Constructing Piecewise-Polynomial Lyapunov Functions on Convex Polytopes Using Handelman's Basis

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Abstract— **We develop a new algorithm for computing the region of attraction of isolated equilibria of nonlinear systems with polynomial vector fields. First, we consider an arbitrary convex polytope that contains the equilibrium in its interior. Then, we decompose the polytope into several convex sub-polytopes with a common vertex at the equilibrium. Then, by using Handelman's theorem, we derive a new set of affine feasibility conditions -solvable by linear programming- on each sub-polytope. Any solution to this feasibility problem yields a piecewise polynomial Lyapunov function on the entire polytope. This is the first result which utilizes Handelman's theorem and decomposition to construct piecewise polynomial Lyapunov functions on arbitrary polytopes. In a computational complexity analysis, we show that for large number of states and large degrees of the Lyapunov function, the complexity of the proposed feasibility problem is less than the complexity of certain semi-definite programs associated with alternative methods based on Sum-of-Squares or Polya's Lemma. Using different types of convex polytopes, we assess the accuracy of the algorithm in estimating the region of attraction of the equilibrium point of the reverse-time Van Der Pol oscillator.**

I. INTRODUCTION

One approach to stability analysis of nonlinear systems is the search for a decreasing Lyapunov function. For those systems with polynomial vector fields, searching for polynomial Lyapunov functions has been shown to be necessary and sufficient for stability on any bounded set [1]. However, searching for a polynomial Lyapunov which proves local stability requires enforcing positivity on a neighborhood of the equilibrium. Unfortunately, while we do have necessary and sufficient conditions for positivity of a polynomial (e.g. Tarski-Seidenberg, Artin), it has been shown that the general problem of determining whether a polynomial is positive is NP-hard [2].

The most well-known approach to determining positivity of a polynomial is to search for a representation as the sum and quotient of squared polynomials [3]. Such a representation is necessary and sufficient for a polynomial to be positive semidefinite. If we leave off the quotient, the the search for a Sum-of-Squares (SOS) is a common sufficient condition for positivity of a polynomial. The advantage of the SOS approach is that verifying the existence of an SOS representation is a semidefinite programming problem [4]. This approach was first articulated in [5]. SOS programming has been used extensively in stability analysis and control

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including stability analysis of nonlinear systems [6], robust stability analysis of switched and hybrid systems [7], and stability analysis of time-delay systems [8].

In addition to the SOS representation of positive polynomials, there exist alternative representation theorems for polynomials which are not globally positive. For example, Polya's Theorem [9] states that every strictly positive homogeneous polynomial on the positive orthant can be represented as a sum of even-powered monomials with positive coefficients. Multiple variants of Polya's theorem have been proposed, e.g., extensions to the multi-simplex or hypercube [10], [**?**], extensions to polynomials with zeros on the boundary of the simplex [11] and an extension to the entire real domain [12].

The downside to the use of SOS (with Positivstellensatz multipliers) or Polya's algorithm for stability analysis of nonlinear systems with many states is computational complexity. Specifically, these methods require us to set up and solve large SDPs. For example, using the SOS algorithm to construct a degree 6 Lyapunov function on the hypercube for a system with 10 states implies an SDP with $\sim 10^8$ variables and $\sim 10^5$ constraints. Although Polya's algorithm implies similar complexity to SOS, the SDPs associated with Polya's algorithm possess a block-diagonal structure. This has allowed some work on parallel computing approaches such as can be found in [13], [14] for robust stability and nonlinear stability, repsectively. However, although Polya's algorithm has been generalized to positivity over simplices and hypercubes; as yet no generalization exists for arbitrary convex polytopes. Therefore, in this paper, we look at Handelman's theorem [15]. Specifically, given an arbitrary convex polytope, Handelman's theorem provides a parameterization of all polynomials that are positive on the given polytope.

Some preliminary work on the use of Handelman's theorem for Lyapunov functions on the hypercube has been suggested in [16] and has also been applied to robust stability of positive linear systems in [17]. In this paper, we consider a new approach to the use of Handelman's theorem for computing regions of attraction of stable equilibria by constructing piecewise-polynomial Lyapunov functions on arbitrary convex polytopes. Specifically, we decompose a given convex polytope into a set of convex sub-polytopes that share a common vertex at the origin. Then, on each sub-polytope, we convert Handelman's conditions to linear programming constraints. Additional constraints are then proposed which ensure continuity of the Lyapunov function. We then show the resulting algorithm has polynomial complexity in the number of states and compare this complexity with algorithms based on SOS and Polya's theorem. Finally, we evaluate the accuracy of our by considering the domain of attraction of the reverse-time Van Der Pol oscillator.

II. Definitions and Notation

In this section, we define convex polytopes, facets of polytopes, decompositions, interfaces, Handelman bases, polytopes and polytopic decomposition.

Definition 1: (Convex Polytope) Given the set of vertices $P := \{p_i \in \mathbb{R}^n, i = 1, \cdots, K\}$, define the convex polytope Γ_P as

$$\Gamma_P := \{x \in \mathbb{R}^n : x = \sum_{i=1}^{K} \mu_i p_i : \mu_i \in [0,1] \text{ and } \sum_{i=1}^{K} \mu_i = 1\}.$$

Every convex polytope can be represented as

$$\Gamma := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\},$$

for some $w_i \in \mathbb{R}^n, u_i \in \mathbb{R}, i = 1, \cdots, K$. Throughout the paper, every polytope that we use contains the origin.

Definition 2: (Facets of a polytope) Given a polytope of the form

$$\Gamma := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\},$$

define the $i-$th facet of the polytope Γ as

$$\zeta^i(\Gamma) := \{x \in \mathbb{R}^n : w_i^T x + u_i = 0 \text{ and } w_j^T x + u_j \geq 0 \text{ for } j \in \{1, \cdots, K\}, j \neq i\}.$$

Definition 3: (D-decomposition) Given a polytope of the form

$$\Gamma := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\},$$

we say $D_\Gamma := \{D_i\}_{i=1,\cdots,L}$ is a D-decomposition of Γ, if $D_i \subset \mathbb{R}^n$ are polytopes of the form

$$D_i := \{x \in \mathbb{R}^n : h_{i,j}^T x + g_{i,j} \geq 0, j = 1, \cdots, m_i\}$$

such that $\cup_{i=1}^{L} D_i = \Gamma$, $\cap_{i=1}^{L} D_i = \{0\}$ and $\text{int}(D_i) \cap \text{int}(D_j) = \emptyset$ for $i, j \in \{1, \cdots, L\}, i \neq j$.

Definition 4: (Collection of adjacent sub-polytopes) Given a D-decomposition $\{D_i\}_{i=1,\cdots,L}$ of polytope Γ, where $D_i := \{x \in \mathbb{R}^n : h_{i,j}^T x + g_{i,j} \geq 0,\ j = 1, \cdots m_i\}$, for any i, we denote the polytopes adjacent to D_i as

$$\{H_j^i\}_{j=1,\cdots,e_i} :=$$
$$\{H_j^i \in D_\Gamma : h_{j,k_1}^T x + g_{j,k_1} = h_{i,k_2}^T x + g_{i,k_2} \text{ for some } k_1, k_2 \in \mathbb{N}\}$$

Definition 5: (The Handelman basis associated with a polytope) Given a polytope of the form

$$\Gamma := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0,\ i = 1, \cdots, K\},$$

we define the set of Handelman bases, indexed by

$$\alpha \in E_d := \{\alpha \in \mathbb{N}^K : |\alpha|_1 \leq d\}, \text{ as}$$

$$B(\Gamma) := \{\lambda_\alpha(x) : \lambda_\alpha(x) = \prod_{i=1}^{K} (w_i^T x + u_i)^{\alpha_i},\ \alpha \in E_d\}.$$

Definition 6: (Restriction of a polynomial to a facet) Given a polytope of the form $\Gamma := \{x \in \mathbb{R}^n : w_i^T x + u_i,\ i = 1, \cdots, K\}$, and a polynomial $f(x)$ of the form

$$f(x) = \sum_{\alpha \in E_d} b_\alpha \prod_{i=1}^{K} (w_i^T x + u_i)^{\alpha_i},$$

define the **restriction of $f(x)$ to the k-th facet of Γ** as the function

$$f|_k(x) := \sum_{\alpha \in E_d : \alpha_k = 0} b_\alpha \prod_{i=1}^{K} (w_i^T x + u_i)^{\alpha_i}.$$

Definition 7: (Upper Dini Derivative) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map. Then, define the upper Dini derivative of a function $V : \mathbb{R}^n \to \mathbb{R}$ in the direction $f(x)$ as

$$D^+(V, f(x)) = \limsup_{h \to 0^+} \frac{V(x + hf(x)) - V(x)}{h}.$$

It can be shown that for a continuously differentiable $V(x)$,

$$D^+(V, f(x)) = \langle \nabla V(x), f(x) \rangle.$$

Finally, given a polynomial of the form

$$f(x) = \sum_{\alpha \in E_d} b_\alpha \prod_{i=1}^{K} (w_i^T x + u_i)^{\alpha_i},\ x \in \mathbb{R}^n$$

with $N = \sum_{j=0}^{d} \frac{(j+n-1)!}{j!(n-1)!}$ monomials, we will use the map $\mathcal{C} : \mathbb{R}[x] \to \mathbb{R}^N$, $\mathcal{C}(f) := [c_1, \cdots, c_N]$, where c_j is the coefficient of the j-th monomial in $f(x)$ in lexicographical order. A formula for c_j as a function of $\alpha, b_\alpha, u_i, w_i$, for $i = 1, \cdots, K$ is given in the Appendix.

III. Background and Problem Statement

We address the problem of local stability of nonlinear systems of the form

$$\dot{x}(t) = f(x(t)), \tag{1}$$

about the zero equilibrium, where $f : \mathbb{R}^n \to \mathbb{R}^n$. We use the following Lyapunov stability condition.

Theorem 1: For any $\Omega \subset \mathbb{R}^n$ with $0 \in \Omega$, suppose there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ and continuous positive definite functions W_1, W_2, W_3,

$$W_1(x) \leq V(x) \leq W_2(x) \text{ for } x \in \Omega \text{ and}$$
$$D^+(V, f(x)) \leq -W_3(x) \text{ for } x \in \Omega,$$

then System (1) is asymptotically stable on $\{x : \{y : V(y) \leq V(x)\} \subset \Omega\}$.

In this paper, we construct piecewise-polynomial Lyapunov functions which may not have classical derivatives. As such, we use Dini derivatives which are known to exist for piecewise-polynomial functions.

Problem statement: Given the vertices $p_i \in \mathbb{R}^n, i = 1, \cdots, K$, we would like to find the largest positive s such that there exists a polynomial $V(x)$ where $V(x)$ satisfies the conditions of Theorem 1 on the convex polytope $\{x \in \mathbb{R}^n : x = \sum_{i=1}^{K} \mu_i p_i : \mu_i \in [0, s] \text{ and } \sum_{i=1}^{K} \mu_i = s\}$.

Given a convex polytope, the following result [15] parameterizes the set of polynomials which are positive on that polytope using the positive orthant.

Theorem 2: (Handelman's Theorem) Given $w_i \in \mathbb{R}^n, u_i \in \mathbb{R}, i = 1, \cdots, K$, define the polytope $\Gamma := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\}$. If polynomial $f(x) > 0$ for all $x \in \Gamma$, then there exist $b_\alpha \geq 0$, $\alpha \in \mathbb{N}^m$ such that for some $d \in \mathbb{N}$,

$$f(x) = \sum_{\alpha \in E_d} b_\alpha (w_1^T x + u_1)^{\alpha_1} \cdots (w_K^T x + u_K)^{\alpha_K}.$$

Given a D-decomposition $D_\Gamma := \{D_i\}_{i=1,\cdots,L}$ of the form

$$D_i := \{x \in \mathbb{R}^n : h_{i,j}^T x + g_{i,j} \geq 0, j = 1, \cdots, m_i\}$$

of some polytope Γ, we parameterize a cone of piecewise-polynomial Lyapunov functions which are positive on Γ as

$$V(x) = V_i(x) := \sum_{\alpha \in E_d} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_j}, \quad \text{for } x \in D_i \text{ and } i = 1, \cdots, L.$$

We will use a similar parameterization of piecewise-polynomials which are negative on Γ in order to enforce negativity of the derivative of the Lyapunov function. We will also use linear equality constraints to enforce continuity of the Lyapunov function.

IV. Problem setup

We first present some lemmas necessary for the proof of our main result. The following lemma provides a sufficient condition for a polynomial represented in the Handelman basis to vanish at the origin ($V(0) = 0$).

Lemma 1: Let $D_\Gamma := \{D_i\}_{i=1,\cdots,L}$ be a D-decomposition of a polytope Γ, where

$$D_i := \{x \in \mathbb{R}^n : h_{i,j}^T x + g_{i,j} \geq 0, j = 1, \cdots, m_i\}.$$

For every $i = 1, \cdots, L$, define

$$\hat{E}_d := \{\alpha \in \mathbb{N}^{m_i} : |\alpha|_1 \leq d, \ \alpha_j = 0 \text{ for all } j \in J\},$$

where $J := \{j \in \mathbb{N} : g_{i,j} = 0\}$. Let

$$f_i(x) = \sum_{\alpha \in E_d} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_i}, \ i = 1, \cdots L.$$

If $b_{\alpha,i} = 0$ for $i \in \{1, \cdots, L\}$ and $\alpha \in \hat{E}_d$, then $f_i(0) = 0$ for $i = 1, \cdots, L$.

Proof: Note that

$$f_i(x) = \sum_{\alpha \in E_d \setminus \hat{E}_d} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_i} + \sum_{\alpha \in \hat{E}_d} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_i}.$$

For each $\alpha \in E_d \setminus \hat{E}_d$, there exists at least one $k \in \{1, \cdots, m_i\}$ such that $g_{i,k} = 0$ and $\alpha_k > 0$. Thus, at $x = 0$,

$$\sum_{\alpha \in E_d \setminus \hat{E}_d} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_i} = 0. \qquad (*)$$

Now, since $b_{\alpha,i} = 0$ for all $i \in \{1, \cdots, L\}$ and $\alpha \in \hat{E}_d$, it follows that

$$\sum_{\alpha \in \hat{E}_d} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_i} = 0. \qquad (**)$$

Thus, if $b_{\alpha,i} = 0$ for all $i \in \{1, \cdots, L\}$ and $\alpha \in \hat{E}_d$, then from (*) and (**), we have $f_i(0) = 0$. ∎

This Lemma provides a condition which ensures that a piecewise-polynomial function on a D-Decomposition is continuous.

Lemma 2: (Continuity of piecewise polynomial functions) Let $D_\Gamma := \{D_i\}_{i=1,\cdots,L}$ be a D-decomposition of a polytope Γ, where

$$D_i := \{x \in \mathbb{R}^n : h_{i,j}^T x + g_{i,j} \geq 0, j = 1, \cdots, m_i\}, \text{ for } i = 1, \cdots, L$$

and let $\{H_s^i\}$ be the set of all polytopes adjacent to D_i. For $i = 1, \cdots, L$, define

$$f_i(x) = \sum_{\alpha \in E_d} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_i}.$$

If

$$C(f_i|_k) = C(f_q|_l)$$

for every $k \in \{1, \cdots, m_i\}$, $l \in \{1, \cdots, m_q\}$ and $q \in \{1, \cdots, L\}$ such that $D_q \in \{H_s^i\}$ and $\zeta^k(D_i) = \zeta^l(D_q)$, then $f_i(x) = f_q(x)$ for all $x \in D_i \cap D_q$.

Proof: For any $i \in \{1 \cdots, L\}$ and $k \in \{1, \cdots, m_i\}$, we can write

$$f_i(x) = \sum_{\alpha \in E_d : \alpha_k > 0} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_i} + \sum_{\alpha \in E_d : \alpha_k = 0} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_i}.$$

For any $k \in \{1, \cdots, m_i\}$, if $h_{i,k}^T x + g_{i,k} = 0$, then

$$\sum_{\alpha \in E_d : \alpha_k > 0} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_j} = 0.$$

Therefore, if $h_{i,k}^T x + g_{i,k} = 0$, we have

$$f_i(x) = \sum_{\alpha \in E_d : \alpha_k = 0} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_j}.$$

Thus, from Definition 6, if $h_{i,k}^T x + g_{i,k} = 0$, we have

$$f_i(x) = f_i|_k. \ (*).$$

Furthermore, by assumption, for every $k \in \{1, \cdots, m_i\}$, $l \in \{1, \cdots, m_q\}$ and $q \in \{1, \cdots, L\}$ such that $D_q \in \{H_s^i\}$ and $\zeta^k(D_i) = \zeta^l(D_q)$, we have

$$C(f_i|_k) = C(f_q|_l).$$

This implies that $f_i|_k = f_q|_l$ $(**)$.
Lastly, for any $x \in \zeta^l(D_q)$,

$$f_q|_l = f_q(x). \qquad (***)$$

Thus, if

$$C(f_i|_k) = C(f_q|_l)$$

for every $k \in \{1, \cdots, m_i\}$, $l \in \{1, \cdots, m_q\}$ and $q \in \{1, \cdots, L\}$ such that $(k, l, q) \in \{(k, l, q) : D_q \in \{H_s^i\}, \zeta^k(D_i) = \zeta^l(D_q)\}$, then from (*), (**) and (***), $f_i(x) = f_q(x)$ for all $x \in \zeta^l(D_q)$.

Finally, since $\zeta^k(D_i) = \zeta^l(D_q)$, it follows that $f_i(x) = f_q(x)$ for all $x \in D_i \cap D_q$. ■

Theorem 3: (Main result) Let d_f be the degree of the polynomial vector field $f(x)$ of System (1). Given w_i, $h_{i,j} \in \mathbb{R}^n$, u_i, $g_{i,j} \in \mathbb{R}$, define the polytope

$$\Gamma := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\},$$

with D-decomposition $D_\Gamma := \{D_i\}_{i=1,\cdots,L}$, where

$$D_i := \{x \in \mathbb{R}^n : h_{i,j}^T x + g_{i,j} \geq 0, j = 1, \cdots, m_i\}.$$

Suppose for each $i \in \{1, \cdots, L\}$, $\{H_s^i\}$ is the set of polytopes adjacent to each D_i. Then, for some $d \in \mathbb{N}$, if there exist

$$b_{\alpha,i} \geq 0 \quad \text{for} \quad \alpha \in E_d \text{ and } i = 1, \cdots, L \text{ and} \tag{2}$$

$$c_{\beta,i} \leq 0 \quad \text{for} \quad \beta \in E_{d+d_f-1} \text{ and } i = 1, \cdots, L \tag{3}$$

such that

I. $b_{\alpha,i} > 0$ and $c_{\beta,i} < 0$
for at least one $\alpha \in E_d$ and one $\beta \in E_{d+d_f-1}$,

II. $b_{\alpha,i} = 0$
for all $\alpha \in \{\alpha \in \mathbb{N}^{m_i} : |\alpha|_1 \leq d,\ \alpha_j = 0 \text{ for all } j \in J\}$, where $J := \{j \in \mathbb{N} : g_{i,j} = 0\}$,

III. $\mathcal{C}(V_i|_k) = \mathcal{C}(V_q|_l)$
for every $k \in \{1, \cdots, m_i\}$, $l \in \{1, \cdots, m_q\}$ and $q \in \{1, \cdots, L\}$ such that $D_q \in \{H_s^i\}$ and $\zeta^k(D_i) = \zeta^l(D_q)$,

IV. $\mathcal{C}(\langle \nabla V_i, f \rangle) = \mathcal{C}(Z_i)$,

hold for all $i = 1, \cdots, L$, where

$$V_i(x) := \sum_{\alpha \in E_d} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_j} \text{ and} \tag{4}$$

$$Z_i(x) := \sum_{\beta \in E_{d+d_f-1}} c_{\beta,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\beta_j},$$

then the origin is an asymptotically stable equilibrium for System 1.

Proof: Let us choose $V(x)$ as

$$V(x) = V_i(x) = \sum_{\alpha \in E_d} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_j} \tag{5}$$

for $x \in D_i$, $i = 1, \cdots, L$.

In order to show that $V(x)$ is a Lyapunov function for system 1, we need to prove the following:

1) $V_i(x) > 0$ for all $x \in D_i \setminus \{0\}$, $i = 1, \cdots, L$,
2) $D^+(V_i(x), f(x)) < 0$ for all $x \in D_i \setminus \{0\}$, $i = 1, \cdots, L$,
3) $V(0) = 0$,
4) Continuity of $V(x)$.

Then, by Theorem 1, it follows that the system 1 is asymptotically stable at the origin. Now, let us prove items (1)-(4).

Item 1. First we show that $V_i(x) > 0$ for all $x \in D_i \setminus \{0\}$, $i = 1, \cdots, L$. From the definition of the D-decomposition in the Theorem statement, $h_{i,j}^T x + g_{i,j} \geq 0$, for all $x \in D_i$, $j = 1, \cdots, m_i$. From (2), every $b_{\alpha,i} \geq 0$. Thus,

$$V_i(x) = \sum_{\alpha \in E_d} b_{\alpha,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_j} \geq 0$$

for all $x \in D_i \setminus \{0\}$, $i = 1, \cdots, L$. Furthermore, from Condition I, there exists at least one $b_{\alpha,i} > 0$, for $i = 1, \cdots, L$. Hence, $V_i(x) > 0$ for all $x \in D_i \setminus \{0\}$ and $i \in \{1, \cdots, L\}$.

Item 2. Next we show that $D^+(V_i(x), f(x)) < 0$ for all $x \in D_i \setminus \{0\}$, $i = 1, \cdots, L$. From (3), every $c_{\beta,i} \leq 0$. Since $h_{i,j}^T x + g_{i,j} \geq 0$ for all $x \in D_i \setminus \{0\}$, it follows that

$$Z_i(x) = \sum_{\beta \in E_{d+d_f-1}} c_{\beta,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\beta_j} \leq 0$$

for all $x \in D_i \setminus \{0\}$, $i = 1, \cdots, L$. From Condition I, there exists at least one $c_{\beta,i} < 0$ for $i \in \{1, \cdots, L\}$. Thus, $Z_i(x) < 0$ for all $x \in D_i \setminus \{0\}$, $i = 1, \cdots, L$. From Condition IV, $\mathcal{C}(\langle \nabla V_i, f \rangle) = \mathcal{C}(Z_i)$ implies $\langle \nabla V_i(x), f(x) \rangle < 0$ for all $x \in D_i \setminus \{0\}$ and $i \in \{1 \cdots, L\}$. Since $D^+(V_i(x), f(x)) = \langle \nabla V_i(x), f(x) \rangle$ for all $x \in D_i$, it follows that $D^+(V_i(x), f(x)) < 0$ for all $x \in D_i \setminus \{0\}$, $i \in \{1 \cdots, L\}$.

Item 3. Now we show that $V(0) = 0$. By Lemma 1, Condition II implies $V_i(0) = 0$ for $i \in \{1, \cdots, L\}$.

Item 4. Finally, we show that $V(x)$ is continuous for $x \in \Gamma$. Since each $V_i(x)$ is continuous on $\text{int}(D_i)$, we only need to show that $V(x)$ is continuous on $D_i \cap D_q$, for $i, q \in \{(i,q) : D_q \in \{H_s^i\}\}$. By Lemma 2, Condition III implies that $V_i(x) = V_q(x)$ for all $x \in D_q \cap D_i$. ■

The inequality and equality constraints (2), (3), I and II in Theorem 3 are affine in $\{b_{\alpha,i}\}$ and $\{c_{\beta,i}\}$. Furthermore, in the Appendix, we have shown that the equality constraints in III and IV are affine in $\{b_{\alpha,i}\}$ and $\{c_{\beta,i}\}$. Thus, for some $d \in \mathbb{N}$, one can use linear programming to find $\{b_{\alpha,i}\}$ and $\{c_{\beta,i}\}$ that satisfy the constraints (2), (3) and I to IV.

Using Theorem 3 we may now define Algorithm 1, which is illustrated below and computes the set of coefficients $\{b_{\alpha,i}\}$ which define a Lyapunov function of the form (4).

Algorithm 1: Search for piecewise polynomial Lyapunov functions

Inputs:
- Vertices of the polytope: p_i for $i = 1, \cdots, K$
- $h_{i,j}$ and $g_{i,j}$ for $i = 1, \cdots, K$ and $j = 1, \cdots, m_i$
- Coefficients and degree of the polynomial vector field of System (1)
- Maximum degree of the Lyapunov function: d_{max}

for $i = 1, \cdots, L$ **do**
  Calculate the set of polytopes adjacent to D_i, i.e., $\{H_j^i\}$ using Definition 4
  Calculate the set of facets of D_i, i.e., $\{\zeta^j\}$ using Definition 2

while $d < d_{max}$ **do**
  Check the feasibility of the constraints (2), (3) and I to IV in Theorem 3 using an LP solver
  if *feasible* **then**
    Break the while loop
  else
    Set $d = d + 1$

Outputs:
- In case of feasibility: the coefficients $b_{\alpha,i}$ for $\alpha \in E_d$, $i = 1, \cdots, L$ of the piecewise polynomial Lyapunov function $V(x)$ defined in (5).

V. Complexity Analysis

In this section, we analyze and compare the complexity of the LP and SDP problems associated with Algorithm 1,

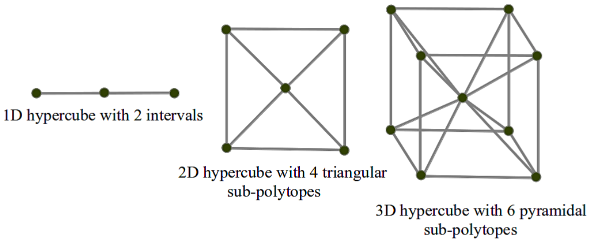

Fig. 1. Decomposition of the hypercube in $1-$,$2-$ and $3-$dimensions

Polya's algorithm in [14] and an SOS approach using Positivstellensatz multipliers. For simplicity, we consider Lyapunov functions defined on a hypercube centered at the origin. Note that we make frequent use of the formula

$$N_{vars} := \sum_{i=0}^{d} \frac{(i+K-1)!}{i!(K-1)!},$$

which gives the number of basis functions in $B(\Gamma)$ for a convex polytope Γ with K facets.

A. Complexity of the LP associated with Handelman's Representation

We consider the following decomposition.

Assumption 1: We perform the analysis on an $n-$dimensional hypercube, centered at the origin. The hypercube is decomposed into $L = 2n$ sub-polytopes such that each sub-polytope has $m = 2n-1$ facets. Fig. 1 shows the $1-$, $2-$ and $3-$dimensional decomposed hypercube.

Let n be the number of states in System (1). Let d_f be the degree of the polynomial vector field in System (1). Suppose we use Algorithm 1 to search for a Lyapunov function of degree d_V. Then, the number of decision variables in the LP associated with Algorithm 1 is

$$N_{vars}^{H} = L\left(\sum_{d=0}^{d_V} \frac{(d+m-1)!}{d!(m-1)!} + \sum_{d=0}^{d_V+d_f-1} \frac{(d+m-1)!}{d!(m-1)!} - (d_V+1)\right), \tag{6}$$

where the first term is the number of b_α coefficients, the second term is the number of c_β coefficients and the third term is the number of b_α coefficients that are set to zero in Condition II of Theorem 3. By substituting for L and m in (6), from Assumption 1 we have

$$N_{vars}^{H} = 2n\left(\sum_{d=0}^{d_V} \frac{(d+2n-2)!}{d!(2n-2)!} + \sum_{d=0}^{d_V+d_f-1} \frac{(d+2n-2)!}{d!(2n-2)!} - d_V - 1\right).$$

Then, for large number of states, i.e., large n,

$$N_{vars}^{H} \sim 2n\left((2n-2)^{d_V} + (2n-2)^{d_V+d_f-1}\right) \sim n^{d_V+d_f}$$

Meanwhile, the number of constraints in the LP is

$$N_{cons}^{H} = N_{vars}^{H} + L\left(\sum_{d=0}^{d_V} \frac{(d+n-1)!}{d!(n-1)!} + \sum_{d=0}^{d_V+d_f-1} \frac{(d+n-1)!}{d!(n-1)!}\right), \tag{7}$$

where the first term is total number of inequality constraints associated with the positivity of b_α and negativity of c_β, the second term is the number of equality constraints on the coefficients of the Lyapunov function required to ensure continuity (see condition III of Theorem 3) and the third term is the number of equality constraints associated with negativity of the Lie derivative of the Lyapunov function (see condition IV of Theorem 3). By substituting for L in (7), from Assumption 1 for large n we get

$$N_{cons}^{H} \sim n^{d_V+d_f} + 2n(n^{d_V} + n^{d_V+d_f-1}) \sim n^{d_V+d_f}.$$

The complexity of LP using interior-point algorithms is approximately $O(N_{vars}^2 N_{cons})$ [18]. Therefore the computational cost of solving the LP associated with Algorithm 1 is $\sim n^{3(d_V+d_f)}$.

B. Complexity of the SDP associated with Polya's algorithm

Before giving our analysis, we briefly review Polya's algorithm [19] as applied to positivity of a polynomial on the hypercube. First, given a polynomial $T(x)$, for every variable $x_i \in [l_i, u_i]$, we define an auxiliary variable y_i such that the pair (x_i, y_i) lies on the simplex. Then, by using the procedure in [19], we construct a homogeneous version of T, defined as $\tilde{T}(x,y)$ so that $\tilde{T}(x,y) = T(x)$ for $(x_i, y_i) \in \Delta_i$. Finally, if for some $e \geq 0$ (Polya's exponent) the coefficients of $(x_1 + y_1 + \cdots + x_n + y_n)^e \tilde{T}(x,y)$ are positive, then $T(x)$ is positive on the hypercube $[l_1, u_1] \times \cdots \times [l_n, u_n]$.

In [14], we used this approach to construct Lyapunov functions defined on the hypercube. This algorithm used semidefinite programming to search for the coefficients of a matrix-valued polynomial $P(x)$ which defined a Lyapunov function as $V(x) = x^T P(x) x$. In [14], we determined that the number of decision variables in the associated SDP was

$$N_{vars}^{P} = \frac{n(n+1)}{2} \sum_{d=0}^{d_V-2} \frac{(d+n-1)!}{d!(n-1)!}.$$

The number of constraints in the SDP was

$$N_{cons}^{P} = \frac{n(n+1)}{2}\left((d_V+e-1)^n + (d_V+d_f+e-2)^n\right),$$

where e is Polya's exponent mentioned earlier. Then, for large n,

$$N_{vars}^{P} \sim n^{d_V} \quad \text{and} \quad N_{cons}^{P} \sim (d_V+d_f+e-2)^n.$$

Since solving an SDP with an interior-point algorithm typically requires approximately $O(N_{cons}^3 + N_{var}^3 N_{cons} + N_{var}^2 N_{cons}^2)$ operations [18], the computational cost of solving the SDP associated with Polya's algorithm is estimated as $\sim (d_V + d_f + e - 2)^{3n}$.

C. Complexity of the SDP associated with SOS algorithm

To find a Lyapunov function for (1) over the polytope $\Gamma = \left\{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i \in \{1, \cdots, K\}\right\}$

using the SOS approach with Positivstellensatz multipliers [20], we search for a polynomial $V(x)$ and SOS polynomials $s_i(x)$ and $t_i(x)$ such that for any $\varepsilon > 0$

$$V(x) - \varepsilon x^T x - \sum_{i=1}^{K} s_i(x)(w_i^T x + u_i) \text{ is SOS} \quad \text{and}$$
$$-\langle \nabla V(x), f(x) \rangle - \varepsilon x^T x - \sum_{i=1}^{K} t_i(x)(w_i^T x + u_i) \text{ is SOS}.$$

Suppose we choose the degree of the $s_i(x)$ to be $d_V - 2$ and the degree of the $t_i(x)$ to be $d_V + d_f - 2$. Then, it can be shown that the total number of decision variables in the SDP associated with the SOS approach is

$$N_{vars}^S = \frac{N_1(N_1+1)}{2} + K\frac{N_2(N_2+1)}{2} + K\frac{N_3(N_3+1)}{2}, \quad (8)$$

where N_1 is the number of monomials in a polynomial of degree $d_V/2$, N_2 is the number of monomials in a polynomial of degree $(d_V-2)/2$ and N_3 is the number of monomials in a polynomial of degree $(d_V+d_f-2)/2$ calculated as

$$N_1 = \sum_{d=1}^{d_V/2} \frac{(d+n-1)!}{(d)!(n-1)!},$$

$$N_2 = \sum_{d=0}^{(d_V-2)/2} \frac{(d+n-1)!}{(d)!(n-1)!} \quad \text{and} \quad N\text{-} - \sum^{(d_V+d_f-2)/2} (d+n-1)!$$

The first terms in (8) is the nu
variables associated with the poly
and third terms are the number o
polynomials s_i and t_i respectively.
number of constraints in the SDP

$$N_{cons}^S = N_1 + KN_2 +$$

where

$$N_4 = \sum_{d=0}^{(d_V+d_f)/2} \frac{(d-}{(d)}$$

The first term in (9) is the number
with positivity of $V(x)$, the secon
number of constraints associated w
nomials s_i and t_i, respectively. The fourth term is the number of constraints associated with negativity of the Lie derivative. By substituting $K = 2n$ (For the case of a hypercube), for large n we have

$$N_{vars}^S \sim N_3^2 \sim n^{d_V+d_f-1} \text{ and}$$

$$N_{cons}^S \sim KN_3 + N_4 \sim nN_3 + N_4 \sim n^{0.5(d_V+d_f)}.$$

Finally, using an interior-point algorithm with complexity $O(N_{cons}^3 + N_{var}^3 N_{cons} + N_{var}^2 N_{cons}^2)$ to solve the SDP associated the SOS algorithm requires approximately $\sim n^{3.5(d_V+d_f)-3}$ operations. As an additional comparison, we also considered the SOS algorithm for global stability analysis, which does not use Positivstellensatz multipliers. For a large number of states, we have

$$N_{vars}^S \sim n^{0.5d_V} \quad \text{and} \quad N_{cons}^S \sim n^{0.5(d_V+d_f)}.$$

In this case, the complexity of the SDP is approximately $\sim n^{1.5(d_V+d_f)} + n^{2d_V+d_f}$.

D. Comparison of the Complexities

We draw the following conclusions from our complexity analysis.

1. For large number of states, the complexity of the LP associated with Algorithm 1 and the SDP associated with SOS are both **polynomial** in the number of states, whereas the complexity of the SDP associated with Polya's algorithm grows **exponentially** in the number of states. For a large number of states and large degree of the Lyapunov polynomial, the LP has the least computational complexity.

2. The complexity of the LP associated with Algorithm 1 scales linearly with the number of sub-polytopes L.

3. In Fig. 2, we show the number of decision variables and constraints for the LP and SDPs using different degrees of the Lyapunov function and different degrees of the vector field. The figure shows that in general, the SDP associated with Polya's algorithm has the least number of variables and the greatest number of constraints, whereas the SDP associated with SOS has the greatest number of variables and the least number of constraints.

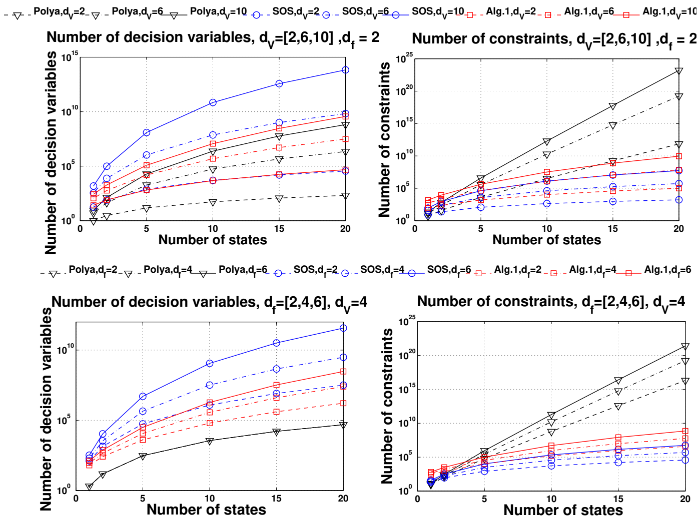

Fig. 2. Number of decision variables and constraints of the optimization problems associated with Algorithm 1, Polya and SOS for different degrees of the Lyapunov function and the vector field $f(x)$

VI. Numerical Results

In this section, we test the accuracy of our algorithm in approximating the region of attraction of a locally-stable nonlinear system known as the reverse-time Van Der Pol oscillator. The system is defined as

$$\dot{x}_1 = -x_2, \; \dot{x}_2 = x_1 + x_2(x_1^2 - 1). \quad (10)$$

We considered the following class of scaled polytopes:

$$\Gamma^s := \{x \in \mathbb{R}^2 : x = \sum_{i=1}^K \mu_i p_i : \mu_i \in [0,s] \text{ and } \sum_{i=1}^K \mu_i = s\}$$

where the vertices p_i are defined as

1) Parallelogram (Γ_P^s): vertices at
$$p_1 = \begin{bmatrix} -1.31 \\ 0.18 \end{bmatrix}, p_2 = \begin{bmatrix} 0.56 \\ 1.92 \end{bmatrix}, p_3 = \begin{bmatrix} -0.56 \\ -1.92 \end{bmatrix} p_4 = \begin{bmatrix} 1.31 \\ -0.18 \end{bmatrix}$$

2) A Square (Φ_Q^s): vertices at
$$p_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, p_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, p_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} p_4 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

3) A Diamond (Θ_R^s): vertices at
$$p_1 = \begin{bmatrix} -1.41 \\ 0 \end{bmatrix}, p_2 = \begin{bmatrix} 0 \\ 1.41 \end{bmatrix}, p_3 = \begin{bmatrix} 1.41 \\ 0 \end{bmatrix} p_4 = \begin{bmatrix} 0 \\ -1.41 \end{bmatrix}$$

We decompose the parallelogram and the diamond into 4 triangles and the decompose the square into 4 squares. We solve the following optimization problem for Lyapunov functions of degree $d = 2,4,6,8$.

$$\max_{s\in\mathbb{R}^+, b_{\alpha,i}, c_{\beta,i}} \quad s$$
$$\text{s.t.} \quad b_{\alpha,i} \geq 0, c_{\beta,i} \leq 0 \text{ for } \alpha \in E_d, \beta \in E_{d+2}, i = 1,\cdots,4$$
and Conditions I to IV of Theorem 3 hold for
$$V_i(x) = \sum_{\alpha\in E_d} b_{\alpha,i} \prod_{j=1}^{3} (h_{i,j}^T x + g_{i,j})^{\alpha_j}, i = 1,\cdots,4,$$
$$Z_i(x) := \sum_{\beta\in E_{d+d_f-1}} c_{\beta,i} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\beta_j}, i = 1,\cdots,4,$$
$$\text{on } \Gamma_P^s := \{x \in \mathbb{R}^2 : x = \sum_{i=1}^{K} \mu_i p_i : \mu_i \in [0,s] \text{ and } \sum_{i=1}^{K} \mu_i = s\}.$$

To solve this problem, we use a bisection search on s in an outer-loop and our LP solver in the inner loop. Fig. 3 illustrates the largest Γ_P^s
$$\Gamma_P^* := \{x \in \mathbb{R}^n : x = \sum_{i=1}^{K} \mu_i p_i : \mu_i \in [0,s^*] \text{ and } \sum_{i=1}^{K} \mu_i = s^*\}$$
and the largest level-set of $V_i(x)$ inscribed in Γ_P^*, for different degrees of $V_i(x)$. Similarly, we solved the same optimization problem replacing Γ_P^s with the square Φ_Q^s and diamond Θ_R^s. In all cases, increasing d resulted in a larger maximum inscribed sub-level set of $V(x)$. We obtained the best results using the parallelogram Γ_P^s which achieved the scaling factor $s^* = 1.639$. The maximum scaling factor for Φ_Q was $s^* = 1.800$ and the maximum scaling factor for Θ_R was $s^* = 1.666$.

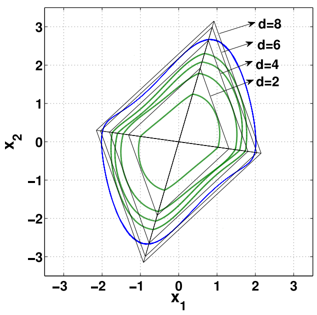

Fig. 3. Largest level sets of Lyapunov functions of different degrees and their associated parallelograms

VII. Conclusion and Future Work

In this paper, we propose an algorithm for stability analysis of nonlinear systems with polynomial vector fields. The algorithm searches for piecewise polynomial Lyapunov functions defined on convex polytope and represented in the Handelman basis and is expressed as a linear program. We also show that the resulting linear program has polynomial complexity in the number of states. We further improve the effectiveness of the algorithm by exploring the best polytopic domain for a given region of attraction. This work can also be potentially applied to stability analysis of switched systems and controller synthesis.

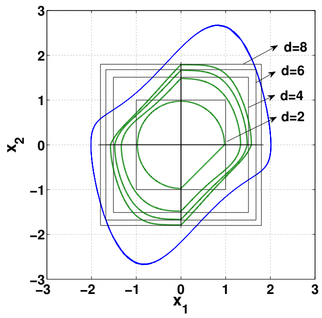

Fig. 4. Largest level sets of Lyapunov functions of different degrees and their associated squares

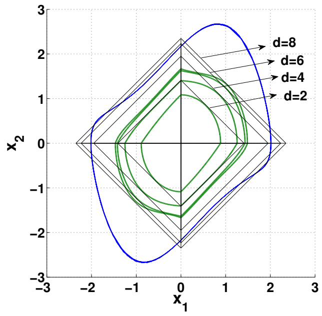

Fig. 5. Largest level sets of Lyapunov functions of different degrees and their associated diamonds

VIII. Appendix: Derivation of the Coefficient Map

In this section, we derive a closed form formula for the coefficient map $\mathcal{C} : \mathbb{R}[x] \to \mathbb{R}^N$ (used in Section IV). For any polynomial of the form
$$V(x) = \sum_{\alpha\in E_d} b_\alpha \prod_{i=1}^{K} (w_i^T x + u_i)^{\alpha_i},$$
we derive $\mathcal{C}(V)$ in terms of w_i, u_i, α and b_α. We show that the map $\mathcal{C}$ is affine in b_α.

Given $w_i \in \mathbb{R}^n$ and $b_\alpha, u_i \in \mathbb{R}$, Let
$$V(x) = \sum_{\alpha\in E_d} b_\alpha \prod_{i=1}^{K} (w_i^T x + u_i)^{\alpha_i},$$
where $E_d := \{\alpha \in \mathbb{N}^K : |\alpha|_1 \leq d\}$. Let $X := (x_1,\cdots,x_n,y)$. Then, we homogenize $V(x)$ as
$$\tilde{V}(X) = \sum_{\alpha\in E_d} b_\alpha \prod_{i=1}^{K} (w_i^T x + u_i y)^{\alpha_i}.$$
by defining the new variable y. By employing the multinomial theorem, we get
$$\tilde{V}(X) = \sum_{\alpha\in E_d} b_\alpha \prod_{i=1}^{K} \left(\sum_{l_{i,j}\in\mathbb{N}:|l_i|_1=\alpha_i} \left(\frac{\alpha_i! w_{i,1}^{l_{i,1}} \dots w_{i,n}^{l_{i,n}} u_i^{l_{i,n+1}}}{l_{i,1}! \dots l_{i,n+1}!} \right) x_1^{l_{i,1}} \dots x_n^{l_{i,n}} y^{l_{i,n+1}} \right).$$
By defining $p(\alpha_i, l_i, w_i, u_i) := \frac{\alpha_i! w_{i,1}^{l_{i,1}} \dots w_{i,n}^{l_{i,n}} u_i^{l_{i,n+1}}}{l_{i,1}! \dots l_{i,n+1}!}$ and expanding $\tilde{V}(X)$, we have

$\tilde{V}(X) =$

$$\sum_{\alpha \in E_d} b_\alpha \left(\sum_{\substack{l_{1,j} \in \mathbb{N}: \\ |l_1|_1 = \alpha_1}} \cdots \sum_{\substack{l_{K,j} \in \mathbb{N}: \\ |l_K|_1 = \alpha_K}} \left(\prod_{i=1}^{K} p(\alpha_i, l_i, w_i, u_i) \right) y^{\gamma_{n+1}} x_1^{\gamma_1} \cdots x_n^{\gamma_n} \right),$$

where $\gamma_i = \sum_{m=1}^{K} l_{m,i}$ for $i = 1, \cdots, n$. Since $F(x_1, ..., x_n, 1) = V(x)$, by substituting $y = 1$ we have

$$V(x) = \sum_{\alpha \in E_d} \left(b_\alpha \sum_{\substack{l_{1,j} \in \mathbb{N}: \\ |l_1|_1 = \alpha_1 = \alpha_1}} \cdots \sum_{\substack{l_{K,j} \in \mathbb{N}: \\ |l_K|_1 = \alpha_K}} \left(\prod_{i=1}^{K} p(\alpha_i, l_i, w_i, u_i) \right) x_1^{\gamma_1} \cdots x_n^{\gamma_n} \right).$$

Thus, for every $\{l_{i,j} \in \mathbb{N}\}_{\substack{i=1,\cdots,K, \\ j=1,\cdots,n}}$ the coefficient of the monomial $x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ is

$$\sum_{\substack{\alpha \in E_d: \\ |l_1|_1 = \alpha_1 \\ \vdots \\ |l_K|_1 = \alpha_K}} \left(\sum_{\substack{l_{1,n+1} \in \{0, \cdots, d - \alpha_1\} \\ \vdots \\ l_{K,n+1} \in \{0, \cdots, d - \alpha_K\}}} b_\alpha \left(\prod_{i=1}^{K} p(\alpha_1, l_i, w_i, u_i) \right) \right).$$

Thus, for any polynomial $V(x)$ of the form

$$V(x) = \sum_{\alpha \in E_d} b_\alpha \prod_{i=1}^{K} (w_i^T x + u_i)^{\alpha_i}, x \in \mathbb{R}^n,$$

with $N = \sum_{j=0}^{d} \frac{(j+n-1)!}{j!(n-1)!}$ monomials, we define the coefficient map $\mathcal{C} : \mathbb{R}[x] \to \mathbb{R}^N$ as $\mathcal{C}(V) := [c_1, \cdots, c_N]$, where

$$c_k = \sum_{\substack{\alpha \in E_d: \\ |l_1|_1 = \alpha_1 \\ \vdots \\ |l_K|_1 = \alpha_K}} \left(\sum_{\substack{l_{1,n+1} \in \{0, \cdots, d - \alpha_1\} \\ \vdots \\ l_{K,n+1} \in \{0, \cdots, d - \alpha_K\}}} b_\alpha \left(\prod_{i=1}^{K} p(\alpha_1, l_i, w_i, u_i) \right) \right). \tag{11}$$

for some $l_{p,j} \in \mathbb{N}$ such that $x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ with $\gamma_j = \sum_{p=1}^{K} l_{p,j}$, is the $k-$th monomial in $V(x)$ using the lexicographical ordering, and where

$$p(\alpha_i, l_i, w_i, u_i) = \frac{\alpha_i! w_{i,1}^{l_{i,1}} ... w_{i,n}^{l_{i,n}} u_i^{l_{i,n+1}}}{l_{i,1}! ... l_{i,n+1}!}.$$

Notice that, from (11), the coefficients of $V(x)$ are affine in $\{b_\alpha\}$. Similarly, it can be shown that the coefficients of $\langle \nabla V(x), f(x) \rangle$ are also affine in $\{b_\alpha\}$. However, for the sake of brevity, we do not derive a formula for the coefficients of $\langle \nabla V(x), f(x) \rangle$.

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