bmatrix} 0 \\ -1.41 \end{bmatrix}$$

We decompose the parallelogram and the diamond into 4 triangles and the decompose the square into 4 squares. We would like to solve the following optimization problem for $d = 2,4,6,8$ as the degree of the Lyapunov function.

$$\max_{s\in\mathbb{R}^+, b_{\alpha,i}\geq 0, c_{\beta,i}\leq 0} \quad s$$

s.t. conditions I to IV of Theorem 3 hold for

$$V_i(x) = \sum_{\alpha\in E_d} b_{\alpha,i} \prod_{j=1}^{3} (h_{i,j}^T x + g_{i,j})^{\alpha_j}, i = 1,\cdots,4,$$

$$Z_i(x) := \sum_{\beta\in E_{d+d_f-1}} c_{\beta,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\beta_j}, i = 1,\cdots,4,$$

and system (8) for all

$$x \in \{x \in \mathbb{R}^2 : x = \sum_{i=1}^{K} \mu_i p_i : \mu_i \in [0,s] \text{ and } \sum_{i=1}^{K} \mu_i = s\}.$$

We solve the above problem using a bisection search on s in an outer-loop, and an LP solver in the inner loop. We repeat the bisection search for $V_i(x)$ of degrees $d = 2,4,6,8$. In Fig. 3, for each d, we have shown

$$\Gamma_P^* := \{x \in \mathbb{R}^n : x = \sum_{i=1}^{K} \mu_i p_i : \mu_i \in [0,s^*] \text{ and } \sum_{i=1}^{K} \mu_i = s^*\}$$

and the largest level-set of $V_i(x)$ inscribed in Γ_P^*. Similarly, we solved the same optimization problem with square Φ_Q and diamond Θ_R as the sets of analysis. In all cases, increasing d results in a larger level-set of $V_i(x)$. We obtained the largest level-set using the parallelogram Γ_P with the scaling factor $s^* = 1.639$. The maximum scaling factor for Φ_Q is $s^* = 1.800$ and the maximum scaling factor for Θ_R is $s^* = 1.666$.

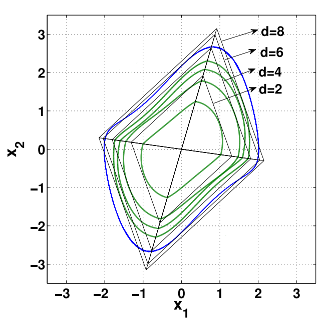

Fig. 3. Largest level sets of Lyapunov functions of different degrees and their associated parallelograms

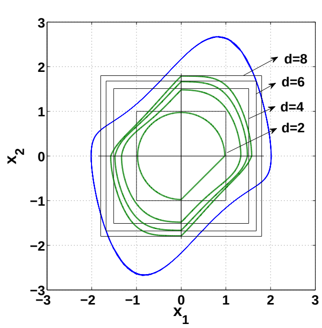

Fig. 4. Largest level sets of Lyapunov functions of different degrees and their associated squares

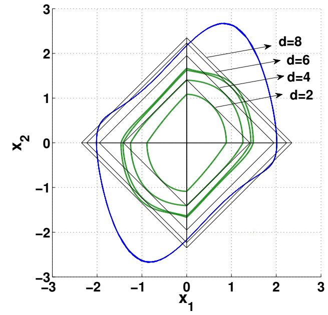

Fig. 5. Largest level sets of Lyapunov functions of different degrees and their associated diamonds

VII. Conclusion and future work

In this paper, we introduce an algorithm to analyze the local stability of nonlinear systems with polynomial vector fields. The algorithm searches for the coefficients of a piecewise polynomial Lyapunov function defined on a subdivided convex polytope and represented in the Handelman basis. We show that these coefficients can be obtained by solving a sequence of linear programs. We also show that the complexities of the linear programs scale polynomially in the number of states. In the numerical examples, we show the accuracy of the algorithm in approximating the region of attraction inscribed within different polytopes. As future work, we would like to apply this algorithm to stability analysis of switched systems and controller synthesis problems. We will also develop a decentralized algorithm to setup and solve the linear program associated with Theorem 3.

VIII. Appendix: Derivation of the Coefficient map

Given $w_i \in \mathbb{R}^n$ and $b_\alpha, u_i \in \mathbb{R}$, Let

$$V(x) = \sum_{\alpha\in E_d} b_\alpha \prod_{i=1}^{K} (w_i^T x + u_i)^{\alpha_i},$$

where $E_d := \{\alpha \in \mathbb{N}^K : |\alpha|_1 \leq d\}$. Let $X := (x_1,\cdots,x_n,y)$. Then, we define the homogeneous representation of $V(x)$ as

$$\tilde{V}(X) = \sum_{\alpha\in E_d} b_\alpha \prod_{i=1}^{K} (w_i^T x + u_i y)^{\alpha_i}.$$

By employing the multinomial theorem, we get

$$\tilde{V}(X) = \sum_{\alpha\in E_d} b_\alpha \prod_{i=1}^{K} \left(\sum_{l_{i,j}\in\mathbb{N}: \sum_{j=1}^{n+1} l_{i,j}=\alpha_i} \left(\frac{\alpha_i! w_{i,1}^{l_{i,1}} \ldots w_{i,n}^{l_{i,n}} u_i^{l_{i,n+1}}}{l_{i,1}! \ldots l_{i,n+1}!} \right) x_1^{l_{i,1}} \ldots x_n^{l_{i,n}} y^{l_{i,n+1}} \right).$$

By letting $p(\alpha_i, l_i, w_i, u_i) = \frac{\alpha_i! w_{i,1}^{l_{i,1}} \ldots w_{i,n}^{l_{i,n}} u_i^{l_{i,n+1}}}{l_{i,1}! \ldots l_{i,n+1}!}$ and expanding $\tilde{V}(X)$, we have

$$\tilde{V}(X) = \sum_{\alpha\in E_d} b_\alpha \left(\sum_{\substack{l_{1,j}\in\mathbb{N}: \\ \sum_{j=1}^{n+1} l_{1,j}=\alpha_1}} \cdots \sum_{\substack{l_{K,j}\in\mathbb{N}: \\ \sum_{j=1}^{n+1} l_{K,j}=\alpha_K}} \left(\prod_{i=1}^{K} p(\alpha_i, l_i, w_i, u_i) \right) y^{\sum_{i=1}^{K} l_{i,n+1}} x_1^{\sum_{m=1}^{K} l_{m,1}} \cdots x_n^{\sum_{m=1}^{K} l_{m,n}} \right).$$

Note that $F(x_1,...,x_n,1)=V(x)$. Thus, by substituting $y=1$ we have

$$V(x)=\sum_{\alpha\in E_d}\left(b_\alpha \sum_{\substack{l_{1,j}\in\mathbb{N}:\\ \sum_{j=1}^{n+1} l_{1,j}=\alpha_1}} \cdots \sum_{\substack{l_{K,j}\in\mathbb{N}:\\ \sum_{j=1}^{n+1} l_{K,j}=\alpha_K}} \left(\prod_{i=1}^{K} p(\alpha_i,l_i,w_i,u_i)\right) x_1^{\sum_{m=1}^{K} l_{m,1}}\cdots x_n^{\sum_{m=1}^{K} l_{m,n}}\right).$$

Thus, for every $\{l_{i,j}\in\mathbb{N}\}_{\substack{i=1,\cdots,K\\ j=1,\cdots,n}}$, the coefficient of the monomial $x_1^{\sum_{m=1}^{K} l_{m,1}}\cdots x_n^{\sum_{m=1}^{K} l_{m,n}}$ is

$$\sum_{\substack{\alpha\in E_d:\\ \sum_{j=1}^{n} l_{1,j}=\alpha_1\\ \vdots\\ \sum_{j=1}^{n} l_{K,j}=\alpha_K}}\left(\sum_{\substack{l_{1,n+1}\in\{0,\cdots,d-\alpha_1\}\\ \vdots\\ l_{K,n+1}\in\{0,\cdots,d-\alpha_K\}}} b_\alpha\left(\prod_{i=1}^{K} p(\alpha_1,l_i,w_i,u_i)\right)\right).$$

Thus, for any polynomial $V(x)$ of the form

$$V(x)=\sum_{\alpha\in E_d} b_\alpha \prod_{i=1}^{K}(w_i^T x+u_i)^{\alpha_i}, x\in\mathbb{R}^n,$$

with $N=\sum_{j=0}^{d}\frac{(j+n-1)!}{j!(n-1)!}$ monomials, we define the coefficient map $\mathcal{C}:\mathbb{R}[x]\to\mathbb{R}^N$ as $\mathcal{C}(f):=[c_1,\cdots,c_N]$, where

$$c_k=\sum_{\substack{\alpha\in E_d:\\ \sum_{j=1}^{n} l_{1,j}=\alpha_1\\ \vdots\\ \sum_{j=1}^{n} l_{K,j}=\alpha_K}}\left(\sum_{\substack{l_{1,n+1}\in\{0,\cdots,d-\alpha_1\}\\ \vdots\\ l_{K,n+1}\in\{0,\cdots,d-\alpha_K\}}} b_\alpha\left(\prod_{i=1}^{K} p(\alpha_1,l_i,w_i,u_i)\right)\right) \quad (9)$$

for some $l_{p,j}\in\mathbb{N}$ such that $x_1^{\gamma_1}\cdots x_n^{\gamma_n}$ with $\gamma_j=\sum_{p=1}^{K} l_{p,j}$, is the $k-$th monomial in $V(x)$ using the lexicographical ordering, and where

$$p(\alpha_i,l_i,w_i,u_i)=\frac{\alpha_i! w_{i,1}^{l_{i,1}}\ldots w_{i,n}^{l_{i,n}} u_i^{l_{i,n+1}}}{l_{i,1}!\ldots l_{i,n+1}!}.$$

Notice that, from (9), the coefficients of $V(x)$ are affine in $\{b_\alpha\}$. Similarly, it can be shown that the coefficients of $\langle\nabla V(x),f(x)\rangle$ are affine in $\{b_\alpha\}$. However, for the sake of brevity, we do not derive a formula for the coefficients of $\langle\nabla V(x),f(x)\rangle$ in terms of $\{b_\alpha\}$.

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