LMI parametrization of Lyapunov Functions for Infinite-Dimensional Systems: A Toolbox

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Abstract— In this paper, we present an algorithmic approach to the construction of Lyapunov functions for infinite-dimensional systems. This paper unifies and extends many previous results which have appeared in conference and journal format. The unifying principle is that a linear matrix parametrization of operators in Hilbert space inevitably leads to a linear parametrization of positive forms using positive semidefinite matrices via squared representations. For linear systems, these positive forms are defined by positive linear operators and define quadratic Lyapunov functions. For nonlinear systems, the forms are defined by nonlinear operators and will define non-quadratic Lyapunov functions. Special cases of these results include operators defined by multipliers and kernels which are: polynomial; piecewise-polynomial; or semi-separable and apply to systems with delay; multiple spatial domains; or mixed boundary conditions. We also introduce a set of efficient software tools for creating these functionals. Finally, we illustrate the approach with numerical examples.

I. INTRODUCTION

Recently, there has been substantial interest in algorithmic approaches to analysis and controller synthesis for both time-delay systems and partial-differential equations. For linear systems, this problem can be addressed in both the time-domain and in the frequency domain. For the purposes of this paper, however, we will only focus on time-domain approaches and, in particular, on Lyapunov-based methods.

It is commonly understood that the difficulty with manipulating infinite-dimensional systems lies in the complicated dependence of the solution on initial conditions. Indeed, the term "infinite-dimensional system" refers to the infinite number of parameters which influence the dynamics of the solution map. For time-delay systems, the initial condition is understood as history of evolution over a previous period of the maximum delay. For partial-differential equations, the initial condition is a combination of the boundary conditions and the distributed state. Unlike for ordinary-differential equations, for infinite-dimensional systems, construction of the solution as an explicit function of the initial condition is often a difficult if not impossible task. For this reason, (with notable exceptions) most of the analysis and controller synthesis work in this area has focused on projection of the infinite-dimensional system onto a finite-dimensional subspace of ordinary differential equations by choosing a suitable finite basis for the set of solutions. This approach is

reasonable given the vast body of literature on discretization and numerical simulation of PDEs. We would argue, however, that lumped-parameter approaches make less sense in the context of Lyapunov theory, wherein we are concerned not with reconstructing the solution, but with deriving *properties* of the solution.

The use of LMI conditions for constructing Lyapunov-Krasovskii functions for infinite-dimensional systems and time-delay systems in particular is not new. Of these, the most relevant are those which are based on the so-called "complete-quadratic functional" - a form based on converse Lyapunov theory [1]. Examples include the piecewise linear approach [2] and the delay-partitioning approach [3]. An interesting (semi-LMI) approach to numerical reconstruction of the Lyapunov function can be found in [4]. The literature on LMI-based Lyapunov methods for PDE systems is more limited, but some recent work can be found in [5]. Using squared representations for delayed and PDE systems has been studied by the author and collaborators in, e.g. [6] and [7]. This paper is a consolidation and extension of many of these earlier results.

A Lyapunov function for both finite and infinite-dimensional systems is a positive map with domain containing the set of initial conditions for which a unique solution exists, D. For ODEs, this domain may be $\mathbb{R}^n$. For time-delay systems with maximal delay τ, the set of initial conditions is often defined as as subspace of $X = \mathbb{R}^n \times \mathcal{C}[-\tau, 0]$. For the purposes of this paper, we suppose that this domain is a Hilbert space, H with $X \subset H$. For $\mathbb{R}^n$, this is clear. For infinite-dimensional systems, the inner-product will typically be L_2 or Sobolev (e.g. W_2). In this framework, any Lyapunov function $V(x)$ may be represented as $V(x) = \langle \mathcal{R}x, \mathcal{R}x \rangle_H$, where $\mathcal{R} : X \to H$ is a bounded, injective, possibly nonlinear operator. When $\mathcal{R}$ is linear, we refer to $\mathcal{P} = \mathcal{R}^*\mathcal{R} \geq 0$ as a Lyapunov operator in that it defines a valid Lyapunov function $V(x) = \langle x, \mathcal{P}x \rangle_H$.

In this paper, we show how to use positive matrices to parameterize positive Lyapunov operators $\mathcal{P} = \mathcal{R}^*\mathcal{R}$ and functions $V(x) = \langle \mathcal{R}x, \mathcal{R}x \rangle_H$ under the constraint that $\mathcal{R}$ lie in one of several subspaces of bounded, injective, possibly nonlinear operators. Specifically, for linear systems, we will consider the case where $\mathcal{R}$ is a combined multiplier and integral operator defined by multipliers and kernels which

are: polynomial; piecewise-polynomial; or semi-separable. This result is also extended to the case when $\mathcal{R}$ is nonlinear.

II. NOTATION

Standard notation includes the Hilbert spaces $L_2[X]$ of operators square integrable on X and $W_2[X] := \{x : x, \dot{x} \in L_2[X]\}$. We will omit the $[X]$ when the domain is clear from context. $\mathcal{C}[X]$ denotes the continuous functions on X. S^n denotes the symmetric matrices of dimension $n \times n$. $I_n \in \mathbb{S}^n$ denotes the identity matrix.

III. SEMIDEFINITE PROGRAMMING AND LMIS

Semidefinite programming (SDP) is the optimization of a linear objective subject to matrix positivity constraints. A general form is

$$\max_{X \in \mathbb{R}^{n \times n}} \text{trace}(CX), \text{ subject to: } L_1(X) \geq B_1, \ L_2(X) = B_2$$

where the L_i can be arbitrary linear transformations. A Linear Matrix Inequality (LMI) is the feasibility problem associated with an SDP. Efficient algorithms exist for the solution of SDPs and LMIs and implementations include [8] [9] [10]. For large SDP and LMI problems, limited parallel implementations exist, such as [11]. Because the goal of this paper is parametrization of Lyapunov functions using positive matrices, the ability to construct and solve relatively large SDP and LMI problems is a prerequisitve for the successful utilization of these results. Generally speaking, the complexity of the LMIs/SDPs is $O(n^6)$ where n is as defined above.

IV. STATE-SPACE FOR INFINITE-DIMENSIONAL SYSTEMS

The use of Lyapunov-Krasovskii functionals can be simplified by considering stability in the semigroup framework - a generalization of the concept of differential equations. See [12] and [13]. Although the results of this paper do not require the semigroup architecture, we adopt this notation in order to simplify the concepts and avoid unnecessary notation. Sometimes known as the 'flow map', a 'strongly continuous semigroup' is an operator, $S(t) : H \to H$, defined by the Hilbert space H, which represents the evolution of the state of the system so that for any solution x, $x(t + s) = S(s)x(t)$. Note that for a given H, the semigroup may not exist even if the solution exists for any initial conditions in H. Associated with a semigroup on H is an operator $\mathcal{A}$, called the 'infinitesimal generator' which satisfies

$$\frac{d}{dt}S(t)\phi = \mathcal{A}S(t)\phi$$

for any $\phi \in X$. The space X is often referred to as the domain of the generator $\mathcal{A}$, and is the space on which the generator is defined and need not be a closed subspace of H. In this paper we will refer to X as the 'state-space'. Although there are very few converse Lyapunov theorems for general classes of infinite-dimensional systems, it is known [12] that a strongly continuous semigroup defined by a linear operator $\dot{x} = \mathcal{A}x$ on Hilbert space H is exponentially stable if and only if there exists a positive operator $\mathcal{P}$ such that

$$\langle \mathcal{A}x, \mathcal{P}x \rangle_H + \langle \mathcal{A}x, \mathcal{P}\mathcal{A}x \rangle_H \leq -\epsilon \|x\|_H$$

for all $x \in X$. Unfortunately, this result tells us very little about the properties of the operator $\mathcal{P}$. A notable exception to this is the case of time-delay systems.

A. A Converse Lyapunov Theorem for Time-Delay Systems

Linear discrete-delay systems are defined by differential equations of the form:

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{K} A_i x(t - \tau_i) \quad \text{for all } t \geq 0,$$
$$x(t) = \phi(t) \quad \text{for } t \in [-\tau_K, 0] \quad (1)$$

where $A_i \in \mathbb{R}^{n \times n}$, $\phi \in \mathcal{C}[-\tau, 0]$ and $0 < \tau_1 < \tau_2 < \cdots < \tau_K = \tau$. For any solution x and any time $t \geq 0$, we can define the 'state' of System (1) as $x_t \in \mathcal{C}[-\tau, 0]$, where $x_t(s) = x(t + s)$. For linear discrete-delay systems of Form (1), the system has a unique solution for any $\phi \in \mathcal{C}[-\tau, 0]$ and global, local, asymptotic and exponential stability are all equivalent.

For systems of Form (1), it is known that stability is equivalent to the existence of a quadratic Lyapunov-Krasovskii functional of the form

$$V(\phi) = \int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds$$
$$+ \int_{-\tau}^{0} \int_{-\tau}^{0} \phi(s)^T N(s, \theta)\phi(\theta) \, ds \, d\theta, \quad (2)$$

where the Lie (upper-Dini) derivative of the functional is negative along any solution of (1). Furthermore, the unknown functions M and N may be assumed to be continuous in their respective arguments everywhere except possibly at points $\{\tau_1, \cdots, \tau_{K-1}\}$. To express this result in the Semigroup framework, we define $H := \{\mathbb{R}^n \times L_2\}$ and

$$\mathcal{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(s) := \begin{bmatrix} A_0 x_1 + \sum_{i=1}^{K} A_i x_2(-\tau_i) \\ \dot{x}_2(s) \end{bmatrix}.$$

The state-space is $X := \{ \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in H : x_2 \in \mathcal{C}[-\tau, 0] \text{ and } x_2(0) = x_1 \}$. Using these definitions of $\mathcal{A}$, H and X, the "complete-quadratic" Lyapunov functional (2) can be represented as

$$V(\phi) = \left\langle \begin{bmatrix} \phi(0) \\ \phi \end{bmatrix}, \mathcal{P} \begin{bmatrix} \phi(0) \\ \phi \end{bmatrix} \right\rangle_{L_2}$$

where

$$\mathcal{P} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := M(s) \begin{bmatrix} x_1 \\ x_2(s) \end{bmatrix} + \int_{-\tau}^{0} N(s, \theta) \begin{bmatrix} x_1 \\ x_2(\theta) \end{bmatrix}. \quad (3)$$

That is, the Lyapunov functional is defined by a multiplier and integral operator whose multiplier and kernel are unknown. Likewise, the derivative of the functional can be represented as

$$\dot{V}(\phi) = \left\langle \begin{bmatrix} \phi(0) \\ \phi \end{bmatrix}, \mathcal{P}\mathcal{A} \begin{bmatrix} \phi(0) \\ \phi \end{bmatrix} \right\rangle_{Z} + \left\langle \mathcal{A} \begin{bmatrix} \phi(0) \\ \phi \end{bmatrix}, \mathcal{P} \begin{bmatrix} \phi(0) \\ \phi \end{bmatrix} \right\rangle_{L_2}.$$
$$(4)$$

Thus, for time-delay systems, our challenge is to parameterize positive operators in the form of (3) and (4).

V. THE SOS JOINT POSITIVITY CONDITIONS

In this paper, we introduce a new method for enforcing positivity of a combined multiplier and integral operator such as defined in (3). This result has two parts. In the first, we observe that a positive operator will always have a square root. We assume that this square root is also of the form of operator (3) with functions M and N polynomial of bounded degree. Under this assumption, we give necessary and sufficient conditions for the positivity of (3). The second part (discussed in Section VIII) is to create slack variables which account for the limited equivalence between the multiplier and integral operator when the state-space has special structure. To begin, consider a simple operator of the form

$$(\mathcal{P}x)(s) := M(s)x(s) + \int_\Gamma N(s,\theta)x(\theta). \qquad (5)$$

where we assume M and N are square-integrable on region of integration Γ.

Theorem 1: For any functions $Z_1 : \Gamma \to \mathbb{R}^{m_1 \times n}$ and $Z_2 : \Gamma \times \Gamma \to \mathbb{R}^{m_2 \times n}$, square integrable on Γ, suppose that

$$M(s) = Z_1(s)^T Q_{11} Z_1(s) \qquad (6)$$
$$N(s,\theta) = Z_1(s)Q_{12}Z_2(s,\theta) + Z_2(\theta,s)^T Q_{21} Z_1(\theta)$$
$$+ \int_\Gamma Z_2(\omega,s)^T Q_{22} Z_2(\omega,\theta)\, d\omega \qquad (7)$$

where

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \geq 0.$$

Then for $\mathcal{P}$ as defined in Equation (5), $\langle x, \mathcal{P}x \rangle_{L_2} \geq 0$ for all $x \in L_2[\Gamma]$.

Proof: Since $Q \geq 0$, there exist matrices $D \in \mathbb{R}^{m_1+m_2 \times m_1}$ and $H \in \mathbb{R}^{m_1+m_2 \times m_2}$ such that

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} = \begin{bmatrix} D^T D & D^T H \\ H^T D & H^T H \end{bmatrix} = \begin{bmatrix} D^T \\ H^T \end{bmatrix} \begin{bmatrix} D & H \end{bmatrix}$$

Now define the operator

$$(\mathcal{R}x)(s) := DZ_1(s)x(s) + \int_\Gamma HZ_2(s,\theta)x(\theta)\, d\theta. \qquad (8)$$

Then $\langle x, \mathcal{P}x \rangle_{L_2} = \langle \mathcal{R}x, \mathcal{R}x \rangle_{L_2}$, as can be seen by the following progression where $y = \mathcal{R}x$.

$$\langle \mathcal{R}x, \mathcal{R}x \rangle_{L_2} = \langle y, \mathcal{R}x \rangle_{L_2}$$
$$= \int_\Gamma \left(y(s)^T DZ_1(s)x(s) + \int_\Gamma y(s)^T HZ_2(s,\theta)x(\theta)\, d\theta \right) ds$$
$$= \int_{-\tau}^0 x(s)^T Z_1(s)^T D^T DZ_1(s)x(s)ds$$
$$+ \int_\Gamma \int_\Gamma x(t)^T Z_2(s,t)^T H^T DZ_1(s)x(s)dsdt$$
$$+ \int_\Gamma \int_\Gamma x(s)^T Z_1(s)^T D^T HZ_2(s,\theta)x(\theta)d\theta ds$$
$$+ \int_\Gamma \int_\Gamma \int_\Gamma x(a)^T Z_2(s,a)^T H^T HZ_2(s,\theta)x(\theta)\, da\, d\theta ds$$

By the definition of D and H we have

$$\langle \mathcal{R}x, \mathcal{R}x \rangle_{L_2} = \int_{-\tau}^0 x(s)^T Z_1(s)^T Q_{11} Z_1(s)x(s)ds$$
$$+ \int_\Gamma \int_\Gamma x(s)^T Z_2(\theta,s)^T Q_{21} Z_1(\theta)x(\theta)d\theta ds$$
$$+ \int_\Gamma \int_\Gamma x(s)^T Z_1(s)^T Q_{12} Z_2(s,\theta)x(\theta)d\theta ds$$
$$+ \int_\Gamma \int_\Gamma x(s)^T \int_\Gamma Z_2(\omega,s)^T Q_{22} Z_2(\omega,\theta)d\omega x(\theta)\, ds\, d\theta$$
$$= \int_{-\tau}^0 x(s)^T M(s)x(s)ds + \int_\Gamma \int_\Gamma x(s)^T N(s,\theta)x(\theta)d\theta ds$$
$$= \langle x, \mathcal{P}x \rangle_{L_2}$$

$\blacksquare$

Theorem 1 gives a linear parametrization of a cone of positive operators using positive semidefinite matrices. Note that there are few constraints on the functions Z_1 and Z_2. These functions serve as the basis for the multipliers and kernels found in the square root of $\mathcal{P}$. The class of multipliers and kernels defined by Theorem 1 is thus determined by these functions. We consider certain choices of Z_1 and Z_2 shortly.

A. Strict Positivity

The conditions of Theorem 1 are positive semidefinite. Lyapunov functions, however, must be positive definite in some norm. In this case, the conditions of Theorem 1 can be readily modified for strict positivity as follows.

Corollary 2: For integrable Z_1 and Z_2, and $\epsilon > 0$, suppose that

$$M(s) = Z_1(s)^T Q_{11} Z_1(s) + \epsilon I$$
$$N(s,\theta) = Z_1(s)Q_{12}Z_2(s,\theta) + Z_2(\theta,s)^T Q_{21} Z_1(\theta)$$
$$+ \int_\Gamma Z_2(\omega,s)^T Q_{22} Z_2(\omega,\theta)\, d\omega$$

where $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \geq 0$. Then for $\mathcal{P}$ as defined in Equation (5), $\langle x, \mathcal{P}x \rangle_{L_2} \geq \|x\|_{L_2}$ for all $x \in L_2[\Gamma]$.

VI. NON-QUADRATIC LYAPUNOV FUNCTIONS

Theorem 1 can be extended to non-quadratic Lyapunov functions, as seen by the following lemma.

Lemma 3: For any functions $Z_1 : \Gamma \times \mathbb{R}^n \to \mathbb{R}^{m_1}$ and $Z_2 : \Gamma \times \Gamma \times \mathbb{R}^n \to \mathbb{R}^{m_2}$, square integrable on Γ, suppose

$$V(x) = \int_\Gamma Z_1(s,x(s))^T Q_{11} Z_1(s,x(s))ds$$
$$+ \int_\Gamma \int_\Gamma Z_1(s,x(s))Q_{12}Z_2(s,\theta,x(\theta))dsd\theta$$
$$+ \int_\Gamma \int_\Gamma Z_2(\theta,s,x(s))^T Q_{21} Z_1(\theta,x(\theta))dsd\theta$$
$$+ \int_\Gamma \int_\Gamma \int_\Gamma Z_2(\omega,s,x(s))^T Q_{22} Z_2(\omega,\theta,x(\theta))\, d\omega dsd\theta,$$

where

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \geq 0.$$

Then $V(x) \geq 0$ for all $x \in L_2[\Gamma]$.

Proof: Similar to the progression in the proof of Theorem 1, define the nonlinear operator

$$(\mathcal{R}x)(s) := DZ_1(s,x(s)) + \int_\Gamma HZ_2(s,\theta,x(\theta))\, d\theta.$$

Note that, at the expense of additional computational complexity, we could improve and simplify these conditions by letting V have the form

$$\int_\Gamma \int_\Gamma \int_\Gamma Z(s,t,x(s),x(t))^T Q Z(s,\theta,x(s),x(\theta))\, dt d\theta ds$$

for $Q \geq 0$ and integrable function Z. In this case, we would recover the squared representation $V(x) = \langle \mathcal{R}x, \mathcal{R}x \rangle$ where

$$(\mathcal{R}x)(s) := \int_\Gamma H Z(s,\theta,x(s),x(\theta))\, d\theta. \qquad (9)$$

However, in multivariate systems, the number of terms in Z would make computation prohibitively expensive.

A. Sum-of-Squares

In the context of analysis of finite-dimensional systems, Lemma 3 is well-known under the title Sum-of-Squares.

Definition 4: A polynomial $p(x)$ is **Sum-of-Squares** (SOS), denoted $p \in \Sigma_s$, if there exist a polynomials g_i such that $p(x) = \sum_{i=1}^M g_i(x)^2$.

Clearly, any SOS polynomial is positive semidefinite and although there exist many positive polynomials which are not SOS, the set of SOS polynomials has been shown to approximate the set of positive polynomials to arbitrary accuracy. Naturally, this is a special case of Lemma 3 where for p of degree $2d$, $Z_1(x)$ is the vector of monomials in x of degree d or less, $Z_2 = 0$, and $H = \mathbb{R}^M$.

Corollary 5: For a polynomial of degree $2d$, let $Z(x)$ be the vector of monomials in x of degree d or less. Then $p \in \Sigma_s$ if and only if $V(x) = Z(x)^T Q Z(x)$ for some $Q \geq 0$.

Proof: (Necessity) Since the g_i are polynomial of degree d, there exists a matrix H such that $g(x) = HZ(x)$ where g is the vector of polynomials g_i. Now let $\mathcal{R}$ be given by $\mathcal{R}x := HZ(x) = g(x)$. Let $Q = H^T H$. Then

$$\langle \mathcal{R}x, \mathcal{R}x \rangle_{R^M} = Z(x)^T Q Z(x) = g(x)^T g(x) = \sum_{i=1}^M g_i(x)^2.$$

Thus, although it is NP-hard to determine whether a given polynomial is positive, determining whether a polynomial is SOS is reducible to a semidefinite programming constraint on the coefficients of the polynomial p. This constraint may be implemented in a straightforward manner through the use of Matlab toolboxes such as SOSTOOLS [14], Gloptipoly [15] or SOSOPT [16].

VII. SPECIAL CASES

We now consider the implications of Theorem 1 for certain classes of basis functions Z_1 and Z_2.

A. Note on Multiple Spacial Domains

Variables s and θ in $Z_1(s)$ and $Z_2(s,\theta)$ need not be scalar and the domain of integration need not be an interval. This applies to each of the special cases to follow.

B. Matrix-Valued Polynomials

We first consider the case where we desire M and N to be matrix-valued polynomials of degree $2d$. First define $Z_d(s)$ as a vector whose elements form a basis for the polynomials in variables s of degree d or less. e.g. The vector of monomials. Then define

$$Z_{1p}(s) = Z_d(s) \otimes I_n, \qquad Z_{2p}(s,\theta) = Z_d(s,\theta) \otimes I_n. \quad (10)$$

If $Z_1(s) = Z_{1p}(s)$ and $Z_2(s,\theta) = Z_{2p}(s,\theta)$ and M and N are defined as in Equations (6) and (7), then M and N are polynomial matrices ($\mathbb{R}^{n \times n}$) of degree $2d$.

C. Matrix-Valued Piecewise-Polynomials

As noted in the section on converse Lyapunov theory for time-delay systems, it is often conservative to assume continuity of the functions M and N. For delay systems, we know that these functions can be discontinuous at points of delay. To define multipliers and kernels with discontinuities at known points, we divide the region of integration Γ into countable disjoint subregions Γ_i on which continuity holds and assume the functions are polynomial on these subregions. To do this, we introduce the indicator functions (not to be confused with the identity matrix)

$$I_i(t) = \begin{cases} 1 & t \in \Gamma_i \\ 0 & \text{otherwise,} \end{cases} \qquad i = 1, \cdots, K$$

and the vector of indicator functions $J = \begin{bmatrix} I_1 & \cdots & I_K \end{bmatrix}^T$,

$$Z_{1pc}(s) = Z_{1p}(s) \otimes J(s), \quad Z_{2pc}(s,\theta) = Z_{2p}(s,\theta) \otimes J(s) \otimes J(\theta).$$

Lemma 6: If $Z_1(s) = Z_{1pc}(s)$ and $Z_2(s,\theta) = Z_{2pc}(s,\theta)$ and M and N are defined as in Equations (6) and (7), then M and N are piecewise-polynomial matrices ($\mathbb{R}^{n \times n}$) of degree $2d$ with possible discontinuities at the boundary of the Γ_i. In this case, the functions M and N can be defined piecewise as

$$M(s) = \Big\{ M_i(s) \quad s \in \Gamma_i$$

where

$$M_i = Z_d(s)^T Q_{11,ii} Z_d(s)$$

where $Q_{11,i,j}$ is the i,jth block of Q_{11}. Likewise,

$$N(s,\theta) = \Big\{ N_{ij}(s,\theta) \quad s \in \Gamma_i \text{ and } \theta \in \Gamma_j$$

where

$$N_{ij} = Z_{1p}(s) Q_{12,i,(i-1)K+j} Z_{1p}(s,\theta)$$
$$+ Z_{2p}(\theta,s)^T Q_{21,(j-1)K+i,j} Z_{1p}(\theta)$$
$$+ \sum_{k=1}^K \int_{\Gamma_k} Z_{2p}(\omega_k,s)^T Q_{22,i+(k-1)K,j+(k-1)K} Z_{2p}(\omega_k,\theta)\, d\omega_k$$

Proof: The proof of this lemma is long, but not sophisticated. First observe the structure of $Z_{1,pc}$

$$Z_{1pc}(s) = \begin{bmatrix} Z_{1p}(s) I_1(s) \\ \vdots \\ Z_{1p}(s) I_K(s) \end{bmatrix}$$

Now, since $I_i(s) I_j(s) = 0$ for $i \neq j$ and $I_i(s) I_i(s) = I_i(s)$,

$$M(s) = Z_1(s)^T Q_{11} Z_1(s)$$

$$= \begin{bmatrix} Z_{1p}(s)I_1(s) \\ \vdots \\ Z_{1p}(s)I_K(s) \end{bmatrix}^T Q_{11} \begin{bmatrix} Z_{1p}(s)I_1(s) \\ \vdots \\ Z_{1p}(s)I_K(s) \end{bmatrix}$$

$$= \sum_{i,j=1}^K Z_{1p}(s)[Q_{11}]_{ij} Z_{1p}(s) I_i(s) I_j(s)$$

$$= \sum_{i=1}^K Z_{1p}(s)[Q_{11}]_{ii} Z_{1p}(s) I_i(s)$$

Therefore

$$M(s) = \left\{ M_i(s) \quad s \in \Gamma_i. \right.$$

Similarly, we expand $N(s,t)$ using the structure of Z_2. First, we divide N as $N = N_1 + N_2 + \int_\Gamma N_3 \, d\theta$:

$$N_1(s,t) = Z_1(s)Q_{12}Z_2(s,t), \ N_2(s,t) = Z_2(t,s)^T Q_{21} Z_1(t),$$

$$N_3(s,t) = \int_\Gamma Z_2(\omega,s)^T Q_{22} Z_2(\omega,t) \, d\omega.$$

Recall the *ceil* and *mod* functions

$$c(i) = \min_{j \geq i, \, j \in \mathbb{N}} j, \qquad m(i,j) = i - j \max_{k \leq i/j, \, k \in \mathbb{N}} k.$$

$$N_1(s,t) = Z_1(s)Q_{12}Z_2(s,t)$$

$$= \begin{bmatrix} Z_{1p}(s)I_1(s) \\ \vdots \\ Z_{1p}(s)I_K(s) \end{bmatrix} Q_{12} \begin{bmatrix} Z_{2p}(s,t)I_1(s)I_1(t) \\ Z_{2p}(s,t)I_1(s)I_2(t) \\ \vdots \\ Z_{2p}(s,t)I_1(s)I_K(t) \\ Z_{2p}(s,t)I_2(s)I_1(t) \\ \vdots \\ Z_{2p}(s,t)I_K(s)I_K(t) \end{bmatrix}$$

$$= \sum_{i=1}^K \sum_{j=1}^{K^2} Z_{1p}(s)[Q_{12}]_{i,j} Z_{2p}(s,t) I_i(s) I_{c(\frac{j}{K})}(s) I_{j-m(j,K)}(t)$$

$$= \sum_{i=1}^K \sum_{j=1}^K Z_{1p}(s)[Q_{12}]_{i,(i-1)K+j} Z_{2p}(s,t) I_i(s) I_j(t)$$

Similarly,

$$N_2(s,t) = Z_2(t,s)^T Q_{21} Z_1(t)$$

$$= \sum_{i=1}^K \sum_{j=1}^K Z_{2p}(t,s)[Q_{21}]_{(i-1)K+j,i} Z_{1p}(t) I_i(t) I_j(s)$$

$$= \sum_{i=1}^K \sum_{j=1}^K Z_{2p}(t,s)[Q_{21}]_{(j-1)K+i,j} Z_{1p}(t) I_i(s) I_j(t)$$

Finally,

$$N_3(s,t) = Z_2(\theta,s)^T Q_{22} Z_2(\theta,t)$$

$$= \begin{bmatrix} Z_{2p}(\theta,s)I_1(\theta)I_1(s) \\ Z_{2p}(\theta,s)I_1(\theta)I_2(s) \\ \vdots \\ Z_{2p}(\theta,s)I_1(\theta)I_K(s) \\ Z_{2p}(\theta,s)I_2(\theta)I_1(s) \\ \vdots \\ Z_{2p}(\theta,s)I_K(s)I_K(\theta) \end{bmatrix} Q_{22} \begin{bmatrix} Z_{2p}(\theta,t)I_1(\theta)I_1(t) \\ Z_{2p}(\theta,t)I_1(\theta)I_2(t) \\ \vdots \\ Z_{2p}(\theta,t)I_1(\theta)I_K(t) \\ Z_{2p}(\theta,t)I_2(\theta)I_1(t) \\ \vdots \\ Z_{2p}(\theta,t)I_K(\theta)I_K(t) \end{bmatrix}$$

$$= \sum_{i,j=1}^{K^2} Z_{2p}(\theta,s)[Q_{22}]_{i,j} Z_{2p}(\theta,t) \cdot$$

$$I_{c(\frac{i}{K})}(\theta) I_{m(i,K)}(s) I_{c(\frac{j}{K})}(\theta) I_{m(j,K)}(t)$$

$$= \sum_{i,j,k=1}^K Z_{2p}(\theta,s)[Q_{22}]_{\substack{i+(k-1)K, \\ j+(k-1)K}} Z_{2p}(\theta,t) I_k(\theta) I_i(s) I_j(t).$$

Thus

$$\int_\Gamma N_3 d\theta$$

$$= \sum_{i,j,k=1}^K \int_{\Gamma_k} Z_{2p}(\theta,s)[Q_{22}]_{\substack{i+(k-1)K, \\ j+(k-1)K}} Z_{2p}(\theta,t) d\theta I_i(s) I_j(t)$$

We conclude that

$$N(s,\theta) = \left\{ N_{i,j}(s,\theta) \quad s \in \Gamma_i \text{ and } \theta \in \Gamma_j. \right.$$

■

Note that this proof implies that many blocks of Q do not appear directly in the Lyapunov functional. This feature can be exploited to improve computational performance.

D. Semi-Separable Functions

Semi-separable kernels are often preferable to separable kernels in that they can define operators with infinite-dimensional image space. The use of semi-sepable kernels without joint positivity was first used to define Lyapunov-Krasovskii functionals in [17]. A discussion of the advantages of this class of operators can be found therein.

Now define the indicator function

$$I_s(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

If t is multidimensional (e.g. $t \in \mathbb{R}^n$), then the inequality is understood to represent a complete ordering on Γ (e.g. $t \geq 0$ if $c^T t \geq 0$ for arbitrary vector c). Now define the basis vectors

$$Z_{1ss}(s) = Z_{1p}(s), \ Z_{2ss}(s,\theta) = \begin{bmatrix} Z_p(s,\theta)I(s-\theta) \\ Z_p(s,\theta)I(\theta-s) \end{bmatrix}$$

If $Z_1 = Z_{1ss}$ and $Z_2 = Z_{2ss}$ and M and N are defined as in Equations (6) and (7), then M is a polynomial matrix and N is a semi-separable polynomial matrix ($\mathbb{R}^{n \times n}$) of degree $2d$. If Z_{1pc} and Z_{2pc} are substituted for Z_{1p} and Z_{2p}, then the matrices are semiseparable and piecewise continuous.

Lemma 7: For a complete ordering $\geq$, define the sets $\Gamma_{s+} := \{\theta : \theta - s \geq 0\}$ and $\Gamma_{s-} := \{\theta : s - \theta \geq 0\}$. Suppose $Z_1 = Z_{1ss}$ and $Z_2 = Z_{2ss}$ and M and N are

defined as in Equations (6) and (7) where we partition the matrix $Q \geq 0$ as

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}.$$

Then M is a polynomial matrix and N is a semi-separable polynomial matrix, both of degree $2d$ where

$$M(s) = Z_1(s)^T Q_{11} Z_1(s), \quad N(s,t) = \begin{cases} N_1(s,t) & s \geq t \\ N_2(s,t) & s < t, \end{cases}$$

where

$$\begin{aligned} N_1(s,t) = & Z_1(s)^T Q_{12} Z_2(s,t) + Z_2(t,s) Q_{31} Z_1(t) \\ & + \int_{\Gamma_{s+}} Z(\theta,s)^T Q_{22} Z(\theta,t) d\theta \\ & + \int_{\Gamma_{t+} \cap \Gamma_{s-}} Z(\theta,s)^T Q_{32} Z(\theta,t) d\theta \\ & + \int_{\Gamma_{t-}} Z(\theta,s)^T Q_{33} Z(\theta,t) d\theta. \end{aligned}$$

and

$$\begin{aligned} N_2(s,t) = & Z_1(s)^T Q_{13} Z_2(s,t) + Z_2(t,s) Q_{21} Z_1(t) \\ & + \int_{\Gamma_{t+}} Z(\theta,s)^T Q_{22} Z(\theta,t) d\theta \\ & + \int_{\Gamma_{s+} \cap \Gamma_{t-}} Z(\theta,s)^T Q_{23} Z(\theta,t) d\theta \\ & + \int_{\Gamma_{s-}} Z(\theta,s)^T Q_{33} Z(\theta,t) d\theta. \end{aligned}$$

Proof: To conserve space, for this proof only, we denote $Z(s) = Z_1(s)$, $Z(s,t) = Z_2(s,t)$ and $I(t) = I_s(t)$. Now, expanding the expressions for M and N in Equations (6) and (7), we obtain

$$M(s) = Z(s) Q_{11} Z(s)$$

and

$$\begin{aligned} N(s,t) = & Z(s)^T Q_{12} Z(s,t) I(s-t) \\ & + Z(s)^T Q_{13} Z(s,t) I(t-s) \\ & + Z(t,s)^T Q_{21} Z(t) I(t-s) + Z(t,s)^T Q_{31} Z(t) I(s-t) \\ & + \int_{\Gamma} Z(\theta,s)^T Q_{22} Z(\theta,t) I(\theta-s) I(\theta-t) d\theta \\ & + \int_{\Gamma} Z(\theta,s)^T Q_{23} Z(\theta,t) I(\theta-s) I(t-\theta) d\theta \\ & + \int_{\Gamma} Z(\theta,s)^T Q_{32} Z(\theta,t) I(s-\theta) I(\theta-t) d\theta \\ & + \int_{\Gamma} Z(\theta,s)^T Q_{33} Z(\theta,t) I(s-\theta) I(t-\theta) d\theta \end{aligned}$$

Noting the identity $1 = I(\theta - s) + I(s - \theta)$, we obtain

$$\begin{aligned} & \int_{\Gamma} Z(\theta,s)^T Q_{22} Z(\theta,t) I(\theta-s) I(\theta-t) d\theta \\ & = \int_{\Gamma_{s+}} Z(\theta,s)^T Q_{22} Z(\theta,t) d\theta I(s-t) \\ & \quad + \int_{\Gamma_{t+}} Z(\theta,s)^T Q_{22} Z(\theta,t) d\theta I(t-s) \end{aligned}$$

which holds since

$$I(\theta-s) I(\theta-t) I(s-t) = \begin{cases} 1 & \theta \geq s \geq t \\ 0 & \text{otherwise} \end{cases}$$

$$I(\theta-s) I(\theta-t) I(t-s) = \begin{cases} 1 & \theta \geq t \geq s \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, $I(\theta - s) I(t - \theta) I(s - t) = 0$ and

$$I(\theta-s) I(t-\theta) I(t-s) = \begin{cases} 1 & t \geq \theta \geq s \\ 0 & \text{otherwise} \end{cases}$$

yields

$$\begin{aligned} & \int_{\Gamma} Z(\theta,s)^T Q_{23} Z(\theta,t) I(\theta-s) I(t-\theta) d\theta \\ & = \int_{\Gamma_{s+} \cap \Gamma_{t-}} Z(\theta,s)^T Q_{23} Z(\theta,t) d\theta I(t-s). \end{aligned}$$

Again, $I(s - \theta) I(\theta - t) I(t - s) = 0$ and

$$I(s-\theta) I(\theta-t) I(s-t) = \begin{cases} 1 & s \geq \theta \geq t \\ 0 & \text{otherwise} \end{cases}$$

yields

$$\begin{aligned} & \int_{\Gamma} Z(\theta,s)^T Q_{32} Z(\theta,t) I(s-\theta) I(\theta-t) d\theta \\ & = \int_{\Gamma_{t+} \cap \Gamma_{s-}} Z(\theta,s)^T Q_{23} Z(\theta,t) d\theta I(s-t). \end{aligned}$$

Finally,

$$I(s-\theta) I(t-\theta) I(s-t) = \begin{cases} 1 & s \geq t \geq \theta \\ 0 & \text{otherwise} \end{cases}$$

$$I(s-\theta) I(t-\theta) I(t-s) = \begin{cases} 1 & t \geq s \geq \theta \\ 0 & \text{otherwise} \end{cases}$$

yields

$$\begin{aligned} & \int_{\Gamma} Z(\theta,s)^T Q_{33} Z(\theta,t) I(s-\theta) I(t-\theta) d\theta \\ & = \int_{\Gamma_{t-}} Z(\theta,s)^T Q_{33} Z(\theta,t) d\theta I(s-t) \\ & \quad + \int_{\Gamma_{s-}} Z(\theta,s)^T Q_{33} Z(\theta,t) d\theta I(t-s). \end{aligned}$$

Grouping the terms which multiply $I(s - t)$ and $I(t - s)$ separately, we obtain the equality in the Lemma statement. ∎

Note that, if desired, this approach can be extended to a more generalized partition of Γ. However, the integrals in this case are more complicated.

VIII. SPACING FUNCTIONS AND MIXED STATE-SPACE

The result in Theorem 1 as stated applies to the space $L_2(\Gamma)$. However, as seen in (3), for delay systems, the state lies in the subspace $\mathbb{R}^n \times L_2(\Gamma)$. For such systems, the positivity conditions can be improved through the use of spacing functions.

Theorem 8: For any integrable functions $Z_1 : \Gamma \to \mathbb{R}^{m_1 \times 2n}$ and $Z_2 : \Gamma \times \Gamma \to \mathbb{R}^{m_2 \times 2n}$, suppose that

$$M(s) = Z_1(s)^T Q_{11} Z_1(s)$$
$$+ \begin{bmatrix} T(s) + \frac{1}{\tau} \int_\Gamma \int_\Gamma R_{11}(\omega, t) d\omega dt & \int_\Gamma R_{12}(\omega, s) d\omega \\ \int_\Gamma R_{21}(s, \omega) d\omega & 0 \end{bmatrix}$$
$$N(s, \theta) = Z_1(s) Q_{12} Z_2(s, \theta) + Z_2(\theta, s)^T Q_{21} Z_1(\theta)$$
$$+ \int_\Gamma Z_2(\omega, s)^T Q_{22} Z_2(\omega, \theta) \, d\omega - \begin{bmatrix} R_{11}(s, \theta) & R_{12}(s, \theta) \\ R_{21}(s, \theta) & 0 \end{bmatrix}$$
$$\int_\Gamma T(s) = 0$$

where $\tau = \int_\Gamma ds$ and

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \geq 0$$

Then for $\mathcal{P}$ as defined in Equation (5), $\langle x, \mathcal{P}x \rangle \geq 0$ for all $x = (c, y) \in \mathbb{R}^n \times \mathcal{C}[I]$.

Proof: The proof is straightforward

$$\langle x, \mathcal{P}x \rangle = \int_\Gamma \begin{bmatrix} c \\ y(s) \end{bmatrix}^T M(s) \begin{bmatrix} c \\ y(s) \end{bmatrix} ds$$
$$+ \int_\Gamma \int_\Gamma \begin{bmatrix} c \\ y(s) \end{bmatrix}^T N(s, \theta) \begin{bmatrix} c \\ y(\theta) \end{bmatrix} d\theta ds$$
$$= \int_\Gamma y(s)^T Z_1(s)^T Q_{11} Z_1(s) y(s) ds$$
$$+ \int_\Gamma \int_\Gamma y(s)^T Z_2(\theta, s)^T Q_{21} Z_1(\theta) y(\theta) d\theta ds$$
$$+ \int_\Gamma \int_\Gamma y(s)^T Z_1(s)^T Q_{12} Z_2(s, \theta) y(\theta) d\theta ds$$
$$+ \int_\Gamma \int_\Gamma y(s)^T \int_\Gamma Z_2(\omega, s)^T Q_{22} Z_2(\omega, \theta) \, d\omega y(\theta) \, ds \, d\theta$$

which was shown to be positive in Theorem 1 for $Q \geq 0$. ∎

The use of the spacing function T was introduced in [6] as part of necessary and sufficient conditions for positivity of the multiplier operator on mixed state-space. Theorem 8 extends this concept through the use of the shifting functions R_{ij} which account for equivalence between multiplier and integral operators when acting on $\mathbb{R}^n$.

IX. A MATLAB TOOLBOX

To assist with the application of these results, we have created a library of functions for the synthesis of the Lyapunov functions described in this paper. These libraries make use of modified versions of the SOSTOOLS and MULTIPOLY toolboxes coupled with either SeDuMi or SDPT. A complete package can be downloaded from the website `http://control.asu.edu/software`. Key examples of functions included are:

`sosjointpos_mat_ker.m`
- Declares a positive polynomial multiplier, kernel pair.

`sosjointpos_mat_ker_ndelay.m`
- Declares a positive piecewise-polynomial multiplier, kernel pair.

`sosjointpos_mat_ker_semisep.m`
- Declares a positive semiseparable multiplier/kernel pair.

`sosmateq.m`
- Declare a matrix-valued equality constraint.

The functions are implemented within the pvar framework of SOSTOOLS and the user must have some familiarity with this relatively intuitive language to utilize these functions. Note also that the entire toolbox and supporting modified implementations of SOSTOOLS and MULTIPOLY must be added to the path for these functions to execute.

X. STABILITY OF TIME-DELAY SYSTEMS

Although the conditions for positivity which appear in this paper may seem complex, to some extent, this complexity is hidden for the user. That, is the user need only consider M and N to be functions of the desired class. The toolbox will then ensure that M and N define positive operators. The difficulty for the user is to find the derivative of the Lyapunov function and ensure it is defined by multiplier and integral operators which can then be constrained to be negative using the toolbox. The simplest application of the parametrization discussed above is to systems with delay. In this section, we give an example of this. These results can be viewed as an extension of previous work developed in [6] [18] [17].

Our first step is to define the class of Lyapunov functions to be used. Let

$$\Xi := \{(M, N) : \begin{smallmatrix} M \text{ and } N \text{ satisfy the conditions of} \\ \text{Theorem 8 with } Z_1 = Z_{1pc} \text{ and } Z_2 = Z_{2pc}. \end{smallmatrix}\}$$

The next step is to define the linear map between (M, N) and the multiplier and kernel which define the derivative of the Lyapunov function as defined in (2). For convenience, define the jump values of M and N at the discontinuities as as $\Delta M(\tau_i) = M_i(-\tau_i) - M_{i+1}(-\tau_i)$ and

$$\Delta N_j(\tau_i, t) = N_{i,j}(-\tau_i, t) - N_{i+1,j}(-\tau_i, t).$$

Then we have

Definition 9: Define the map L by $(D, E) = L(M, N)$ if

$$D(t) = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14,i}(t) \\ T & D_{22} & D_{23} & D_{24,i}(t) \\ T & *^T & D_{33} & D_{34,i}(t) \\ T & *^T & *^T & D_{44,i}(t) \end{bmatrix} \quad t \in [-\tau_i, -\tau_{i-1}]$$

$$D_{11} = A_0^T M_{11} + M_{11} A_0 + \frac{1}{\tau} \big(M_{12}(0) + M_{21}(0) + M_{22}(0) \big),$$
$$D_{12} = \begin{bmatrix} M_{11} A_1 & \cdots & M_{11} A_{K-1} \end{bmatrix}$$
$$\qquad - \frac{1}{\tau} \begin{bmatrix} \Delta M_{12}(\tau_1) & \cdots & \Delta M_{12}(\tau_{k-1}) \end{bmatrix},$$
$$D_{13} = M_{11} A_K - \frac{1}{\tau} \big(M_{12}(-\tau) \big),$$
$$D_{22} = \frac{1}{\tau} \operatorname{diag} \big(-\Delta M_{22}(\tau_1), \ldots, -\Delta M_{22}(\tau_{K-1}) \big),$$
$$D_{23} = 0, \quad D_{33} = \frac{-1}{\tau} M_{22}(-\tau),$$
$$D_{14,i}(t) = N_i(0, t) + A_0^T M_{12,i}(t) - \dot{M}_{12,i}(t),$$
$$D_{24,i}(t) = \begin{bmatrix} -\Delta N_i(\tau_1, t) + A_1^T M_{12,i}(t) \\ \vdots \\ -\Delta N_i(\tau_{K-1}, t) + A_{K-1}^T M_{12,i}(t) \end{bmatrix},$$
$$D_{34,i}(t) = A_k^T M_{12,i}(t) - N_{K,i}(-\tau, t),$$
$$D_{44,i}(t) = -\dot{M}_{22,i}(t)$$

and

$$E(s,t) = \frac{\partial N_{ij}(s,t)}{\partial s} + \frac{\partial N_{ij}(s,t)}{\partial t} \quad \begin{matrix} s \in [-\tau_i, -\tau_{i-1}] \\ t \in [-\tau_j, -\tau_{j-1}] \end{matrix}$$

Theorem 10: Suppose there exist $\epsilon > 0$ such that $(M - \epsilon I, N) \in \Xi$ and $-L(M, N) \in \Xi$. Then the system defined by Equation (1) is exponentially stable.

Proof: If V is defined as in (2), then

$$\dot{V}(x) = \int_{-\tau}^{0} \begin{bmatrix} x(0) \\ \vdots \\ x(-\tau_K) \\ x(s) \end{bmatrix}^T D(s) \begin{bmatrix} x(0) \\ \vdots \\ x(-\tau_K) \\ x(s) \end{bmatrix}^T$$

$$+ \int_{-\tau}^{0} \int_{-\tau}^{0} x(s) E(s,t) x(t) ds dt \le 0$$

where $(D, E) = L(M, N)$. ∎

To illustrate how these conditions can be efficiently coded using the Matlab toolbox, we give a pseudocode implmentation of the conditions of Theorem 10.

```
1) [M,N]=sosjointpos_mat_ker_ndelay
2) [D,E]=L(M, N)
3) [Q,R]=sosjointpos_mat_ker_ndelay
4) sosmateq((D,E) + (Q,R))
```

For brevity, the pseudocode does not include the spacing functions of Theorem 8. See the complete solver in `solver_ndelay_nd_joint.m` for a full implementation of the algorithm for multiple delays.

XI. NUMERICAL RESULTS

The conditions of Theorem 10 are implemented in the file `solver_ndelay_joint.m`, available with the rest of the supporting functions described previously.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & .1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x(t - \frac{\tau}{2}) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t - \tau)$$

The maximum and minimum values of τ for this system are listed in Table XI, where SOS [6] refers to the SOS stability test without joint positivity and $SOS - joint$ refers to the conditions of Theorem 10.

	SOS [6]		SOS-joint	
d	$\tau_{\min}$	$\tau_{\max}$	$\tau_{\min}$	$\tau_{\max}$
1	.20247	1.354	.20247	1.3711
2	.20247	1.3722	.20247	1.3722

Because previous SOS results for stability of time-delay systems were asymptotically exact, the numerical validation here should not be particularly surprising. However, as expected , the convergence is faster as a function of the polynomial degree.

XII. CONCLUSION

In this paper, we have given a primer on how to construct Lyapunov functions for infinite-dimensional systems. Specifically, we have shown how several classes of positive operators may be parameterized using positive matrices and constructed efficient algorithms to implement these results.

Notably absent from this discussion is an application of these results to PDE systems. This is partially a lack of space and partially because such work is non-trivial and will vary from application to application. Indeed, our results should not be seen as a general solution for analysis of infinite-dimensional systems, but rather a reference for those working in the field on specific applications and who may find the representations and algorithms useful. We also mention, the joint positivity results of this paper have been used to create asymptotically-exact dual stability conditions using the results of [19]. However, discussion of these results is beyond the scope of this paper.

REFERENCES

[1] R. Datko, "Lyapunov functionals for certain linear delay differential equations in a Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 76, no. 1, pp. 37–57, 1980.

[2] K. Gu, V. L. Kharitonov, and J. Chen, *Stability of Time-Delay Systems*. Birkhauser, 2003.

[3] F. Gouaisbaut and D. Peaucelle, "A note on stability of time delay systems," in *Proceedings of 5th IFAC Symposium on Robust Control Design (Rocond 06)*, 2006.

[4] S. Mondie and V. Kharitonov, "Exponential estimates for retarded time-delay systems: an LMI approach," *IEEE Transactions on Automatic Control*, vol. 50, no. 2, pp. 268–273, 2005.

[5] E. Fridman and Y. Orlov, "An LMI approach to H_∞ boundary control of semilinear parabolic and hyperbolic systems," *Automatica*, vol. 45, no. 9, pp. 2060–2066, 2009.

[6] M. M. Peet, A. Papachristodoulou, and S. Lall, "Positive forms and stability of linear time-delay systems," *SIAM Journal on Control and Optimization*, vol. 47, no. 6, 2009.

[7] A. Papachristodoulou and M. M. Peet, "On the analysis of systems described by classes of partial differential equations," in *Proceedings of the IEEE Conference on Decision and Control*, Dec. 2006, pp. 747–752.

[8] B. Borchers, "CSDP, a C library for semidefinite programming," *Optimization Methods and Software*, vol. 11, no. 2, pp. 613–623, 1999.

[9] J. Lofberg, "Yalmip: A toolbox for modeling and optimization in matlab," in *Computer Aided Control Systems Design, 2004 IEEE International Symposium on*, 2004, pp. 284–289.

[10] K. Toh, M. Todd, and R. Tutuncu, "SDPT3 — a Matlab software package for semidefinite programming," *Optimization Methods and Software*, vol. 11, pp. 545–581, 1999.

[11] B. Borchers and J. G. Young, "Implementation of a primal-dual method for SDP on a shared memory parallel architecture," *Computational Optimization and Applications*, vol. 37, no. 3, pp. 355–369, 2007.

[12] R. Curtain and H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*. Springer-Verlag, 1995.

[13] A. Bensoussan, G. D. Prato, M. C. Delfour, and S. K. Mitter, *Representation and Control of Infinite Dimensional Systems Volume I*. Birkhäuser, 1992.

[14] S. Prajna, A. Papachristodoulou, and P. A. Parrilo, "Introducing SOSTOOLS: a general purpose sum of squares programming solver," *Proceedings of the IEEE Conference on Decision and Control*, 2002.

[15] D. Henrion, J. B. Lasserre, and J. Loefberg, "Gloptipoly 3: moments, optimization and semidefinite programming," *Optimization Methods and Software*, vol. 24, no. 4-5, pp. 761–779, 2009.

[16] P. Seiler, "Sosopt: A toolbox for polynomial optimization," December 2010, university of Minnesota.

[17] M. M. Peet and A. Papachristodoulou, "Using polynomial semi-separable kernels to construct infinite-dimensional Lyapunov functions," in *Proceedings of the IEEE Conference on Decision and Control*, 2008.

[18] ——, "Joint positivity of Lyapunov operators for linear time-delay systems," in *Proceedings of the IEEE Conference on Decision and Control*, Dec. 2007.

[19] M. Peet, "Full-state feedback of delayed systems using sos: A new theory of duality," in *11th IFAC Workshop on Time-Delay Systems*, 2013.