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POLYNOMIAL OPTIMIZATION WITH APPLICATIONS TO STABILITY ANALYSIS AND CONTROL - ALTERNATIVES TO SUM OF SQUARES

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Abstract. In this paper, we explore the merits of various algorithms for polynomial optimization problems, focusing on alternatives to sum of squares programming. While we refer to advantages and disadvantages of Quantifier Elimination, Reformulation Linear Techniques, Blossoming and Groebner basis methods, our main focus is on algorithms defined by Polya's theorem, Bernstein's theorem and Handelman's theorem. We first formulate polynomial optimization problems as verifying the feasibility of semi-algebraic sets. Then, we discuss how Polya's algorithm, Bernstein's algorithm and Handelman's algorithm reduce the intractable problem of feasibility of semi-algebraic sets to linear and/or semi-definite programming. We apply these algorithms to different problems in robust stability analysis and stability of nonlinear dynamical systems. As one contribution of this paper, we apply Polya's algorithm to the problem of H_∞ control of systems with parametric uncertainty. Numerical examples are provided to compare the accuracy of these algorithms with other polynomial optimization algorithms in the literature.

1. **Introduction.**

Consider problems such as portfolio optimization, structural design, local stability of nonlinear ordinary differential equations, control of time-delay systems and control of systems with uncertainties. These problems can all be formulated as *polynomial optimization* or *optimization of polynomials*. In this paper, we survey how computation can be applied to polynomial optimization and optimization of polynomials. One example of polynomial optimization is $\beta^* = \min_{x\in\mathbb{R}^n} p(x)$, where $p : \mathbb{R}^n \to \mathbb{R}$ is a multi-variate polynomial. In general, since $p(x)$ is not convex, this is not a convex optimization problem. It is well-known that polynomial

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optimization is NP-hard [1]. We refer to the dual problem to polynomial optimization as optimization of polynomials, e.g., the dual optimization of polynomials to $\beta^* = \min_{x\in\mathbb{R}^n} p(x)$ is

$$\begin{aligned} \beta^* = \max_{y\in\mathbb{R}} \; & y \\ \text{subject to} \quad & p(x) - y \geq 0 \text{ for all } x \in \mathbb{R}^n. \end{aligned} \tag{1}$$

This problem is convex, yet NP-hard.

One approach to find lower bounds on the optimal objective β^* is to apply Sum of Squares (SOS) *programming* [7, 8]. A polynomial p is SOS if there exist polynomials q_i such that $p(x) = \sum_{i=1}^r q_i(x)^2$. The set $\{q_i \in \mathbb{R}[x], i = 1, \cdots, r\}$ is called an SOS *decomposition* of $p(x)$, where $\mathbb{R}[x]$ is the ring of real polynomials. An SOS program is an optimization problem of the form

$$\begin{aligned} \min_{y\in\mathbb{R}^m} \quad & c^T y \\ \text{subject to} \quad & A_{i,0}(x) + \sum_{j=1}^m y_j A_{i,j}(x) \text{ is SOS, } i = 1, \cdots, k, \end{aligned} \tag{2}$$

where $c \in \mathbb{R}^m$ and $A_{i,j} \in \mathbb{R}[x]$ are given. If $p(x)$ is SOS, then clearly $p(x) \geq 0$ on $\mathbb{R}^n$. While verifying $p(x) \geq 0$ on $\mathbb{R}^n$ is NP-hard, checking whether $p(x)$ is SOS - hence non-negative - can be done in polynomial time [7]. It was first shown in [7] that verifying the existence of a SOS decomposition is a Semi-Definite Program. Fortunately, there exist several algorithms [9, 10, 11] and solvers [12, 14, 13] that solve SDPs to arbitrary precision in polynomial time. To find lower bounds on $\beta^* = \min_{x\in\mathbb{R}^n} p(x)$, consider the SOS program

$$y^* = \max_{y\in\mathbb{R}^m} y \;\text{ subject to }\; p(x) - y \text{ is SOS.}$$

Clearly $y^* \leq \beta^*$. By performing a bisection search on y and semi-definite programming to verify $p(x) - y$ is SOS, one can find y^*. SOS programming can also be used to find lower bounds on the global minimum of polynomials over a semi-algebraic set $S := \{x \in \mathbb{R}^n : g_i(x) \geq 0, h_j(x) = 0\}$ generated by $g_i, h_j \in \mathbb{R}[x]$. Given problem (1) with $x \in S$, *Positivstellensatz* results [15, 16, 17] define a sequence of SOS programs whose objective values form a sequence of lower bounds on the global minimum β^*. It is shown that under certain conditions on S [16], the sequence of lower bounds converges to the global minimum. See [18] for a comprehensive discussion on the Positivstellensatz.

In this paper, we explore the merits of some of the alternatives to SOS programming. There exist several results in the literature that can be applied to polynomial optimization; e.g., Quantifier Elimination (QE) algorithms [19] for testing the feasibility of semi-algebraic sets, Reformulation Linear Techniques (RLTs) [20, 21] for linearizing polynomial optimizations, Polya's result [2] for positivity on the positive orthant, Bernstein's [22, 23] and Handelman's [24] results for positivity on simplices and convex polytopes, and other results based on Groebner bases [3] and Blossoming [4]. We will discuss Polya's, Bernstein's and Handelman's results in more depth. The discussion of the other results are beyond the scope of this paper, however the ideas behind these results can be summarized as follows.

QE algorithms apply to First-Order Logic (FOR) formulae, e.g.,

$$\forall x\, \exists y\, (f(x,y) \geq 0 \Rightarrow ((g(a) < xy) \wedge (a > 2)),$$

to eliminate the *quantified* variables x and y (preceded by quantifiers $\forall, \exists$) and construct an equivalent FOR formula in terms of the *unquantified* variable a. The key result underlying QE algorithms is Tarski-Seidenberg theorem [5]. The theorem implies that for every formula of the form $\forall x \in \mathbb{R}^n \, \exists y \in \mathbb{R}^m (f_i(x, y, a) \geq 0)$, where $f_i \in \mathbb{R}[x, y, a]$, there exists an equivalent quantifier-free formula of the form $\wedge_i (g_i(a) \geq 0) \vee_j (h_j(a) \geq 0)$ with $g_i, h_j \in \mathbb{R}[a]$. QE implementations [25, 26] with a bisection search yields the exact solution to optimization of polynomials, however the complexity scales double exponentially in the dimension of variables x, y.

RLT was initially developed to find the convex hull of feasible solutions of zero-one linear programs [27]. It was later generalized to address polynomial optimizations of the form $\min_x p(x)$ subject to $x \in [0,1]^n \cap S$ [20]. RLT constructs a $\delta-$hierarchy of linear programs by performing two steps. In the first step (reformulation), RLT introduces the new constraints $\prod_i x_i \prod_j (1 - x_j) \geq 0$ for all $i, j : i + j = \delta$. In the second step (linearization), RTL defines a linear program by replacing every product of variables x_i by a new variable. By increasing δ and repeating the two steps, one can construct a $\delta-$hierarchy of lower bounding linear programs. A combination of RLT and branch-and-bound partitioning of $[0,1]^n$ was developed in [21] to achieve tighter lower bounds on the global minimum. For a survey of different extensions of RLT see [6].

Groebner bases can be used to reduce a polynomial optimization to the problem of finding the roots of univariate polynomials [28]. First, one needs to construct the system of polynomial equations $\nabla L(x, \lambda, \mu) = [f_1(x, \lambda, \mu), \cdots, f_N(x, \lambda, \mu)] = 0$, where $L := p(x) + \sum_i \lambda_i g_i(x) + \sum_i \mu_i h_i(x)$ is the Lagrangian. It is well-known that the set of solutions to $\nabla L(x, \lambda, \mu) = 0$ is the set of extrema of the polynomial optimization $\min_{x \in S} p(x)$. Using the elimination property [3] of the Groebner bases, the minimal Groebner basis of the ideal of $f_1, \cdots, f_N$ defines a triangular-form system of polynomial equations. This system can be solved by calculating one variable at a time and back-substituting into other polynomials. The most computationally expensive part is the calculation of the Groebner basis, which in the worst case scales double-exponentially in the number of decision variables.

The blossoming approach involves mapping the space of polynomials to the space of multi-affine functions (polynomials that are affine in each variable). By using this map and the diagonal property of blossoms [4], one can reformulate any polynomial optimization $\min_{x \in S} p(x)$ as an optimization of multi-affine functions. In [30], it is shown that the dual to optimization of multi-affine functions over a hypercube is a linear program. The optimal objective value of this linear program is a lower bound on the minimum of $p(x)$ over the hypercube.

While the discussed algorithms have advantages and disadvantages, we focus on Polya's, Bernstein's and Handelman's results on parameterization of positive polynomials. Polya's theorem yields a basis to parameterize the cone of polynomials that are positive on the positive orthant. Bernstein's and Handelman's theorems yield a basis to parameterize the space of polynomials that are positive on simplices and convex polytopes. Similar to SOS programming, one can find Polya's, Bernstein's and Handelman's parameterizations by solving a sequence of Linear Programs (LPs) and/or SDPs. However, unlike the SDPs associated with SOS programming, the SDPs associated with these theorems have a block-diagonal structure. This structure has been exploited in [29] to design parallel algorithms for optimization of polynomials with large degrees and number of variables. Unfortunately, unlike SOS programming, Bernstein's, Handelman's and the original Polya's theorems do not parameterize polynomials with zeros in the positive orthant. Yet, there exist some variants of Polya's theorem which considers zeros at the corners [31] and edges [32] of simplices. Moreover, there exist other variants of Polya's theorem which provide certificates of positivity on hypercubes [33, 34], intersection of semi-algebraic sets and the positive orthant [35] and the entire $\mathbb{R}^n$ [36], or apply to polynomials with rational exponents [37].

We organize this paper as follows. In Section 2, we place Polya's, Bernstein's, Handelman's and the Positivstellensatz results in the broader topic of research on polynomial positivity. In Section 3, we first define polynomial optimization and optimization of polynomials. Then, we formulate optimization of polynomials as the problem of verifying the feasibility of semi-algebraic sets. To verify the feasibility of different semi-algebraic sets, we present algorithms based on the different variants of Polya's, Bernstein's, Handelman's and Positivstellensatz results. In Section 4, we discuss how these algorithms apply to robust stability analysis [38, 29, 39] and nonlinear stability [41, 42, 43, 44]. Finally, one contribution of this paper is to apply Polya's algorithm to the problem of H_∞ control synthesis for systems with parametric uncertainties.

2. **Background on positivity of polynomials.** In 1900, Hilbert published a list of mathematical problems, one of which was: For every non-negative $f \in \mathbb{R}[x]$, does there exist some non-zero $q \in \mathbb{R}[x]$ such that $q^2 f$ is a sum of squares? In other words, is every non-negative polynomial a sum of squares of rational functions? This question was motivated by his earlier works [48, 49], in which he proved: 1- Every non-negative bi-variate degree 4 homogeneous polynomial is a SOS of three polynomials. 2- Every bi-variate non-negative polynomial is a SOS of four rational functions. 3- Not every homogeneous polynomial with more than two variables and degree greater than 5 is SOS of polynomials. Eighty years later, Motzkin constructed a non-negative degree 6 polynomial with three variables which is not SOS [50]:

$$M(x_1, x_2, x_3) = x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 x_3^2 + x_3^6.$$

Robinson [51] generalized Motzkin's example as follows. Polynomials of the form $(\prod_{i=1}^n x_i^2) f(x_1, \cdots, x_n) + 1$ are not SOS if polynomial f of degree $< 2n$ is not SOS. Hence, although the non-homogeneous Motzkin polynomial $M(x_1, x_2, 1) = x_1^2 x_2^2 (x_1^2 + x_2^2 - 3) + 1$ is non-negative it is not SOS.

In 1927, Artin answered Hilbert's problem in the following theorem [52].

Theorem 2.1. *(Artin's theorem) A polynomial $f \in \mathbb{R}[x]$ satisfies $f(x) \geq 0$ on $\mathbb{R}^n$ if and only if there exist SOS polynomials N and $D \neq 0$ such that $f(x) = \frac{N(x)}{D(x)}$.*

Although Artin settled Hilbert's problem, his proof was neither constructive nor gave a characterization of the numerator N and denominator D. In 1939, Habicht [54] showed that if f is positive definite and can be expressed as $f(x_1, \cdots, x_n) = g(x_1^2, \cdots, x_n^2)$ for some polynomial g, then one can choose the denominator $D = \sum_{i=1}^n x_i^2$. Moreover, he showed that by using $D = \sum_{i=1}^n x_i^2$, the numerator N can be expressed as a sum of squares of monomials. Habicht used Polya's theorem [53] to obtain the above characterizations for N and D.

Theorem 2.2. *(Polya's theorem) Suppose a homogeneous polynomial p satisfies $p(x) > 0$ for all $x \in \{x \in \mathbb{R}^n : x_i \geq 0, \sum_{i=1}^n x_i \neq 0\}$. Then $p(x)$ can be expressed as*

$$p(x) = \frac{N(x)}{D(x)},$$

where $N(x)$ and $D(x)$ are homogeneous polynomials with all positive coefficients. For every homogeneous $p(x)$ and some $e \geq 0$, the denominator $D(x)$ can be chosen as $(x_1 + \cdots + x_n)^e$.

Suppose f is homogeneous and positive on the positive orthant and can be expressed as $f(x_1, \cdots, x_n) = g(x_1^2, \cdots, x_n^2)$ for some homogeneous polynomial g. By using Polya's theorem $g(y) = \frac{N(y)}{D(y)}$, where $y := (y_1, \cdots, y_n)$ and polynomials N and D have all positive coefficients. By Theorem 2.2 we may choose $D(y) = \left(\sum_{i=1}^n y_i\right)^e$. Then $\left(\sum_{i=1}^n y_i\right)^e g(y) = N(y)$. Now let $x_i = \sqrt{y_i}$, then $\left(\sum_{i=1}^n x_i^2\right)^e f(x_1, \cdots, x_n) = N(x_1^2, \cdots, x_n^2)$. Since N has all positive coefficients, $N(x_1^2, \cdots, x_n^2)$ is a sum of squares of monomials. Unlike the case of positive definite polynomials, it is shown that there exists no single SOS polynomial $D \neq 0$ which satisfies $f = \frac{N}{D}$ for every positive semi-definite f and some SOS polynomial N [55].

As in the case of positivity on $\mathbb{R}^n$, there has been an extensive research regarding positivity of polynomials on bounded sets. A pioneering result on local positivity is Bernstein's theorem (1915) [56]. Bernstein's theorem uses the polynomials $h_{i,j} = (1+x)^i(1-x)^j$ as a basis to parameterize univariate polynomials which are positive on $[-1, 1]$.

Theorem 2.3. *(Bernstein's theorem) If a polynomial $f(x) > 0$ on $[-1, 1]$, then there exist $c_{i,j} > 0$ such that*

$$f(x) = \sum_{\substack{i,j \in N \\ i+j=d}} c_{i,j}(1+x)^i(1-x)^j$$

for some $d > 0$.

Reference [57] uses Goursat's transform of f to find an upper bound on d. The bound is a function of the minimum of f on $[-1, 1]$. However, computing the minimum itself is intractable. In 1988, Handelman [58] used products of affine functions as a basis (the Handelman basis) to extend Bernstein's theorem to multivariate polynomials which are positive on convex polytopes.

Theorem 2.4. *(Handelman's Theorem) Given $w_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}$, define the polytope $\Gamma^K := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\}$. If a polynomial $f(x) > 0$ on Γ^K, then there exist $b_\alpha \geq 0$, $\alpha \in \mathbb{N}^K$ such that for some $d \in \mathbb{N}$,*

$$f(x) = \sum_{\substack{\alpha \in \mathbb{N}^K \\ \alpha_1 + \cdots + \alpha_K \leq d}} b_\alpha (w_1^T x + u_1)^{\alpha_1} \cdots (w_K^T x + u_K)^{\alpha_K}. \tag{3}$$

In [22], first the *standard triangulation* of a simplex (the convex hull of vertices in $\mathbb{R}^n$) is developed to decompose an arbitrary simplex into sub-simplices. Then, an algorithm is proposed to ensure positivity of a polynomial f on the simplex by finding an expression of Form (3) for f on each sub-simplex. An upper bound on the degree d in (3) was provided in [23] as a function of the minimum of f on the simplex, the number of variables of f, the degree of f and the maximum of certain [23] affine combinations of the coefficients b_α. Reference [22] also provides a bound on d as a function of $\max_\alpha b_\alpha$ and the minimum of f over the polytope.

An extension of Handelman's theorem was made by Schweighofer [59] to verify non-negativity of polynomials over compact semi-algebraic sets. Schweighofer used the cone of polynomials in (4) to parameterize any polynomial f which has the following properties:

1. f is non-negative over the compact semi-algebraic set S
2. $f = q_1 p_1 + q_2 p_2 + \cdots$ for some q_i in the cone (4) and for some $p_i > 0$ over $S \cap \{x \in \mathbb{R}^n : f(x) = 0\}$

Theorem 2.5. *(Schweighofer's theorem) Suppose*

$$S := \{x \in \mathbb{R}^n : g_i(x) \geq 0, g_i \in \mathbb{R}[x] \text{ for } i = 1, \cdots, K\}$$

is compact. Define the following set of polynomials which are positive on S.

$$\Theta_d := \left\{ \sum_{\lambda \in \mathbb{N}^K : \lambda_1 + \cdots + \lambda_K \leq d} s_\lambda g_1^{\lambda_1} \cdots g_K^{\lambda_k} : s_\lambda \text{ are SOS} \right\} \tag{4}$$

If $f \geq 0$ on S and there exist $q_i \in \Theta_d$ and polynomials $p_i > 0$ on $S \cap \{x \in \mathbb{R}^n : f(x) = 0\}$ such that $f = \sum_i q_i p_i$ for some d, then $f \in \Theta_d$.

On the assumption that g_i are affine functions, $p_i = 1$ and s_λ are constant, Schweighofer's theorem gives the same parameterization of f as in Handelman's theorem. Another special case of Schweighofer's theorem is when $\lambda \in \{0,1\}^K$. In this case, Schweighofer's theorem reduces to Schmudgen's Positivstellensatz [17]. Schmudgen's Positivstellensatz states that the cone

$$\Lambda_g := \left\{ \sum_{\lambda \in \{0,1\}^K} s_\lambda g_1^{\lambda_1} \cdots g_K^{\lambda_k} : s_\lambda \text{ are } SOS \right\} \subset \Theta_d \tag{5}$$

is sufficient to parameterize every $f > 0$ over the semi-algebraic set S generated by $\{g_1, \cdots, g_K\}$. Unfortunately, the cone Λ_g contains 2^K products of g_i, thus finding a representation of Form (5) for f requires a search for at most 2^K SOS polynomials. Putinar's Positivstellensatz [16] reduces the complexity of Schmudgen's parameterization in the case where the quadratic module of g_i defined in (6) is *Archimedean*.

Theorem 2.6. *(Putinars's Positivstellensatz) Let $S := \{x \in \mathbb{R}^n : g_i(x) \geq 0, g_i \in \mathbb{R}[x] \text{ for } i = 1, \cdots, K\}$ and define*

$$M_g := \left\{ s_0 + \sum_{i=1}^{K} s_i g_i : s_i \text{ are SOS} \right\}. \tag{6}$$

If there exist some $N > 0$ such that $N - \sum_{i=1}^n x_i^2 \in M_g$, then M_g is Archimedean. If M_g is Archimedean and $f > 0$ over S, then $f \in M_g$.

Finding a representation of Form (6) for f, only requires a search for K SOS polynomials using SOS programming. Verifying the Archimedian condition $N - \sum_{i=1}^n x_i^2 \in M_g$ in Theorem 2.6 is also a SOS program. Observe that the Archimedian condition implies the compactness of S. The following theorem, lifts the compactness requirement for the semi-algebraic set S.

Theorem 2.7. *(Stengle's Positivstellensatz) Let $S := \{x \in \mathbb{R}^n : g_i(x) \geq 0, g_i \in \mathbb{R}[x] \text{ for } i = 1, \cdots, K\}$ and define*

$$\Lambda_g := \left\{ \sum_{\lambda \in \{0,1\}^K} s_\lambda g_1^{\lambda_1} \cdots g_K^{\lambda_k} : s_\lambda \text{ are SOS} \right\}.$$

If $f > 0$ on S, then there exist $p, g \in \Lambda_g$ such that $qf = p + 1$.

Notice that the Parameterziation (3) in Handelman's theorem is affine in f and the coefficients b_α. Likewise, the parameterizations in Theorems 2.5 and 2.6, i.e., $f = \sum_\lambda s_\lambda g_1^{\lambda_1} \cdots g_K^{\lambda_K}$ and $f = s_0 + \sum_i s_i g_i$ are affine in f, s_λ and s_i. Thus, one can use convex optimization to find b_α, s_λ, s_i and f. Unfortunately, since the parameterization $qf = p + 1$ in Stengle's Positivstellensatz is non-convex (bilinear in q and f), it is more difficult to verify the feasibility of $qf = p + 1$ compared to Handelman's and Putinar's parameterizations.

For a comprehensive discussion on the Positivstellensatz and other polynomial positivity results in algebraic geometry see [61, 60, 62].

3. **Algorithms for Polynomial Optimization.**

In this Section, we first define polynomial optimization, optimization of polynomials and its equivalent feasibility problem using semi-algebraic sets. Then, we introduce some algorithms to verify the feasibility of different semi-algebraic sets. We observe that combining these algorithms with bisection yields some lower bounds on optimal objective values of polynomial optimization problems.

3.1. **Polynomial Optimization and optimization of polynomials.**

We define *polynomial optimization* problems as

$$\begin{aligned} \beta^* = \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \geq 0 \text{ for } i = 1, \cdots, m \\ & h_j(x) = 0 \text{ for } j = 1, \cdots, r, \end{aligned} \tag{7}$$

where $f, g_i, h_j \in \mathbb{R}[x]$ are given. For example, the integer program

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & p(x) \\ \text{subject to} \quad & a_i^T x \geq b_i \text{ for } i = 1, \cdots, m, \\ & x \in \{-1, 1\}^n, \end{aligned} \tag{8}$$

with given $a_i \in \mathbb{R}^n, b_i \in \mathbb{R}$ and $p \in \mathbb{R}[x]$, can be formulated as a polynomial optimization problem by setting

$$\begin{aligned} & f = p \\ & g_i(x) = a_i^T x - b_i \qquad \text{for } i = 1, \cdots, m \\ & h_j(x) = x_j^2 - 1 \qquad \text{for } j = 1, \cdots, n. \end{aligned}$$

Let $S := \{x \in \mathbb{R}^n : g_i(x) \geq 0, h_j(x) = 0,\ i = 1, \cdots, m,\ j = 1, \cdots, r\}$. We define *Optimization of polynomials* problems as

$$\begin{aligned} \gamma^* = \max_{x \in \mathbb{R}^n} \quad & c^T x \\ \text{subject to} \quad & F(x, y) := F_0(y) + \sum_{i=1}^n x_i F_i(y) \geq 0 \text{ for all } y \in S, \end{aligned} \tag{9}$$

where $c \in \mathbb{R}^n$ and $F_i(y) = \sum_\alpha F_{i,\alpha} y^\alpha$, where $F_{i,\alpha} \in \mathbb{R}^q$ are either given or are decision variables. Optimization of polynomials can be used to find β^* in (7). For example, we can compute the optimal objective value α^* of the polynomial optimization problem

$$\begin{aligned} \alpha^* = \min_{x\in\mathbb{R}^n} \quad & p(x) \\ \text{subject to} \quad & a_i^T x - b_i \geq 0 \qquad \text{for } i = 1,\cdots,m, \\ & x_j^2 - 1 = 0 \qquad \text{for } j = 1,\cdots,n, \end{aligned}$$

by solving the problem

$$\begin{aligned} \alpha^* = \max_{\alpha} \quad & \alpha \\ \text{subject to} \quad & p(x) \geq \alpha \text{ for all } x \in \{-1,1\}^n \\ & a_i^T x \geq b_i \text{ for } i = 1,\cdots,m \text{ and for all } x \in \{-1,1\}^n, \end{aligned} \tag{10}$$

where Problem (10) can be expressed in the Form (9) by setting

$$c = 1, \quad n = 1, \quad k = 0, \quad q = m+1, \quad h_j(y) = y_j^2 - 1 \text{ for } j = 1,\cdots,n$$

$$F_0(y) = \begin{bmatrix} p(y) & 0 & \cdots & 0 \\ 0 & a_1^T y - b_1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a_m^T y - b_m \end{bmatrix}, \quad F_1 = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{bmatrix}.$$

Optimization of polynomials (9) can be formulated as the following feasibility problem.

$$\begin{aligned} &\gamma^* = \min_{\gamma} \;\; \gamma \\ &\text{subject to } S_\gamma := \left\{ x, y \in \mathbb{R}^n : c^T x > \gamma,\, F(x,y) \geq 0,\, g_i(y) \geq 0, h_j(y) = 0 \right\} = \emptyset, \end{aligned} \tag{11}$$

where c, F, g_i and h_j are given. The question of feasibility of a semi-algebraic set is NP-hard [1]. However, if we have a test to verify $S_\gamma = \emptyset$, we can find γ^* by performing bisection on γ. In Section 3.2, we use the results of Section 2 to provide sufficient conditions, in the form of Linear Matrix Inequalities (LMIs), for $S_\gamma = \emptyset$.

3.2. **Algorithms.** In this section, we discuss how to find lower bounds on β^* for different classes of polynomial optimization problems. The results in this section are primarily expressed as methods for verifying $S_\gamma = \emptyset$ and can be used with bisection to solve polynomial optimization problems.

Case 1. Optimization over the standard simplex Δ^n

Define the standard unit simplex as

$$\Delta^n := \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, x_i \geq 0\}. \tag{12}$$

Consider the polynomial optimization problem

$$\gamma^* = \min_{x \in \Delta^n} \quad f(x),$$

where f is a homogeneous polynomial of degree d. If f is not homogeneous, we can homogenize it by multiplying each monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in f by $(\sum_{i=1}^n x_i)^{d - \|\alpha\|_1}$. Notice that since $\sum_{i=1}^n x_i = 1$ for all $x \in \Delta^n$, the homogenized f is equal to f for all $x \in \Delta^n$. To find γ^*, one can solve the following optimization of polynomials problem.

$$\begin{aligned} \gamma^* = \max_{\gamma \in \mathbb{R}} \quad & \gamma \\ \text{s.t. } \; & f(x) \geq \gamma \text{ for all } x \in \Delta^n \end{aligned} \tag{13}$$

It can be shown that Problem (13) is equivalent to the feasibility problem

$$\begin{aligned} \gamma^* = \min_{\gamma \in \mathbb{R}} \quad & \gamma \\ \text{s.t. } \; & S_\gamma := \{x \in \Delta^n : f(x) - \gamma < 0\} = \emptyset. \end{aligned}$$

For a given γ, we use the following version of Polya's theorem to verify $S_\gamma = \emptyset$.

Theorem 3.1. *(Polya's theorem, simplex version) If a homogeneous matrix-valued polynomial F satisfies $F(x) > 0$ for all $x \in \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, x_i \geq 0\}$, then there exists $e \geq 0$ such that all the coefficients of*

$$\left(\sum_{i=1}^n x_i\right)^e F(x)$$

are positive definite.

Given $\gamma \in \mathbb{R}$, it follows from Theorem 3.1 that $S_\gamma = \emptyset$ if there exist $e \geq 0$ such that

$$\left(\sum_{i=1}^n x_i\right)^e \left(f(x) - \gamma \left(\sum_{i=1}^n x_i\right)^d\right) \tag{14}$$

has all positive coefficients. We can compute lower bounds on γ^* by performing bisection on γ. For each γ in bisection, if there exist $e \geq 0$ such that all of the coefficients of (14) are positive, then $\gamma \leq \gamma^*$.

Case 2. Optimization over the hypercube Φ^n:

Given $r_i \in \mathbb{R}$, define the hypercube

$$\Phi^n := \{x \in \mathbb{R}^n : |x_i| \leq r_i, i = 1, \cdots, n\}. \tag{15}$$

Define the set of n-variate *multi-homogeneous* polynomials of degree vector $d \in \mathbb{N}^n$ as

$$\left\{ p \in \mathbb{R}[x, y] : p(x, y) = \sum_{\substack{h,g \in \mathbb{N}^n \\ h+g=d}} p_{h,g} x_1^{h_1} y_1^{g_1} \cdots x_n^{h_n} y_n^{g_n} \right\}.$$

It is shown in [34] that for every polynomial $f(z)$ with $z \in \Phi^n$, there exists a multi-homogeneous polynomial p such that

$$\{f(z) \in \mathbb{R} : z \in \Phi^n\} = \left\{p(x, y) \in \mathbb{R} : (x_i, y_i) \in \Delta^2, i = 1, \cdots, n\right\}. \tag{16}$$

For example, consider $f(z_1, z_2) = z_1^2 + z_2$, with $z_1 \in [-2, 2]$ and $z_2 \in [-1, 1]$. Let $x_1 = \frac{z_1+2}{4} \in [0, 1]$ and $x_2 = \frac{z_2+1}{2} \in [0, 1]$. Then define

$$q(x_1, x_2) := f(4x_1 - 2, 2x_2 - 1) = 16x_1^2 - 16x_1 + 2x_2 + 3$$

By homogenizing q we obtain the multi-homogeneous polynomial

$$\begin{aligned} p(x, y) =& 16x_1^2(x_2 + y_2) - 16x_1(x_1 + y_1)(x_2 + y_2) + 2x_2(x_1 + y_1)^2 \\ & + 3(x_1 + y_1)^2(x_2 + y_2), \quad (x_1, y_1), (x_2, y_2) \in \Delta^2 \end{aligned}$$

with degree vector $d = [2, 1]$. See [34] for an algorithm which computes the multi-homogeneous polynomial p for an arbitrary f defined on a hypercube.

Now consider the polynomial optimization problem

$$\gamma^* = \min_{x \in \Phi^n} f(x).$$

To find γ^*, one can solve the following feasibility problem.

$$\begin{aligned} \gamma^* = &\min_{\gamma \in \mathbb{R}} \quad \gamma \\ &\text{subject to} \quad S_\gamma := \{x \in \Phi^n : f(x) - \gamma < 0\} = \emptyset \end{aligned} \tag{17}$$

For a given γ, one can use the following version of Polya's theorem to verify $S_\gamma = \emptyset$.

Theorem 3.2. *(Polya's theorem, multi-homogeneous version) A multi-homogeneous matrix-valued polynomial F satisfies $F(x, y) > 0$ for all $(x_i, y_i) \in \Delta^2, i = 1, \cdots, n$, if there exist $e \geq 0$ such that all the coefficients of*

$$\left(\prod_{i=1}^{n} (x_i + y_i)^e \right) F(x, y)$$

are positive definite.

First, by using the algorithm in [34] we obtain the multi-homogeneous form p of the polynomial f in (17). Given $\gamma \in \mathbb{R}$, from Theorem 3.2 it follows that $S_\gamma = \emptyset$ in (17) if there exist $e \geq 0$ such that

$$\left(\prod_{i=1}^{n} (x_i + y_i)^e \right) \left(p(x, y) - \gamma \left(\prod_{i=1}^{n} (x_i + y_i)^{d_i} \right) \right) \tag{18}$$

has all positive coefficients, where d_i is the degree of x_i in $p(x, y)$. We can compute lower bounds on γ^* by performing bisection on γ. For each γ in bisection, if there exist $e \geq 0$ such that all of the coefficients of (18) are positive, then $\gamma \leq \gamma^*$.

Case 3. Optimization over the convex polytope Γ^K:

Given $w_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}$, define the convex polytope $\Gamma^K := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\}$. Suppose Γ^K is bounded. Consider the polynomial optimization problem

$$\gamma^* = \min_{x \in \Gamma^K} f(x),$$

where f is a polynomial of degree d_f. To find γ^*, one can solve the feasibility problem.

$$\begin{aligned} \gamma^* = &\min_{\gamma \in \mathbb{R}} \quad \gamma \\ &\text{subject to} \quad S_\gamma := \{x \in \Gamma^K : f(x) - \gamma < 0\} = \emptyset. \end{aligned}$$

Given γ, one can use Handelman's theorem (Theorem 2.4) to verify $S_\gamma = \emptyset$ as follows. Consider the Handelman basis associated with polytope Γ^K defined as

$$B_s := \left\{ \lambda_\alpha \in \mathbb{R}[x] : \lambda_\alpha(x) = \prod_{i=1}^{K} \left(w_i^T x + u_i \right)^{\alpha_i}, \alpha \in \mathbb{N}^K, \sum_{i=1}^{K} \alpha_i \leq s \right\}.$$

Basis B_s spans the space of polynomials of degree s or less, however it is not minimal. Given polynomial $f(x)$ of degree $d_f, \gamma \in \mathbb{R}$ and $d_{\max} \in \mathbb{N}$, if there exist

$$c_\alpha \geq 0 \text{ for all } \alpha \in \{\alpha \in \mathbb{N}^K : \|\alpha\|_1 \leq d\} \tag{19}$$

such that

$$f(x) - \gamma = \sum_{\|\alpha\|_1 \le d} c_\alpha \prod_{i=1}^{K} (w_i^T x + u_i)^{\alpha_i}, \tag{20}$$

for $d = d_f$, then $f(x) - \gamma \ge 0$ for all $x \in \Gamma^K$. Thus $S_\gamma = \emptyset$. Feasibility of Conditions (19) and (20) can be determined using linear programming. If (19) and (20) are infeasible for some d, then one can increase d up to $d_{\max}$. From Handelman's theorem, if $f(x) - \gamma > 0$ for all $x \in \Gamma^K$, then for some $d \ge d_f$, Conditions (19) and (20) hold. However, computing upper bounds for d is difficult [63, 23].

Similar to Cases 1 and 2, we can compute lower bounds on γ^* by performing bisection on γ. For each γ in bisection, if there exist $d \ge d_f$ such that Conditions (19) and (20), then $\gamma \le \gamma^*$.

Case 4: Optimization over compact semi-algebraic sets:

Recall that we defined a semi-algebraic set as

$$S := \{x \in \mathbb{R}^n : g_i(x) \ge 0, i = 1, \cdots, m, \; h_j(x) = 0, j = 1, \cdots, r\}.$$

Suppose S is compact. Consider the polynomial optimization problem

$$\begin{aligned} \gamma^* = \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \ge 0 \text{ for } i = 1, \cdots, m \\ & h_j(x) = 0 \text{ for } j = 1, \cdots, r. \end{aligned}$$

Define the following cone of polynomials which are positive over S.

$$M_{g,h} := \left\{ m \in \mathbb{R}[x] : m(x) = s_0(x) + \sum_{i=1}^{m} s_i(x) g_i(x) + \sum_{i=1}^{r} t_i(x) h_i(x), s_i \in \Sigma_{2d}, t_i \in \mathbb{R}[x] \right\}, \tag{21}$$

where Σ_{2d} denotes the set of SOS polynomials of degree $2d$. From Putinar's Positivstellensatz (Theorem 2.6) it follows that if the Cone(21) is Archimedean, then the solution to the following SOS program is a lower bound on γ^*. Given $d \in \mathbb{N}$, define

$$\begin{aligned} \gamma^d = & \max_{\gamma \in \mathbb{R}, s_i, t_i} \quad \gamma \\ & \text{subject to} \quad f(x) - \gamma = s_0(x) + \sum_{i=1}^{m} s_i(x) g_i(x) + \sum_{i=1}^{r} t_i(x) h_i(x), \; t_i \in \mathbb{R}[x], s_i \in \Sigma_{2d}. \end{aligned} \tag{22}$$

For given $\gamma \in \mathbb{R}$ and $d \in \mathbb{N}$, Problem (22) is the following linear matrix inequality.

$$\begin{aligned} \text{Find} \quad & Q_i \ge 0, \, P_j \quad \text{for } i = 0, \cdots, m \text{ and } j = 1, \cdots, r \\ \text{such that} \quad & f(x) - \gamma = z_d^T(x) \left(Q_0 + \sum_{i=1}^{m} Q_i g_i(x) + \sum_{j=1}^{r} P_j h_j(x) \right) z_d(x), \end{aligned} \tag{23}$$

where $Q_i, P_j \in \mathbb{S}^N$, where $\mathbb{S}^N$ is the subspace of symmetric matrices in $\mathbb{R}^{N \times N}$ and $N := \binom{n+d}{d}$, and where $z_d(x)$ is the vector of monomial basis of degree d or less. See [8, 64] for methods of solving SOS programs. It is shown in [65] that if the Cone (21) is Archimedean, then $\lim_{d \to \infty} \gamma^d = \gamma^*$.

If the Cone (21) is not Archimedean, then we can use Schmudgen's Positivstellensatz to obtain the following SOS program with solution $\gamma^d \leq \gamma^*$.

$$\gamma^d = \max_{\gamma \in \mathbb{R}, s_i, t_i} \quad \gamma$$
$$\text{subject to} \quad f(x) - \gamma = 1 + \sum_{\lambda \in \{0,1\}^m} s_\lambda(x) g_1(x)^{\lambda_1} \cdots g_m(x)^{\lambda_m} + \sum_{i=1}^{r} t_i(x) h_i(x),\ t_i \in \mathbb{R}[x],$$
$$s_\lambda \in \Sigma_{2d}. \quad (24)$$

Case 5: Tests for non-negativity on $\mathbb{R}^n$:

The following theorem [54], gives a test for non-negativity of a class of homogeneous polynomials.

Theorem 3.3. *(Habicht theorem) For every homogeneous polynomial f that satisfies $f(x_1, \cdots, x_n) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $f(x_1, \cdots, x_n) = g(x_1^2, \cdots, x_n^2)$ for some polynomial g, there exist $e \geq 0$ such that all of the coefficients of*

$$\left(\sum_{i=1}^{n} x_i^2 \right)^e f(x_1, \cdots, x_n) \quad (25)$$

are positive. In particular, the product is a sum of squares of monomials.

Based on Habicht's theorem, a test for non-negativity of a homogeneous polynomial f of the form $f(x_1, \cdots, x_n) = g(x_1^2, \cdots, x_n^2)$ is to multiply it repeatedly by $\sum_{i=1}^n x_i^2$. If for some $e \in \mathbb{N}$, the Product (25) has all positive coefficients, then $f \geq 0$.

An alternative test for non-negativity on $\mathbb{R}^n$ is given in the following theorem [36].

Theorem 3.4. *Define $E_n := \{-1, 1\}^n$. Suppose a polynomial $f(x_1, \cdots, x_n)$ of degree d satisfies $f(x_1, \cdots, x_n) > 0$ for all $x \in \mathbb{R}^n$ and its homogenization is positive definite. Then*

1. *there exist $\lambda_e \geq 0$ and coefficients $c_\alpha \in \mathbb{R}$ such that*
$$\left(1 + e^T x\right)^{\lambda_e} f(x_1, \cdots, x_n) = \sum_{\alpha \in I_e} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} \text{ for all } e \in E_n, \quad (26)$$
where $I_e := \{\alpha \in \mathbb{N}^n : \|\alpha\|_1 \leq d + \lambda_e\}$ and $sgn(c_\alpha) = e_1^{\alpha_1} \cdots e_n^{\alpha_n}$.
2. *there exist positive $N, D \in \mathbb{R}[x_1^2, \cdots, x_n^2, f^2]$ such that $f = \frac{N}{D}$.*

Based on Theorem 3.4, we can propose the following test for non-negativity of polynomials over the cone $\Lambda_e := \{x \in \mathbb{R}^n : sgn(x_i) = e_i, i = 1, \cdots, n\}$ for some $e \in E_n$. Multiply a given polynomial f repeatedly by $1 + e^T x$ for some $e \in E_n$. If there exists $\lambda_e \geq 0$ such that $sgn(c_\alpha) = e_1^{\alpha_1} \cdots e_n^{\alpha_n}$, then $f(x) \geq 0$ for all $x \in \Lambda_e$. Since $\mathbb{R}^n = \cup_{e \in E_n} \Lambda_e$, we can repeat the test 2^n times to obtain a test for non-negativity of f over $\mathbb{R}^n$.

The second part of Theorem 3.4 gives a solution to the Hilbert's problem in Section 2. See [36] for an algorithm which computes polynomials N and D.

4. **Applications of polynomial optimization.** In this section, we discuss how the algorithms in Section 3.2 apply to stability analysis and control of dynamical systems. We consider robust stability analysis of linear systems with parametric uncertainty, stability of nonlinear systems, robust controller synthesis for systems with parametric uncertainty and stability of systems with time-delay.

4.1. **Robust stability analysis.** Consider the linear system

$$\dot{x}(t) = A(\alpha)x(t), \tag{27}$$

where $A(\alpha) \in \mathbb{R}^{n\times n}$ is a polynomial and $\alpha \in Q \subset \mathbb{R}^l$ is the vector of uncertain parameters, where Q is compact. From converse Lyapunov theory and existence of polynomial solutions for feasible patameter-dependent LMIs [66] it follows that that System (27) is asymptotically stable if and only if there exist matrix-valued polynomial $P(\alpha) \in \mathbb{S}^n$ such that

$$P(\alpha) > 0 \text{ and } A^T(\alpha)P(\alpha) + P(\alpha)A(\alpha) < 0 \text{ for all } \alpha \in Q. \tag{28}$$

If Q is a semi-algebraic set, then asymptotic stability of System (27) is equivalent to positivity of γ^* in the following optimization of polynomials problem for some $d \in \mathbb{N}$.

$$\gamma^* = \max_{\gamma\in\mathbb{R}, C_\beta \in \mathbb{S}^n} \gamma$$

$$\text{subject to} \begin{bmatrix} \sum_{\beta\in E_d} C_\beta \alpha^\beta & 0 \\ 0 & -A^T(\alpha)\Big(\sum_{\beta\in E_d} C_\beta\alpha^\beta\Big) - \Big(\sum_{\beta\in E_d} C_\beta\alpha^\beta\Big)A(\alpha)\end{bmatrix} - \gamma I \alpha^T\alpha \geq 0, \alpha \in Q, \tag{29}$$

where we have denoted $\alpha_1^{\beta_1}\cdots\alpha_l^{\beta_l}$ by α^β and

$$E_d := \{\beta \in \mathbb{R}^l : \sum_{i=1}^n \beta_i \leq d\}. \tag{30}$$

Given stable systems of Form (27) with different classes of polynomials $A(\alpha)$, we discuss different algorithms for solving (29). Solutions to (29) yield Lyapunov functions of the form $V = x^T(\sum_{\beta\in E_d} C_\beta\alpha^\beta)x$ proving stability of System (27).

Case 1. $A(\alpha)$ **is affine with** $\alpha \in \Delta^l$**:**
Consider the case where $A(\alpha)$ belongs to the polytope

$$\Lambda_l := \left\{ A(\alpha) \in \mathbb{R}^{n\times n} : A(\alpha) = \sum_{i=1}^l A_i\alpha_i, A_i \in \mathbb{R}^{n\times n}, \alpha_i \in \Delta^l \right\},$$

where A_i are the vertices of the polytope and Δ^l is the standard unit simplex defined as in (12). Given $A(\alpha) \in \Lambda_l$, we address the problem of stability analysis of System (27) for all $\alpha \in \Delta^l$.

A sufficient condition for asymptotic stability of System (27) is to find a matrix $P > 0$ such that the Lyapunov inequality $A^T(\alpha)P + PA(\alpha) < 0$ holds for all $\alpha \in \Delta^l$. If $A(\alpha) = \sum_{i=1}^l A_i\alpha_i$, then from convexity of A it follows that the condition

$$A^T(\alpha)P + PA(\alpha) < 0 \text{ for all } \alpha \in \Delta^l$$

is equivalent to positivity of γ^* in the following semi-definite program.

$$\gamma^* = \max_{\gamma\in\mathbb{R}, P\in\mathbb{S}^n} \gamma$$

$$\text{subject to} \begin{bmatrix} P & 0 & \cdots & 0 \\ 0 & -A_1^TP - PA_1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & -A_l^TP - PA_l \end{bmatrix} - \gamma I \geq 0, \tag{31}$$

Any $P \in \mathbb{S}^n$ that satisfies the LMI in (31) for some $\gamma > 0$, yields a Lyapunov function of the form $V = x^T P x$. However for many systems, this class of Lyapunov functions can be conservative (see Numerical Example 1).

More general classes of Lyapunov functions such as parameter-dependent functions of the forms $V = x^T(\sum_{i=1}^{l} P_i \alpha_i)x$ [38, 67, 68] and $V = x^T(\sum_{\beta} P_{\beta \in E_d} \alpha^{\beta})x$ [39, 29] have been utilized in the literature. As shown in [38], given $A_i \in \mathbb{R}^{n\times n}$, $x^T P(\alpha) x$ with $P(\alpha) = \sum_{i=1}^{l} P_i \alpha_i$ is a Lyapunov function for (27) with $\alpha \in \Delta^l$ if the following LMI consitions hold.

$$
\begin{aligned}
&P_i > 0 && \text{for } i = 1, \cdots, l\\
&A_i^T P_i + P_i A_i < 0 && \text{for } i = 1, \cdots, l\\
&A_i^T P_j + A_j^T P_i + P_j A_i + P_i A_j < 0 && \text{for } i = 1, \cdots, l-1,\ j = i+1, \cdots, l
\end{aligned}
$$

In [69], it is shown that given continuous functions $A_i, B_i : \Delta^l \to \mathbb{R}^{n\times n}$ and continuous function $R : \Delta^l \to \mathbb{S}^n$, if there exists a continuous function $X : \Delta^l \to \mathbb{S}^n$ which satisfies

$$
\sum_{i=1}^{N} \left(A_i(\alpha) X(\alpha) B_i(\alpha) + B_i(\alpha)^T X(\alpha) A_i(\alpha)^T \right) + R(\alpha) > 0 \ \text{ for all } \alpha \in \Delta^l, \tag{32}
$$

then there exists a homogeneous polynomial $Y : \Delta^l \to \mathbb{S}^n$ which also satisfies (32). Motivated by this result, [39] uses the class of homogeneous polynomials of the form

$$
P(\alpha) = \sum_{\beta \in I_d} P_\beta \alpha_1^{\beta_1} \cdots \alpha_l^{\beta_l}, \tag{33}
$$

with

$$
I_d := \left\{ \beta \in \mathbb{N}^l : \sum_{i=1}^{l} \beta_i = d \right\} \tag{34}
$$

to provide the following necessary and sufficient LMI condition for stability of System (27). Given $A(\alpha) = \sum_{i=1}^{l} A_i \alpha_i$, System (27) is asymptotically stable for all $\alpha \in \Delta^l$ if and only if there exist some $d \geq 0$ and positive definite $P_\beta \in \mathbb{S}^n, \beta \in I_d$ such that

$$
\sum_{\substack{i=1,\cdots,l\\ \beta_i > 0}} \left(A_i^T P_{\beta - e_i} + P_{\beta - e_i} A_i \right) < 0 \text{ for all } \beta \in I_{d+1}, \tag{35}
$$

where $e_i = [\,0 \cdots 0 \underbrace{1}_{i^{th}} 0 \cdots 0\,] \in \mathbb{N}^l, i = 1, \cdots, l$ form the canonical basis for $\mathbb{R}^l$.

Numerical Example 1: Consider the system $\dot{x}(t) = A(\alpha, \eta)x(t)$ from [40], where $A(\alpha, \eta) = (A_0 + A_1\eta)\alpha_1 + (A_0 + A_2\eta)\alpha_2 + (A_0 + A_3\eta)\alpha_3$, where

$$
A_0 = \begin{bmatrix} -2.4 & -0.6 & -1.7 & 3.1 \\ 0.7 & -2.1 & -2.6 & -3.6 \\ 0.5 & 2.4 & -5.0 & -1.6 \\ -0.6 & 2.9 & -2.0 & -0.6 \end{bmatrix}, A_1 = \begin{bmatrix} 1.1 & -0.6 & -0.3 & -0.1 \\ -0.8 & 0.2 & -1.1 & 2.8 \\ -1.9 & 0.8 & -1.1 & 2.0 \\ -2.4 & -3.1 & -3.7 & -0.1 \end{bmatrix},
$$

$$
A_2 = \begin{bmatrix} 0.9 & 3.4 & 1.7 & 1.5 \\ -3.4 & -1.4 & 1.3 & 1.4 \\ 1.1 & 2.0 & -1.5 & -3.4 \\ -0.4 & 0.5 & 2.3 & 1.5 \end{bmatrix}, A_3 = \begin{bmatrix} -1.0 & -1.4 & -0.7 & -0.7 \\ 2.1 & 0.6 & -0.1 & -2.1 \\ 0.4 & -1.4 & 1.3 & 0.7 \\ 1.5 & 0.9 & 0.4 & -0.5 \end{bmatrix}
$$

and $(\alpha_1, \alpha_2, \alpha_3) \in \Delta^3$, $\eta \geq 0$. We would like to find $\eta^* = \max \eta$ such that $\dot{x}(t) = A(\alpha, \eta)x(t)$ is asymptotically stable for all $\eta \in [0, \eta^*]$.

By performing bisection on η and verifying the inequalities in (35) for each η in the bisection algorithm, we obtained lower bounds on η^* (see Figure 1) using $d = 0, 1, 2$ and 3. For comparison, we have also plotted the lower bounds computed in [40] using the Complete Square Matricial Representation (CSMR) of the Lyapunov inequalities in (28). Both methods found $\max \eta = 2.224$, however the method in [40] used a lower d to find this bound.

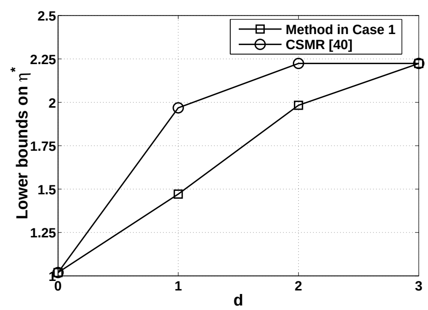

FIGURE 1. lower-bounds for η^* computed using the LMIs in (35) and the method in [40]

Case 2. $A(\alpha)$ **is a polynomial with** $\alpha \in \Delta^l$**:**

Given $A_h \in \mathbb{R}^{n \times n}$ for $h \in I_d$ as defined in (34), we address the problem of stability analysis of System (27) with $A(\alpha) = \sum_{h \in I_d} A_h \alpha_1^{h_1} \cdots \alpha_l^{h_l}$ for all $\alpha \in \Delta^l$. Using Lyapunov theory, this problem can be formulated as the following optimization of polynomials problem.

$$\gamma^* = \max_{\gamma \in \mathbb{R}, P \in \mathbb{R}[\alpha]} \gamma$$
$$\text{subject to } \begin{bmatrix} P(\alpha) & 0 \\ 0 & -A(\alpha)^T P(\alpha) - P(\alpha)A(\alpha) \end{bmatrix} - \gamma I \geq 0 \text{ for all } \alpha \in \Delta^l \tag{36}$$

System (27) is asymptotically stable for all $\alpha \in \Delta^l$ if and only if $\gamma^* > 0$. As in Case 1 of Section 3.2, one can apply bisection algorithm on γ and use Polya's theorem (Theorem 3.1) as a test for feasibility of Constraint (36) to find lower bounds on γ^*. Suppose P and A are homogeneous matrix valued polynomials. Given $\gamma \in \mathbb{R}$, it follows from Theorem 3.1 that the inequality condition in (36) holds for all $\alpha \in \Delta$ if there exist some $e \geq 0$ such that

$$\left(\sum_{i=1}^{l} \alpha_i\right)^e \left(P(\alpha) - \gamma I \left(\sum_{i=1}^{l} \alpha_i\right)^{d_p}\right) \tag{37}$$

and

$$-\left(\sum_{i=1}^{l} \alpha_i\right)^e \left(A(\alpha)^T P(\alpha) + P(\alpha)A(\alpha) + \gamma I \left(\sum_{i=1}^{l} \alpha_i\right)^{d_p + d_a}\right) \tag{38}$$

have all positive coefficients, where d_p is the degree of P and d_a is the degree of A. Let P and A be of the forms

$$P(\alpha) = \sum_{h \in I_{d_p}} P_h \alpha_1^{h_1} \cdots \alpha_l^{h_l}, P_h \in \mathbb{S}^n \quad \text{and} \quad A(\alpha) = \sum_{h \in I_{d_a}} A_h \alpha_1^{h_1} \cdots \alpha_l^{h_l}, A_h \in \mathbb{R}^{n \times n}. \tag{39}$$

By combining (39) with (37) and (38) it follows that for a given $\gamma \in \mathbb{R}$, the inequality condition in (36) holds for all $\alpha \in \Delta^l$ if there exist some $e \geq 0$ such that

$$\left(\sum_{i=1}^l \alpha_i\right)^e \left(\sum_{h \in I_{d_p}} P_h \alpha_1^{h_1} \cdots \alpha_l^{h_l} - \gamma I \left(\sum_{i=1}^l \alpha_i\right)^{d_p}\right) = \sum_{g \in I_{d_p+e}} \left(\sum_{h \in I_{d_p}} f_{g,h} P_h\right) \alpha_1^{g_1} \cdots \alpha_l^{g_l} \tag{40}$$

and

$$\begin{aligned} &-\left(\sum_{i=1}^l \alpha_i\right)^e \left(\left(\sum_{h \in I_{d_a}} A_h^T \alpha^h\right)\left(\sum_{h \in I_{d_p}} P_h \alpha_1^{h_1} \cdots \alpha_l^{h_l}\right) + \left(\sum_{h \in I_{d_p}} P_h \alpha_1^{h_1} \cdots \alpha_l^{h_l}\right)\left(\sum_{h \in I_{d_a}} A_h \alpha^h\right)\right. \\ &\left. +\gamma I \left(\sum_{i=1}^l \alpha_i\right)^{d_p+d_a}\right) = \sum_{q \in I_{d_a+d_p+e}} \left(\sum_{h \in I_{d_p}} M_{h,q}^T P_h + P_h M_{h,q}\right) \alpha_1^{q_1} \cdots \alpha_l^{q_l} \end{aligned} \tag{41}$$

have all positive coefficients, i.e.,

$$\begin{aligned} &\sum_{h \in I_{d_p}} f_{h,g} P_h > 0 \quad \text{for all } g \in I_{d_p+e} \\ &\sum_{h \in I_{d_p}} \left(M_{h,q}^T P_h + P_h M_{h,q}\right) < 0 \quad \text{for all } q \in I_{d_p+d_a+e}, \end{aligned} \tag{42}$$

where we define $f_{h,g} \in \mathbb{R}$ as the coefficient of $P_h \alpha_1^{g_1} \cdots \alpha_l^{g_l}$ after expanding (40). Likewise, we define $M_{h,q} \in \mathbb{R}^{n \times n}$ as the coefficient of $P_h \alpha_1^{q_1} \cdots \alpha_l^{q_l}$ after expanding (41). See [70] for recursive formulae for $f_{h,g}$ and $M_{h,q}$. Feasibility of Conditions (42) can be verified by the following semi-definite program.

$$\max_{\eta \in \mathbb{R}, P_h \in \mathbb{S}_+^n} \eta$$

$$\text{subject to} \begin{bmatrix} \sum_{h \in I_{d_p}} f_{h,g^{(1)}} P_h & 0 & \cdots & & & 0 \\ 0 & \ddots & & & & \\ & & \sum_{h \in I_{d_p}} f_{h,g^{(L)}} P_h & & & \vdots \\ \vdots & & & -\sum_{h \in I_{d_p}} \left(M_{h,q^{(1)}}^T P_h + P_h M_{h,q^{(1)}}\right) & & \\ & & & & \ddots & 0 \\ 0 & \cdots & 0 & & -\sum_{h \in I_{d_p}} \left(M_{h,q^{(M)}}^T P_h + P_h M_{h,q^{(M)}}\right) & \end{bmatrix} - \eta I \geq 0, \tag{43}$$

where we have denoted the elements of I_{d_p+e} by $g^{(i)} \in \mathbb{N}^l, i = 1, \cdots, L$ and have denoted the elements of $I_{d_p+d_a+e}$ by $q^{(i)}, i = 1, \cdots, M$. For any $\gamma \in \mathbb{R}$, if there exist $e \geq 0$ such that SDP (43) is feasible, then $\gamma \leq \gamma^*$. If for a positive γ, there exist $e \geq 0$ such that SDP (43) has a solution $P_h, h \in I_{d_p}$, then

$V = x^T \left(\sum_{h \in I_{d_p}} P_h \alpha_1^{h_1} \cdots \alpha_l^{h_l} \right) x$ is a Lyapunov function proving stability of $\dot{x}(t) = A(\alpha)x(t), \alpha \in \Delta^l$.

SDPs such as (43) can be solved in polynomial time using interior-point algorithms such as the central path primal-dual algorithms in [9, 10, 11]. Fortunately, Problem (43) has block-diagonal structure. Block-diagonal structure in SDP constraints can be used to design massively parallel algorithms, an approach which was applied to Problem (43) in [29].

Numerical Example 2: Consider the system $\dot{x}(t) = A(\alpha)x(t)$, where

$$A(\alpha) = A_1\alpha_1^3 + A_2\alpha_1^2\alpha_2 + A_3\alpha_1\alpha_3^2 + A_4\alpha_1\alpha_2\alpha_3 + A_5\alpha_2^3 + A_6\alpha_3^3,$$

where

$$\alpha \in T_L := \{\alpha \in \mathbb{R}^3 : \sum_{i=1}^{3} \alpha_i = 2L+1, L \le \alpha_i \le 1\}$$

and

$$A_1 = \begin{bmatrix} -0.57 & -0.44 & 0.33 & -0.07 \\ -0.48 & -0.60 & 0.30 & 0.00 \\ -0.22 & -1.12 & 0.08 & -0.24 \\ 1.51 & -0.42 & 0.67 & -1.00 \end{bmatrix} A_2 = \begin{bmatrix} -0.09 & -0.16 & 0.3 & -1.13 \\ -0.15 & -0.17 & -0.02 & 0.82 \\ 0.14 & 0.06 & 0.02 & -1 \\ 0.488 & 0.32 & 0.97 & -0.71 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -0.70 & -0.29 & -0.18 & 0.31 \\ 0.41 & -0.76 & -0.30 & -0.12 \\ -0.05 & 0.35 & -0.59 & 0.91 \\ 1.64 & 0.82 & 0.01 & -1 \end{bmatrix} A_4 = \begin{bmatrix} 0.72 & 0.34 & -0.64 & 0.31 \\ -0.21 & -0.51 & 0.59 & 0.07 \\ 0.27 & 0.49 & -0.84 & -0.94 \\ -1.89 & -0.66 & 0.27 & 0.41 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} -0.51 & -0.47 & -1.38 & 0.17 \\ 1.18 & -0.62 & -0.29 & 0.35 \\ -0.65 & 0.01 & -1.44 & -0.04 \\ -0.74 & -1.22 & 0.60 & -1.47 \end{bmatrix} A_6 = \begin{bmatrix} -0.201 & -0.19 & -0.55 & 0.07 \\ 0.803 & -0.42 & -0.20 & 0.24 \\ -0.440 & 0.01 & -0.98 & -0.03 \\ 0 & -0.83 & 0.41 & -1 \end{bmatrix}.$$

We would like to solve the following optimization problem.

$$\begin{aligned} L^* = \min \quad & L \\ \text{subject to} \quad & \dot{x}(t) = A(\alpha)x(t) \text{ is stable for all } \alpha \in T_L. \end{aligned} \tag{44}$$

We first represent T_L using the unit simplex Δ^3 as follows. Define the map $f : \Delta^3 \to T_L$ as

$$f(\alpha) = [f_1(\alpha), f_2(\alpha), f_3(\alpha)],$$

where $f_i(\alpha) = 2|L|(\alpha_i - |L|)$. Then, we have $\{A(\alpha) : \alpha \in T_L\} = \{A(f(\beta)), \beta \in \Delta^3\}$. Thus, the following optimization problem is equivalent to Problem (44).

$$\begin{aligned} L^* = \min \quad & L \\ \text{subject to} \quad & \dot{x}(t) = A(f(\beta))x(t) \text{ is stable for all } \beta \in \Delta^3. \end{aligned} \tag{45}$$

We solved Problem (45) using bisection on L. For each L, we used Theorem 3.1 to verify the inequality in (36) using Polya's exponents $e = 1$ to 7 and $d_p = 1$ to 4 as degrees of P. Figure 2 shows the computed upper-bounds on L^* for different e and d_p. The algorithm found $\min L = -0.504$.

For comparison, we also the same problem using SOSTOOLS [8] and Putinar's Positivstellensatz (see Case 4 of Section 3.2). By computing a Lyapunov function of degree two in x and degree one in β, SOSTOOLS certified $L = -0.504$ as an upper-bound for L^*. This is the same as the upper-bound computed by Polya's algorithm.

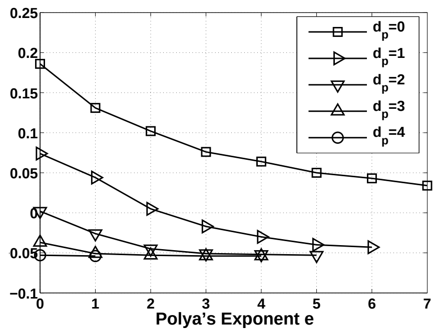

FIGURE 2. Upper-bounds for L^* in Problem (44) for different Polya's exponents e and different degrees of P

Case 3. $A(\alpha)$ **is a polynomial with** $\alpha \in \Phi^l$**:**

Given $A_h \in \mathbb{R}^{n\times n}$ for $h \in E_d$ as defined in (30), we address the problem of stability analysis of System (27) with $A(\alpha) = \sum_{h\in E_d} A_h \alpha_1^{h_1} \cdots \alpha_l^{h_l}$ for all $\alpha \in \Phi^l := \{x \in \mathbb{R}^n : |x_i| \le r_i\}$. From Lyapunov theory, System (27) is asymptotically stable for all $\alpha \in \Phi^l$ if and only if $\gamma^* > 0$ in the following optimization of polynomials problem.

$$\gamma^* = \max_{\gamma\in\mathbb{R}, P\in\mathbb{R}[\alpha]} \gamma$$
$$\text{subject to } \begin{bmatrix} P(\alpha) & 0 \\ 0 & -A(\alpha)^T P(\alpha) - P(\alpha)A(\alpha) \end{bmatrix} - \gamma I \ge 0 \text{ for all } \alpha \in \Phi^l \tag{46}$$

As in Case 2 of Section 3.2, by applying bisection algorithm on γ and using a multi-simplex version of Polya's algorithm (such as Theorem 3.2) as a test for feasibility of Constraint (46) we can compute lower bounds on γ^*. Suppose there exists a multi-homogeneous matrix-valued polynomial Q of degree vector $d_q \in \mathbb{N}^l$ (d_{q_i} is the degree of β_i) such that

$$\{P(\alpha) \in \mathbb{S}^n : \alpha \in \Phi^l\} = \{Q(\beta,\eta) \in \mathbb{S}^n : (\beta_i,\eta_i) \in \Delta^2, i = 1,\cdots,l\}. \tag{47}$$

Likewise, suppose there exists a multi-homogeneous matrix-valued polynomial B of degree vector $d_b \in \mathbb{N}^l$ (d_{b_i} is the degree of β_i) such that

$$\{A(\alpha) \in \mathbb{S}^n : \alpha \in \Phi^l\} = \{B(\beta,\eta) \in \mathbb{S}^n : (\beta_i,\eta_i) \in \Delta^2, i = 1,\cdots,l\}.$$

Given $\gamma \in \mathbb{R}$, it follows from Theorem 3.2 that the inequality condition in (46) holds for all $\alpha \in \Phi^l$ if there exist $e \ge 0$ such that

$$\left(\prod_{i=1}^{l}(\beta_i+\eta_i)^e\right)\left(Q(\beta,\eta) - \gamma I\left(\prod_{i=1}^{l}(\beta_i+\eta_i)^{d_{p_i}}\right)\right) \tag{48}$$

and

$$-\left(\prod_{i=1}^{l}(\beta_i+\eta_i)^e\right)\left(B^T(\alpha,\beta)Q(\beta,\eta) + Q(\beta,\eta)B(\beta,\eta) + \gamma I\left(\prod_{i=1}^{l}(\beta_i+\eta_i)^{d_{pa_i}}\right)\right), \tag{49}$$

have all positive coefficients where d_{p_i} is the degree of α_i in $P(\alpha)$ and d_{pa_i} is the degree of α_i in $P(\alpha)A(\alpha)$. Suppose Q and B are of the forms

$$Q(\beta,\eta) = \sum_{\substack{h,g\in\mathbb{N}^l \\ h+g=d_q}} Q_{h,g}\beta_1^{h_1}\eta_1^{g_1}\cdots\beta_l^{h_l}\eta_l^{g_l} \tag{50}$$

and

$$B(\beta,\eta) = \sum_{\substack{h,g\in\mathbb{N}^l \\ h+g=d_b}} B_{h,g}\beta_1^{h_1}\eta_1^{g_1}\cdots\beta_l^{h_l}\eta_l^{g_l}. \tag{51}$$

By combining (50) and (51) with (48) and (49) we find that for a given $\gamma \in \mathbb{R}$, the inequality condition in (46) holds for all $\alpha \in \Phi^l$ if there exist some $e \geq 0$ such that

$$\begin{aligned} &\sum_{\substack{h,g\in\mathbb{N}^l \\ h+g=d_q}} f_{\{q,r\},\{h,g\}}Q_{h,g} > 0 \quad \text{for all} \quad q,r\in\mathbb{N}^l : q+r = d_q + e\cdot\mathbf{1} \quad \text{and} \\ &\sum_{\substack{h,g\in\mathbb{N}^l \\ h+g=d_q}} M^T_{\{s,t\},\{h,g\}}Q_{h,g} + Q_{h,g}M_{\{s,t\},\{h,g\}} < 0 \text{ for all } s,t\in\mathbb{N}^l : s+t = d_q+d_b+e\cdot\mathbf{1}, \end{aligned} \tag{52}$$

where $\mathbf{1} \in \mathbb{N}^l$ is the vector of ones and where we define $f_{\{q,r\},\{h,g\}} \in \mathbb{R}$ to be the coefficient of $Q_{h,g}\beta^q\eta^r$ after expansion of (48). Likewise, we define $M_{\{s,t\},\{h,g\}} \in \mathbb{R}^{n\times n}$ to be the coefficient of $Q_{h,g}\beta^s\eta^t$ after expansion of (49). See [34] for recursive formulae for calculating $f_{\{q,r\},\{h,g\}}$ and $M_{\{s,t\},\{h,g\}}$. Similar to Case 2, Conditions (52) are an SDP. For any $\gamma \in \mathbb{R}$, if there exist $e \geq 0$ and $\{Q_{h,g}\}$ that satisfy (52), then $\gamma \leq \gamma^*$ as defined in (46). Furthermore, if γ is positive, then $\dot{x}(t) = A(\alpha)x(t)$ is asymptotically stable for all $\alpha \in \Phi^l$.

Numerical Example 3a: Consider the system $\dot{x}(t) = A(\alpha)x(t)$, where

$$\begin{aligned} &A(\alpha) = A_0 + A_1\alpha_1^2 + A_2\alpha_1\alpha_2\alpha_3 + A_3\alpha_1^2\alpha_2\alpha_3^2, \\ &\alpha_1 \in [-1,1],\ \alpha_2 \in [-0.5,0.5],\ \alpha_3 \in [-0.1,0.1], \end{aligned}$$

where

$$A_0 = \begin{bmatrix} -3.0 & 0 & -1.7 & 3.0 \\ -0.2 & -2.9 & -1.7 & -2.6 \\ 0.6 & 2.6 & -5.8 & -2.6 \\ -0.7 & 2.9 & -3.3 & -2.1 \end{bmatrix} A_1 = \begin{bmatrix} 2.2 & -5.4 & -0.8 & -2.2 \\ 4.4 & 1.4 & -3.0 & 0.8 \\ -2.4 & -2.2 & 1.4 & 6.0 \\ -2.4 & -4.4 & -6.4 & 0.18 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -8.0 & -13.5 & -0.5 & -3.0 \\ 18.0 & -2.0 & 0.5 & -11.5 \\ 5.5 & -10.0 & 3.5 & 9.0 \\ 13.0 & 7.5 & 5.0 & -4.0 \end{bmatrix} A_3 = \begin{bmatrix} 3.0 & 7.5 & 2.5 & -8.0 \\ 1.0 & 0.5 & 1.0 & 1.5 \\ -0.5 & -1.0 & 1.0 & 6.0 \\ -2.5 & -6.0 & 8.5 & 14.25 \end{bmatrix}.$$

The problem is to investigate asymptotic stability of this system for all α in the given intervals using the method in Case 3 of Section 4.1. We first represented $A(\alpha)$ over $[-1,1]\times[-0.5,0.5]\times[-0.1,0.1]$ by a multi-homogeneous polynomial $B(\beta,\eta)$ with $(\beta_i,\eta_i) \in \Delta^2$ and with the degree vector $d_b = [2,1,2]$ (see [34] for an algorithm which finds B and see Case 2 of Section (3.2) for an example). Then, by applying Theorem 3.2 (as in (48) and (49)) with $\gamma = 0.1, e = 1$ and $d_p = [1,1,1]$, we set-up the inequalities in (52) with $d_q = [1,1,1]$. By using semi-definite programming, we solved the inequalities and computed the following Lyapunov function as a certificate for asymptotic stability of $\dot{x}(t) = A(\alpha)x(t)$ for all

$\alpha_1 \in [-1,1]$, $\alpha_2 \in [-0.5, 0.5]$, $\alpha_3 \in [-0.1, 0.1]$.

$$V(x,\beta,\eta) = x^T Q(\beta,\eta) x = x^T(\beta_1(Q_1\beta_2\beta_3 + Q_2\beta_2\eta_3 + Q_3\eta_2\beta_3 + Q_4\eta_2\eta_3) \\ + \eta_1(Q_5\beta_2\beta_3 + Q_6\beta_2\eta_3 + Q_7\eta_2\beta_3 + Q_8\eta_2\eta_3))x,$$

where $\beta_1 = 0.5\alpha_1 + 0.5, \beta_2 = \alpha_2 + 0.5, \beta_3 = 5\alpha_3 + 0.5, \eta_1 = 1-\beta_1, \eta_2 = 1-\beta_2, \eta_3 = 1-\beta_3$ and

$$Q_1 = \begin{bmatrix} 5.807 & 0.010 & -0.187 & -1.186 \\ 0.010 & 5.042 & -0.369 & 0.227 \\ -0.187 & -0.369 & 8.227 & -1.824 \\ -1.186 & 0.227 & -1.824 & 8.127 \end{bmatrix} Q_2 = \begin{bmatrix} 7.409 & -0.803 & 1.804 & -1.594 \\ -0.803 & 6.016 & 0.042 & -0.538 \\ 1.804 & 0.042 & 7.894 & -1.118 \\ -1.594 & -0.538 & -1.118 & 8.590 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 6.095 & -0.873 & 0.512 & -1.125 \\ -0.873 & 5.934 & -0.161 & 0.503 \\ 0.512 & -0.161 & 7.417 & -0.538 \\ -1.125 & 0.503 & -0.538 & 6.896 \end{bmatrix} Q_4 = \begin{bmatrix} 5.388 & 0.130 & -0.363 & -0.333 \\ 0.130 & 5.044 & -0.113 & -0.117 \\ -0.363 & -0.113 & 6.156 & -0.236 \\ -0.333 & -0.117 & -0.236 & 5.653 \end{bmatrix}$$

$$Q_5 = \begin{bmatrix} 7.410 & -0.803 & 1.804 & -1.594 \\ -0.803 & 6.016 & 0.042 & -0.538 \\ 1.804 & 0.042 & 7.894 & -1.118 \\ -1.594 & -0.538 & -1.118 & 8.590 \end{bmatrix} Q_6 = \begin{bmatrix} 5.807 & 0.010 & -0.187 & -1.186 \\ 0.010 & 5.042 & -0.369 & 0.227 \\ -0.187 & -0.369 & 8.227 & -1.824 \\ -1.186 & 0.227 & -1.824 & 8.127 \end{bmatrix}$$

$$Q_7 = \begin{bmatrix} 5.388 & 0.130 & -0.363 & -0.333 \\ 0.130 & 5.044 & -0.113 & -0.117 \\ -0.363 & -0.113 & 6.156 & -0.236 \\ -0.333 & -0.117 & -0.236 & 5.653 \end{bmatrix} Q_8 = \begin{bmatrix} 6.095 & -0.873 & 0.512 & -1.125 \\ -0.873 & 5.934 & -0.161 & 0.503 \\ 0.512 & -0.161 & 7.417 & -0.538 \\ -1.125 & 0.503 & -0.538 & 6.896 \end{bmatrix}.$$

Numerical Example 3b: In this example, we used the same method as in Example 3a to find lower bounds on $r^* = \max r$ such that $\dot{x}(t) = A(\alpha)x(t)$ with

$$A(\alpha) = A_0 + \sum_{i=1}^{4} A_i \alpha_i,$$

$$A_0 = \begin{bmatrix} -3.0 & 0 & -1.7 & 3.0 \\ -0.2 & -2.9 & -1.7 & -2.6 \\ 0.6 & 2.6 & -5.8 & -2.6 \\ -0.7 & 2.9 & -3.3 & -2.4 \end{bmatrix} A_1 = \begin{bmatrix} 1.1 & -2.7 & -0.4 & -1.1 \\ 2.2 & 0.7 & -1.5 & 0.4 \\ -1.2 & -1.1 & 0.7 & 3.0 \\ -1.2 & -2.2 & -3.2 & -1.4 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1.6 & 2.7 & 0.1 & 0.6 \\ -3.6 & 0.4 & -0.1 & 2.3 \\ -1.1 & 2 & -0.7 & -1.8 \\ -2.6 & -1.5 & -1.0 & 0.8 \end{bmatrix} A_3 = \begin{bmatrix} -0.6 & 1.5 & 0.5 & -1.6 \\ 0.2 & -0.1 & 0.2 & 0.3 \\ -0.1 & -0.2 & -0.2 & 1.2 \\ -0.5 & -1.2 & 1.7 & -0.1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} -0.4 & -0.1 & -0.3 & 0.1 \\ 0.1 & 0.3 & 0.2 & 0.0 \\ 0.0 & 0.2 & -0.3 & 0.1 \\ 0.1 & -0.2 & -0.2 & 0.0 \end{bmatrix}.$$

is asymptotically stable for all $\alpha \in \{\alpha \in \mathbb{R}^4 : |\alpha_i| \leq r\}$. Table 1 shows the computed lower bounds on r^* for different degree vectors d_q (degree vector of Q in (47)). In all of the cases, we set the Polya's exponent $e = 0$. For comparison, we have also included the lower-bounds computed by the methods of [71] and [72] in Table 1.

4.2. **Nonlinear stability analysis.** Consider nonlinear systems of the form

$$\dot{x}(t) = f(x(t)), \tag{53}$$

TABLE 1. The lower-bounds on r^* computed by the method in Case 3 of Section 4.1 and methods in [71] and [72] - i^{th} entry of d_q is the degree of β_i in (47)

	d_q =[0,0,0,0]	d_q =[0,1,0,1]	d_q =[1,0,1,0]	d_q =[1,1,1,1]	d_q =[2,2,2,2]	Ref.[71]	Ref.[72]
bound on r^*	0.494	0.508	0.615	0.731	0.840	0.4494	0.8739

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a degree d_f polynomial. Suppose the origin is an isolated equilibrium of (53). In this section, we address local stability of the origin in the following sense.

Lemma 4.1. *Consider the System (53) and let $Q \subset \mathbb{R}^{n\times n}$ be an open set containing the origin. Suppose there exists a continuously differentiable function V which satisfies*

$$V(x) > 0 \text{ for all } x \in Q \setminus \{0\}, V(0) = 0 \tag{54}$$

and

$$\langle \nabla V, f(x)\rangle < 0 \text{ for all } x \in Q \setminus \{0\}. \tag{55}$$

Then the origin is an asymptotically stable equilibrium of System (53), meaning that for every $x(0) \in \{x \in \mathbb{R}^n : \{y : V(y) \le V(x)\} \subset Q\}$, $\lim_{t\to\infty} x(t) = 0$.

Since existence of polynomial Lyapunov functions is necessary and sufficient for stability of (53) on any compact set [73], we can formulate the problem of stability analysis of (53) as follows.

$$\gamma^* = \max_{\gamma, c_\beta \in \mathbb{R}} \ \gamma$$

$$\text{subject to} \begin{bmatrix} \sum_{\beta\in E_d} c_\beta x^\beta - x^T x & 0 \\ 0 & -\langle \nabla \sum_{\beta\in E_d} c_\beta x^\beta, f(x)\rangle - \gamma x^T x \end{bmatrix} \ge 0 \text{ for all } x \in Q. \tag{56}$$

Conditions (54) and (55) hold if and only if there exist $d \in \mathbb{N}$ such that $\gamma^* > 0$. In Sections 4.2.1 and 4.2.2, we discuss two alternatives to SOS programming for solving (56). These methods apply Polya's theorem and Handelman's theorem to Problem (56) (as described in Cases 2 and 3 in Section 3.2) to find lower bounds on γ^*. See [43] for a different application of Handelman's theorem and intervals method in nonlinear stability. Also, see [41] for a method of computing continuous piecewise affine Lyapunov functions using linear programming and a *triangulation* scheme for polytopes.

4.2.1. *Application of Handelman's theorem in nonlinear stability analysis.* Recall that every convex polytope can be represented as

$$\Gamma^K := \{x \in \mathbb{R}^n : w_i^T x + u_i \ge 0, i = 1, \cdots, K\} \tag{57}$$

for some $w_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}$. Suppose Γ^K is bounded and the origin is in the interior of Γ^K. In this section, we would like to investigate asymptotic stability of the equilibrium of System (53) by verifying positivity of γ^* in Problem (56) with $Q = \Gamma^K$.

Unfortunately, Handelman's theorem (Theorem 2.4) does not parameterize polynomials which have zeros in the interior of a given polytope. To see this, suppose a polynomial g (g is not identically zero) is zero at $x = a$, where a is in the interior of a polytope $\Gamma^K := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\}$. Suppose there exist $b_\alpha \geq 0, \alpha \in \mathbb{N}^K$ such that for some $d \in N$,

$$g(x) = \sum_{\substack{\alpha \in \mathbb{N}^K \\ \|\alpha_i\|_1 \leq d}} b_\alpha (w_1^T x + u_1)^{\alpha_1} \cdots (w_K^T x + u_K)^{\alpha_K}.$$

Then,

$$g(a) = \sum_{\substack{\alpha \in \mathbb{N}^K \\ \|\alpha_i\|_1 \leq d}} b_\alpha (w_1^T a + u_1)^{\alpha_1} \cdots (w_K^T a + u_K)^{\alpha_K} = 0.$$

From the assumption $a \in \text{int}(\Gamma^K)$ it follows that $w_i^T a + u_i > 0$ for all $i = 1, \cdots, K$. Hence $b_\alpha < 0$ for at least one $\alpha \in \{\alpha \in \mathbb{N}^K : \|\alpha\|_1 \leq d\}$. This contradicts with the assumption that all $b_\alpha \geq 0$.

Based on the above reasoning, one cannot readily use Handelman's theorem to verify the Lyapunov inequalities in (54). In [44], a combination of Handelman's theorem and a decomposition scheme was applied to Problem (56) with $Q = \Gamma^K$. Here we outline a modified version of this result. First, consider the following definitions.

Definition 4.2. Given a bounded polytope of the form $\Gamma^K := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\}$, we call

$$\zeta^i(\Gamma^K) := \{x \in \mathbb{R}^n : w_i^T x + u_i = 0 \text{ and } w_j^T x + u_j \geq 0 \text{ for } j \in \{1, \cdots, K\}, j \neq i\}$$

the $i-$th facet of Γ^K if $\zeta^i(\Gamma^K) \neq \emptyset$.

Definition 4.3. Given a bounded polytope of the form $\Gamma^K := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\}$, we call $D_\Gamma := \{D_i\}_{i=1,\cdots,L}$ a $D-$*decomposition* of Γ^K if

$$D_i := \{x \in \mathbb{R}^n : h_{i,j}^T x + g_{i,j} \geq 0, j = 1, \cdots, m_i\} \quad \text{for some } h_{i,j} \in \mathbb{R}^n,\ g_{i,j} \in \mathbb{R}$$

such that $\cup_{i=1}^L D_i = \Gamma$, $\cap_{i=1}^L D_i = \{0\}$ and $\text{int}(D_i) \cap \text{int}(D_j) = \emptyset$.

Consider System (53) with f of degree d_f. Given $w_i, h_{i,j} \in \mathbb{R}^n$ and $u_i, g_{i,j} \in \mathbb{R}$, let $\Gamma^K := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\}$ with $D-$decomposition $D_\Gamma := \{D_i\}_{i=1,\cdots,L}$, where $D_i := \{x \in \mathbb{R}^n : h_{i,j}^T x + g_{i,j} \geq 0, j = 1, \cdots, m_i\}$. Let $F_i : \mathbb{R}^{N_i} \times \mathbb{N} \to \mathbb{R}^B$ as

$$F_i(y, d) := \left[\sum_{\substack{\alpha \in \mathbb{N}^{m_i} \\ \|\alpha\|_1 \leq d}} p_{\{\lambda^{(1)}, \alpha, i\}} y_\alpha \ , \ \cdots \ , \ \sum_{\substack{\alpha \in \mathbb{N}^{m_i} \\ \|\alpha\|_1 \leq d}} p_{\{\lambda^{(B)}, \alpha, i\}} y_\alpha \right]^T$$

for $i = 1, \cdots, L$, where N_i is the cardinality of $\{\alpha \in \mathbb{N}^{m_i} : \|\alpha\|_1 \leq d\}$ and where we have denoted the elements of $E_d := \{\lambda \in \mathbb{N}^n : \|\lambda\|_1 \leq d\}$ by $\lambda^{(k)}$, $k = 1, \cdots, B$. For any $\lambda^{(k)} \in E_d$, we define $p_{\{\lambda^{(k)}, \alpha, i\}}$ as the coefficient of $y_\alpha x^{\lambda^{(k)}}$ in

$$V_i(x) := \sum_{\substack{\alpha \in \mathbb{N}^{m_i}: \\ \|\alpha\|_1 \leq d}} y_\alpha \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_j}. \tag{58}$$

Let $H_i : \mathbb{R}^{N_i} \times \mathbb{N} \to \mathbb{R}^Q$ as

$$H_i(y,d) := \left[\sum_{\substack{\alpha \in \mathbb{N}^{m_i} \\ \|\alpha\|_1 \le d}} p_{\{\delta^{(1)},\alpha,i\}} y_\alpha \ , \ \cdots \ , \ \sum_{\substack{\alpha \in \mathbb{N}^{m_i} \\ \|\alpha\|_1 \le d}} p_{\{\delta^{(Q)},\alpha,i\}} y_\alpha \right]^T$$

for $i = 1, \cdots, L$, where we have denoted the elements of $\{\delta \in \mathbb{N}^n : \delta = 2e_j$ for $j = 1, \cdots, n\}$ by $\delta^{(k)}, k = 1, \cdots, Q$, where e_j are the canonical basis for $\mathbb{N}^n$.

Let $R_i(y,d) : \mathbb{R}^{N_i} \times \mathbb{N} \to \mathbb{R}^C$ as

$$R_i(y,d) := \left[y_{\beta^{(1)}} \ , \ \cdots \ , \ y_{\beta^{(Z)}} \right]^T,$$

for $i = 1, \cdots, L$, where we have denoted the elements of

$$\{\beta \in \mathbb{N}^{m_i} : \|\beta\|_1 \le d, \beta_j = 0 \text{ for } j \in \{j \in \mathbb{N} : g_{i,j} = 0\}\}$$

by $\beta^{(k)}$, $k = 1, \cdots, C$.

Let $J_{i,k} : \mathbb{R}^{N_i} \times \mathbb{N} \to \mathbb{R}^Z$ as

$$J_{i,k}(y,d) := \left[\sum_{\substack{\alpha \in \mathbb{N}^{m_i} \\ \|\alpha\|_1 \le d, \alpha_k = 0}} p_{\{\mu^{(1)},\alpha,i\}} y_\alpha \ , \ \cdots \ , \ \sum_{\substack{\alpha \in \mathbb{N}^{m_i} \\ \|\alpha\|_1 \le d, \alpha_k = 0}} p_{\{\mu^{(Z)},\alpha,i\}} y_\alpha \right]^T$$

for $i, k = 1, \cdots, L$, where we have denoted the elements of $\{\mu \in \mathbb{N}^n : \|\mu\|_1 \le d, \mu_k = 0\}$ by $\mu^k, k = 1, \cdots, Z$.

Let $G_i : \mathbb{R}^{M_i} \times \mathbb{N} \to \mathbb{R}^P$ as

$$G_i(y,d) := \left[\sum_{\substack{\alpha \in \mathbb{N}^{m_i} \\ \|\alpha\|_1 \le d}} s_{\{\eta^{(1)},\alpha,i\}} y_\alpha \ , \ \cdots \ , \ \sum_{\substack{\alpha \in \mathbb{N}^{m_i} \\ \|\alpha\|_1 \le d}} s_{\{\eta^{(P)},\alpha,i\}} y_\alpha \right]^T$$

for $i = 1, \cdots, L$, where M_i is the cardinality of $\{\alpha \in \mathbb{N}^{m_i} : \|\alpha\|_1 \le d_f + d - 1\}$ and where we have denoted the elements of E_{d+d_f-1} by $\eta^{(k)}$. For any $\eta^{(k)} \in E_{d+d_f-1}$, we define $s_{\{\eta^{(k)},\alpha,i\}}$ as the coefficient of $y_\alpha x^{\eta^{(k)}}$ in $\langle \nabla V_i(x), f(x) \rangle$, where $V_i(x)$ is defined in (58).

Given $i, j \in \{1, \cdots, L\}, i \ne j$, let

$$\Lambda_{i,j} := \left\{ k, l \in \mathbb{N} : k \in \{1, \cdots, m_i\}, l \in \{1, \cdots, m_j\} : \zeta^k(D_i) \ne \emptyset \text{ and } \zeta^k(D_i) = \zeta^l(D_j) \right\}$$

If there exist $d \in \mathbb{N}$ such that $\max \gamma$ in the linear program

$$
\begin{aligned}
&\max_{\gamma \in \mathbb{R}, b_i \in \mathbb{R}^{N_i}, c_i \in \mathbb{R}^{M_i}} \quad \gamma \\
&\text{subject to} \quad b_i \geq \mathbf{0} && \text{for } i = 1, \cdots, L \\
&\qquad c_i \leq \mathbf{0} && \text{for } i = 1, \cdots, L \\
&\qquad R_i(b_i, d) = \mathbf{0} && \text{for } i = 1, \cdots, L \\
&\qquad H_i(b_i, d) \geq \mathbf{1} && \text{for } i = 1, \cdots, L \\
&\qquad H_i(c_i, d + d_f - 1) \leq -\gamma \cdot \mathbf{1} && \text{for } i = 1, \cdots, L \\
&\qquad G_i(b_i, d) = F_i(c_i, d + d_f - 1) && \text{for } i = 1, \cdots, L \\
&\qquad J_{i,k}(b_i, d) = J_{j,l}(b_j, d) && \text{for } i, j = 1, \cdots, L \text{ and } k, l \in \Lambda_{i,j}
\end{aligned} \tag{59}
$$

is positive, then the origin is an asymptotically stable equilibrium for System (53) and

$$
V(x) = V_i(x) = \sum_{\substack{\alpha \in \mathbb{N}^{m_i}: \\ \|\alpha\|_1 \leq d}} b_{i,\alpha} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_j} \quad \text{for } x \in D_i, i = 1, \cdots, L
$$

is a piecewise polynomial Lyapunov function proving stability of System (53).

Numerical Example 4: Consider the following nonlinear system [75].

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -2x_1 - x_2 + x_1 x_2^2 - x_1^5 + x_1 x_2^4 + x_2^5.
\end{aligned}
$$

Using the polytope

$$
\begin{aligned}
\Gamma^4 = \{x_1, x_2 \in \mathbb{R}^2 : &1.428x_1 + x_2 - 0.625 \geq 0, -1.428x_1 + x_2 + 0.625 \geq 0, \\
&1.428x_1 + x_2 + 0.625 \geq 0, -1.428x_1 + x_2 - 0.625 \geq 0\},
\end{aligned} \tag{60}
$$

and $D-$decomposition

$$
\begin{aligned}
D_1 &:= \{x_1, x_2 \in \mathbb{R}^2 : -x_1 \geq 0, x_2 \geq 0, -1.428x_1 + x_2 - 0.625 \geq 0\} \\
D_2 &:= \{x_1, x_2 \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, 1.428x_1 + x_2 + 0.625 \geq 0\} \\
D_3 &:= \{x_1, x_2 \in \mathbb{R}^2 : x_1 \geq 0, -x_2 \geq 0, -1.428x_1 + x_2 + 0.625 \geq 0\} \\
D_4 &:= \{x_1, x_2 \in \mathbb{R}^2 : -x_1 \geq 0, -x_2 \geq 0, 1.428x_1 + x_2 + 0.625 \geq 0\},
\end{aligned}
$$

we set-up the LP in (59) with $d = 4$. The solution to the LP certified asymptotic stability of the origin and yielded the following piecewise polynomial Lyapunov function.

$$V(x)=\begin{cases} 0.543x_1^2+0.233x_2^2+0.018x_2^3-0.074x_1x_2^2-0.31x_1^3 & \\ +0.004x_2^4-0.013x_1x_2^3+0.015x_1^2x_2^2+0.315x_1^4 & \text{if } x\in D_1 \\ \\ 0.543x_1^2+0.329x_1x_2+0.233x_2^2+0.018x_2^3+0.031x_1x_2^2 & \\ +0.086x_1^2x_2+0.3x_1^3+0.004x_2^4+0.009x_1x_2^3+0.015x_1^2x_2^2 & \\ +0.008x_1^3x_2+0.315x_1^4 & \text{if } x\in D_2 \\ \\ 0.0543x_1^2+0.0233x_2^2-0.0018x_2^3+0.0074x_1x_2^2+0.03x_1^3 & \\ +0.004x_2^4-0.013x_1x_2^3+0.015x_1^2x_2^2+0.315x_1^4 & \text{if } x\in D_3 \\ \\ 0.543x_1^2+0.329x_1x_2+0.233x_2^2-0.018x_2^3-0.031x_1x_2^2 & \\ -0.086x_1^2x_2-0.3x_1^3+0.004x_2^4+0.009x_1x_2^3+0.015x_1^2x_2^2 & \\ +0.008x_1^3x_2+0.315x_1^4 & \text{if } x\in D_4 \end{cases} \tag{61}$$

Figure 3 shows the largest level set of $V(x)$ inscribed in the polytope Γ^4. See Numerical Example 5 for a comparison of this method with the method in Section 4.2.2.

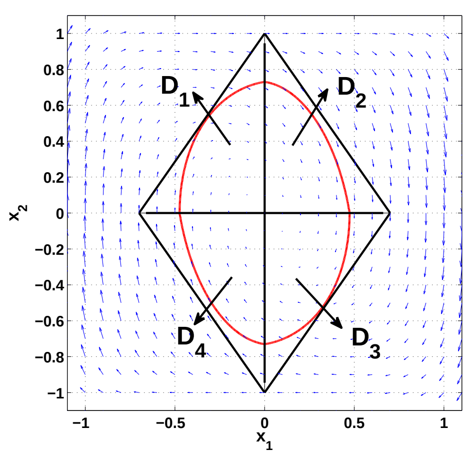

FIGURE 3. The largest level-set of Lyapunov function (61) inscribed in Polytope (60)

4.2.2. *Application of Polya's theorem in nonlinear stability analysis.* In this section, we discuss an algorithm based on a multi-simplex version of Polya's theorem (Theorem 3.2) to verify local stability of nonlinear systems of the form

$$\dot{x}=A(x)x(t), \tag{62}$$

where $A(x)\in\mathbb{R}^{n\times n}$ is a matrix-valued polynomial and $A(0)\neq 0$.

Unfortunately, Polya's theorem does not parameterize polynomials which have zeros in the interior of the unit simplex (see [31] for an elementary proof of this). From the same reasoning as in [31] it follows that the multi-simplex version of Polya's theorem (Theorem 3.2) does not parameterize polynomials which have zeros in the interior of a multi-simplex. On the other hand, if $f(z)$ in (16) has a zero in the interior of Φ^n, then any multi-homogeneous polynomial $p(x,y)$ that satisfies (16) has a zero in the interior of the multi-simplex $\Delta^2 \times \cdots \times \Delta^2$. One way to enforce the condition $V(0) = 0$ in (54) is to search for coefficients of a matrix-valued polynomial P which defines a Lyapunov function of the form $V(x) = x^T P(x) x$. It can be shown that $V(x) = x^T P(x) x$ is a Lyapunov function for System (62) if and only if γ^* in the following optimization of polynomials problem is positive.

$$\begin{aligned} \gamma^* = & \max_{\gamma \in \mathbb{R}, P \in \mathbb{R}[x]} \ \gamma \\ & \text{subject to} \begin{bmatrix} P(x) & 0 \\ 0 & -Q(x) \end{bmatrix} - \gamma I \geq 0 \quad \text{for all } x \in \Phi^n, \end{aligned} \tag{63}$$

where

$$Q(x) = A^T(x)P(x) + P(x)A(x) + \frac{1}{2}\left(A^T(x) \begin{bmatrix} x^T \frac{\partial P(x)}{\partial x_1} \\ \vdots \\ x^T \frac{\partial P(x)}{\partial x_n} \end{bmatrix} + \begin{bmatrix} x^T \frac{\partial P(x)}{\partial x_1} \\ \vdots \\ x^T \frac{\partial P(x)}{\partial x_n} \end{bmatrix}^T A(x) \right).$$

As in Case 2 of Section 3.2, by applying bisection algorithm on γ and using Theorem 3.2 as a test for feasibility of Constraint (63) we can compute lower bounds on γ^*. Suppose there exists a multi-homogeneous matrix-valued polynomial S of degree vector $d_s \in \mathbb{N}^n$ (d_{s_i} is the degree of y_i) such that

$$\{P(x) \in \mathbb{S}^n : x \in \Phi^n\} = \{S(y,z) \in \mathbb{S}^n : (y_i, z_i) \in \Delta^2, i = 1, \cdots, n\}.$$

Likewise, suppose there exist multi-homogeneous matrix-valued polynomials B and C of degree vectors $d_b \in \mathbb{N}^n$ and $d_c = d_s \in \mathbb{N}^n$ such that

$$\{A(x) \in \mathbb{R}^{n\times n} : x \in \Phi^n\} = \{B(y,z) \in \mathbb{R}^{n\times n} : (y_i, z_i) \in \Delta^2, i = 1, \cdots, n\}$$

and

$$\left\{ \left[\tfrac{\partial P(x)}{\partial x_1} x, \cdots, \tfrac{\partial P(x)}{\partial x_n} x \right] \in \mathbb{R}^{n\times n} : x \in \Phi^n \right\} = \left\{ C(y,z) \in \mathbb{R}^{n\times n} : (y_i, z_i) \in \Delta^2, i = 1, \cdots, n \right\}.$$

Given $\gamma \in \mathbb{R}$, it follows from Theorem 3.2 that the inequality condition in (63) holds for all $\alpha \in \Phi^l$ if there exist $e \geq 0$ such that

$$\left(\prod_{i=1}^n (y_i + z_i)^e \right) \left(S(y,z) - \gamma I \left(\prod_{i=1}^n (y_i + z_i)^{d_{p_i}} \right) \right) \tag{64}$$

and

$$\begin{aligned} & \left(\prod_{i=1}^n (y_i + z_i)^e \right) \big(B^T(y,z) S(y,z) + S(y,z) B(y,z) \\ & \qquad + \frac{1}{2} \left(B^T(y,z) C^T(y,z) + C(y,z) B(y,z) \right) - \gamma I \left(\prod_{i=1}^n (y_i + z_i)^{d_{q_i}} \right) \big) \end{aligned} \tag{65}$$

have all positive coefficients, where d_{p_i} is the degree of x_i in $P(x)$ and d_{q_i} is the degree of x_i in $Q(x)$. Suppose S, B and C have the following forms.

$$S(y,z) = \sum_{\substack{h,g \in \mathbb{N}^l \\ h+g = d_s}} S_{h,g} y_1^{h_1} z_1^{g_1} \cdots y_n^{h_l} z_n^{g_l}, \tag{66}$$

$$B(y,z) = \sum_{\substack{h,g \in \mathbb{N}^l \\ h+g = d_b}} B_{h,g} y_1^{h_1} z_1^{g_1} \cdots y_n^{h_l} z_n^{g_l} \tag{67}$$

$$C(y,z) = \sum_{\substack{h,g\in\mathbb{N}^l \\ h+g=d_c}} C_{h,g} y_1^{h_1} z_1^{g_1} \cdots y_n^{h_l} z_n^{g_l}, \tag{68}$$

By combining (66), (67) and (68) with (64) and (65) it follows that for a given $\gamma \in \mathbb{R}$, the inequality condition in (63) holds for all $\alpha \in \Phi^n$ if there exist some $e \geq 0$ such that

$$\sum_{\substack{h,g\in\mathbb{N}^l \\ h+g=d_s}} f_{\{q,r\},\{h,g\}} S_{h,g} > 0 \quad \text{for all } \; q,r \in \mathbb{N}^l : q+r = d_s + e\cdot\mathbf{1} \quad \text{and}$$

$$\sum_{\substack{h,g\in\mathbb{N}^l \\ h+g=d_s}} M^T_{\{u,v\},\{h,g\}} S_{h,g} + S_{h,g} M_{\{u,v\},\{h,g\}} + N^T_{\{u,v\},\{h,g\}} C^T_{h,g} + C_{h,g} N_{\{u,v\},\{h,g\}} < 0$$

$$\text{for all } u,v \in \mathbb{N}^l : u+v = d_s + d_b + e\cdot\mathbf{1}, \tag{69}$$

where similar to Case 3 of Section 4.1, we define $f_{\{q,r\},\{h,g\}}$ to be the coefficient of $S_{h,g} y^q z^r$ after combining (66) with (64). Likewise, we define $M_{\{u,v\},\{h,g\}}$ to be the coefficient of $S_{h,g} y^u z^v$ and $N_{\{u,v\},\{h,g\}}$ to be the coefficient of $C_{h,g} y^u z^v$ after combining (67) and (68) with (65). Conditions (69) are an SDP. For any $\gamma \in \mathbb{R}$, if there exist $e \geq 0$ and $\{S_{h,g}\}$ such that Conditions (69) hold, then γ is a lower bound for γ^* as defined in (63). Furthermore, if γ is positive, then origin is an asymptotically stable equilibrium for System (62).

Numerical Example 5: Consider the reverse-time Van Der Pol oscillator defined as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 + x_2(x_1^2 - 1) \end{bmatrix} = A(x)x,$$

where $A(x) = \begin{bmatrix} 0 & -1 \\ 1 & x_1^2 - 1 \end{bmatrix}$. By using the method in Section 4.2.2, we solved Problem (63) using the hypercubes

$$\begin{aligned}
\Phi_1^2 &= \{x \in \mathbb{R}^2 : |x_1| \leq 1, |x_2| \leq 1\} \\
\Phi_2^2 &= \{x \in \mathbb{R}^2 : |x_1| \leq 1.5, |x_2| \leq 1.5\} \\
\Phi_3^2 &= \{x \in \mathbb{R}^2 : |x_1| \leq 1.7, |x_2| \leq 1.8\} \\
\Phi_4^2 &= \{x \in \mathbb{R}^2 : |x_1| \leq 1.9, |x_2| \leq 2.4\}
\end{aligned} \tag{70}$$

and $d_p = 0, 2, 4, 6$ as the degrees of $P(x)$. For each hypercube Φ_i^2 in (70), we computed a Lyapunov function of the form $V_i(x) = x^T P_i(x) x$. In Figure 4, we have plotted the largest level-set of V_i, inscribed in Φ_i^2 for $i = 1, \cdots, 4$. For all the cases, we used the Polya's exponent $e = 1$.

We also used the method in Section 4.2.1 to solve (see Figure 5) the same problem using the polytopes

$$\Gamma_s := \left\{ x \in \mathbb{R}^2 : x = \sum_{i=1}^4 \rho_i v_i : \rho_i \in [0,s], \sum_{i=1}^4 \rho_i = s \right\}$$

with $s = 0.83, 1.41, 1.52, 1.64$, where

$$v_1 = \begin{bmatrix} -1.31 \\ 0.18 \end{bmatrix}, v_2 = \begin{bmatrix} 0.56 \\ 1.92 \end{bmatrix}, v_3 = \begin{bmatrix} -0.56 \\ -1.92 \end{bmatrix} \text{ and } v_4 = \begin{bmatrix} 1.31 \\ -0.18 \end{bmatrix}.$$

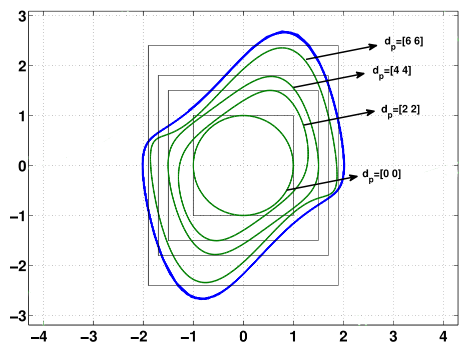

FIGURE 4. Level-sets of the Lyapunov functions $V(x) = x^T P(x) x$ computed by the method in Section 4.2.2 - d_p is the degree of $P(x)$

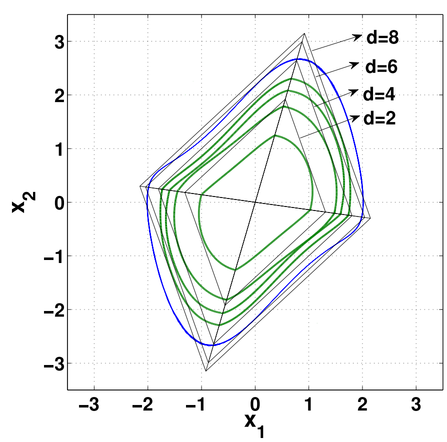

FIGURE 5. Level-sets of the Lyapunov functions computed for Van Der Pol oscillator using the method in Section 4.2.1 - d is the degree of the Lyapunov functions

From Figures 4 and 5 we observe that in both methods, computing larger invariant subsets of the region of attraction of the origin requires an increase in the degree of Lyapunov functions.

4.3. **Robust H_∞ control synthesis.** Consider the uncertain plant G with the state-space realization

$$\dot{x}(t) = A(\alpha)x(t) + \begin{bmatrix} B_1(\alpha) & B_2(\alpha) \end{bmatrix} \begin{bmatrix} \omega(t) \\ u(t) \end{bmatrix}$$

$$\begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} C_1(\alpha) \\ C_2(\alpha) \end{bmatrix} x(t) + \begin{bmatrix} D_{11}(\alpha) & D_{12}(\alpha) \\ D_{21}(\alpha) & D_{22}(\alpha) \end{bmatrix} \begin{bmatrix} \omega(t) \\ u(t) \end{bmatrix}, \tag{71}$$

where $\alpha \in Q \subset \mathbb{R}^l$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $\omega(t) \in \mathbb{R}^p$ is the external input and $z(t) \in \mathbb{R}^q$ is the external output. According to [76] there exists a state feedback gain $K(\alpha) \in \mathbb{R}^{m\times n}$ such that

$$\|S(G, K(\alpha))\|_{H_\infty} \leq \gamma, \text{ for all } \alpha \in Q,$$

if and only if there exist $P(\alpha) > 0$ and $R(\alpha) \in \mathbb{R}^{m\times n}$ such that $K(\alpha) = R(\alpha)P^{-1}(\alpha)$ and

$$\begin{bmatrix} \begin{bmatrix} A(\alpha) & B_2(\alpha)\end{bmatrix}\begin{bmatrix} P(\alpha) \\ R(\alpha)\end{bmatrix} + \begin{bmatrix} P(\alpha) & R^T(\alpha)\end{bmatrix}\begin{bmatrix} A^T(\alpha) \\ B_2^T(\alpha)\end{bmatrix} & \star & \star \\ B_1^T(\alpha) & -\gamma I & \star \\ \begin{bmatrix} C_1(\alpha) & D_{12}(\alpha)\end{bmatrix}\begin{bmatrix} P(\alpha) \\ R(\alpha)\end{bmatrix} & D_{11}(\alpha) & -\gamma I \end{bmatrix} < 0, \tag{72}$$

for all $\alpha \in Q$, where $\gamma > 0$ and $S(G, K(\alpha))$ is the map from the external input ω to the external output z of the closed loop system with a static full state feedback controller. The symbol $\star$ denotes the symmetric blocks in the matrix inequality. To find a solution to the robust H_∞-optimal static state-feedback controller problem with optimal feedback gain $K(\alpha) = P(\alpha)R^{-1}(\alpha)$, one can solve the following optimization of polynomials problem.

$$\gamma^* = \min_{P,R\in\mathbb{R}[\alpha],\gamma\in\mathbb{R}} \gamma$$

subject to

$$\begin{bmatrix} -P(\alpha) & \star & \star & \star \\ 0 & \begin{bmatrix} A(\alpha) & B_2(\alpha)\end{bmatrix}\begin{bmatrix} P(\alpha) \\ R(\alpha)\end{bmatrix} + \begin{bmatrix} P(\alpha) & R^T(\alpha)\end{bmatrix}\begin{bmatrix} A^T(\alpha) \\ B_2^T(\alpha)\end{bmatrix} & \star & \star \\ 0 & B_1^T(\alpha) & 0 & \star \\ 0 & \begin{bmatrix} C_1(\alpha) & D_{12}(\alpha)\end{bmatrix}\begin{bmatrix} P(\alpha) \\ R(\alpha)\end{bmatrix} & D_{11}(\alpha) & 0 \end{bmatrix} - \gamma \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} < 0$$

$$\text{for all } \alpha \in Q. \tag{73}$$

In Problem (73), if $Q = \Delta^l$ as defined in (12), then we can apply Polya' theorem (Theorem 3.1) as in the algorithm in Case 1 of Section 3.2 to find a $\gamma \leq \gamma^*$ and P and R which satisfy the inequality in (73). Suppose $P, A, B_1, B_2, C_1, D_{11}$ and D_{12} are homogeneous polynomials. If any of these polynomials is not homogeneous, use the procedure in Case 1 of Section 3.2 to homogenize it. Let

$$F(P(\alpha), R(\alpha)) := \begin{bmatrix} -P(\alpha) & \star & \star & \star \\ 0 & \begin{bmatrix} A(\alpha) & B_2(\alpha)\end{bmatrix}\begin{bmatrix} P(\alpha) \\ R(\alpha)\end{bmatrix} + \begin{bmatrix} P(\alpha) & R^T(\alpha)\end{bmatrix}\begin{bmatrix} A^T(\alpha) \\ B_2^T(\alpha)\end{bmatrix} & \star & \star \\ 0 & B_1^T(\alpha) & 0 & \star \\ 0 & \begin{bmatrix} C_1(\alpha) & D_{12}(\alpha)\end{bmatrix}\begin{bmatrix} P(\alpha) \\ R(\alpha)\end{bmatrix} & D_{11}(\alpha) & 0 \end{bmatrix},$$

and denote the degree of F by d_f. Given $\gamma \in \mathbb{R}$, the inequality in (73) holds if there exist $e \geq 0$ such that all of the coefficients in

$$\left(\sum_{i=1}^{l} \alpha_i\right)^e \left(F(P(\alpha), R(\alpha)) - \gamma \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \left(\sum_{i=1}^{l} \alpha_i\right)^{d_f} \right) \tag{74}$$

are negative-definite. Let P and R be of the forms

$$P(\alpha) = \sum_{h\in I_{d_p}} P_h \alpha_1^{h_1} \cdots \alpha_l^{h_l}, P_h \in \mathbb{S}^n \text{ and } R(\alpha) = \sum_{h\in I_{d_r}} R_h \alpha_1^{h_1} \cdots \alpha_l^{h_l}, R_h \in \mathbb{R}^{n\times n}, \tag{75}$$

where I_{d_p} and I_{d_r} are defined as in (34). By combining (75) with (74) it follows from Theorem (3.1) that for a given γ, the inequality in (73) holds, if there exist $e \geq 0$ such that

$$\sum_{h \in I_{d_p}} \left(M_{h,q}^T P_h + P_h M_{h,q} \right) + \sum_{h \in I_{d_r}} \left(N_{h,q}^T R_h^T + R_h N_{h,q} \right) < 0 \quad \text{for all } q \in I_{d_f+e}, \tag{76}$$

where we define $M_{h,q} \in \mathbb{R}^{n \times n}$ as the coefficient of $P_h \alpha_1^{q_1} \cdots \alpha_l^{q_l}$ after combining (75) with (74). Likewise, $N_{h,q} \in \mathbb{R}^{n \times n}$ is the coefficient of $R_h \alpha_1^{q_1} \cdots \alpha_l^{q_l}$ after combining (75) with (74). For given $\gamma > 0$, if there exist $e \geq 0$ such that the LMI (76) has a solution, say $P_h, h \in I_{d_p}$ and $R_g, g \in I_{d_r}$, then

$$K(\alpha) = \left(\sum_{h \in I_{d_p}} P_h \alpha_1^{h_1} \cdots \alpha_l^{h_l} \right) \left(\sum_{g \in I_{d_r}} R_g \alpha_1^{g_1} \cdots \alpha_l^{g_l} \right)^{-1}$$

is a feedback gain for an H_∞-suboptimal static state-feedback controller for System (71). By performing bisection on γ and solving (76) form each γ in the bisection, one may find an H_∞-optimal controller for System (71).

In Problem (73), if $Q = \Phi^l$ as defined in (15), then by applying the algorithm in Case 2 of section 3.2 to Problem (73), we can find a solution P, Q, γ to (73), where $\gamma \leq \gamma^*$. See Case 3 of Section 4.1 and Section 4.2.2 for similar applications of this theorem.

If $Q = \Gamma^l$ as defined in (57), then we can use Handelman's theorem (Theorem 2.4) as in the algorithm in Case 3 of section 3.2 to find a solution to Problem (73). We have provided a similar application of Handelman's theorem in Section 4.2.1.

If Q is a compact semi-algebraic set, then for given $d \in \mathbb{N}$, one can apply the Positivstellensatz results in Case 4 of Section 3.2 to the inequality in (73) to obtain a SOS program of the Form (22). A solution to the SOS program yields a solution to Problem (73).

5. **Conclusion.** SOS programming, moment's approach and their applications in polynomial optimization have been well-served in the literature. To promote diversity in commonly used algorithms for polynomial optimization, we dedicated this paper to some of the alternatives to SOS programming. In particular, we focused on the algorithms defined by Polya's theorem, Bernstein's theorem and Handelman's theorem. We discussed how these algorithms apply to polynomial optimization problems with decision variables on simplices, hypercubes and arbitrary convex polytopes. Moreover, we demonstrated some of the applications of Polya's and Handelman's algorithms in stability analysis of nonlinear systems and stability analysis and H_∞ control of systems with parametric uncertainty. For most of these applications, we have provided numerical examples to compare the conservativeness of Polya's and Handelman's algorithms with other algorithms in the literature such as SOS programming.

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