Output Feedback Control of Inhomogeneous Parabolic PDEs with Point Actuation and Point Measurement using SOS and Semi-Separable Kernels

Aditya Gahlawat [1]   Matthew M. Peet [2]

[1] Illinois Institute of Technology, Chicago, USA
GIPSA Laboratoire, Université Grenoble Alpes, Grenoble, France

[2] Cybernetics Systems and Controls Laboratory, Arizona State University, Tempe, USA

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Contribution

- We provide a polynomial time algorithmic approach to output feedback controller synthesis for parabolic PDEs.
  - Akin to LMI framework for ODEs.
- No model reduction is required at the design stage.
- Lyapunov functions and controller/observer gains are parametrized by polynomials.
- Polynomials are in turn parametrized by positive semi-definite matrices.
- Controller synthesis problem reduced to a semi-definite programming feasibility problem.
  - Can be solved by interior point algorithms.

System Under Consideration

We consider the following class of possibly unsteady parabolic PDEs

$$w_t(t,x) = a(x)w_{xx}(t,x) + b(x)w_x(t,x) + c(x)w(t,x) + f(t,x).$$

The boundary conditions are of the form

$$w_x(t,0) = 0, \quad w(t,1) = u(t).$$

- The coefficients a, b and c are polynomially distributed in the spatial variable $x \in [0,1]$.
- $u(t) \in \mathbb{R}$ is the *boundary (point) control input*.
- $f \in L_2(0,\infty; L_2(0,1))$ is the *exogenous input*.

Goal

Using the *boundary (point) measurement*

$$y(t) = w_x(t, 1),$$

design control input $u(t)$ such that there exists a finite $\gamma > 0$ satisfying

$$\|w\|_{L_2(0,\infty;L_2(0,1))} \leq \gamma \|f\|_{L_2(0,\infty;L_2(0,1))}.$$

To achieve this goal we

- Construct a *Luenberger observer* whose state $\hat{w}$ estimates the system state w.
- Construct quadratic Lyapunov functions for the observer and the error dynamics $e = \hat{w} - w$.
- Construct a controller $\mathcal{F} \in \mathcal{L}\left(H^2(0,1), \mathbb{R}\right)$ such that if

  $$u(t) = \mathcal{F}\hat{w}(t, \cdot),$$

  then

  $$\|w\|_{L_2(0,\infty;L_2(0,1))} \leq \gamma \|f\|_{L_2(0,\infty;L_2(0,1))}.$$

Luenberger Observer Design

We wish to design the following Luenberger observer

$$\hat{w}_t(t,x) = \underbrace{a(x)\hat{w}_{xx}(t,x) + b(x)\hat{w}_x(t,x) + c(x)\hat{w}(t,x)}_{\text{Copy of plant}}$$
$$+ \underbrace{O_1(x)\left(\hat{y}(t) - y(t)\right)}_{\text{Output injection}},$$
$$\hat{w}_x(t,0) = 0, \quad \hat{w}(t,1) = u(t) + \underbrace{O_2\left(\hat{y}(t) - y(t)\right)}_{\text{Output injection}}.$$

- $O_1(x)$ and O_2 are the observer gains to be determined.

With the observer dynamics, the error dynamics $e = \hat{w} - w$ are given by

$$e_t(t,x) = a(x)e_{xx}(t,x) + b(x)e_x(t,x) + c(x)e(t,x)$$
$$+ O_1(x)e_x(t,1) - f(t,x),$$
$$e_x(t,0) = 0, \quad e(t,1) = O_2 e_x(t,1).$$

We wish to design

- Observer gains $O_1(x)$ and O_2.
- Controller $\mathcal{F}$ which defines $u(t) = \mathcal{F}\hat{w}(\cdot,t)$.

Quadratic Lyapunov Functions

We wish to construct quadratic Lyapunov functions for the observer and error dynamics to achieve our goal.

- For systems whose dynamics are governed by ODEs:
  - Quadratic Lyapunov functions are parametrized by Positive Semi-Definite (PSD) matrices.
  - Controller and observer gains are defined by matrices.
- Such a parametrization renders the search for the Lyapunov functions and gains as a Semi-Definite Programming (SDP) feasibility problem.
  - Can be solved in polynomial time.

Question: Can we use SDP to solve our PDE problem?

Answer: Yes!

- Use Lyapunov functions parametrized by positive self-adjoint operators on $L_2(0, 1)$ which are in turn parametrized by PSD matrices.

Positive Operators Parametrized by PSD Matrices

We use operators of the form

$$\left(\mathcal{P}z\right)(x) = M(x)z(x) + \int_0^x K_1(x,\xi)z(\xi)d\xi + \int_x^1 K_2(x,\xi)z(\xi)d\xi.$$

- Infinite dimensional generalizations of populated matrices.

  Question: How to enforce positivity of such operators.

  Answer: Define M, K_1 and K_2 to be polynomials defined using a positive semidefinite matrix.

Parametrization of Choice

- Given any scalar $\epsilon > 0$ and a PSD matrix U such that

$$U = \begin{bmatrix} U_{11} + \epsilon I & U_{12} & U_{13} \\ \star^T & U_{22} & U_{23} \\ \star^T & \star^T & U_{33} \end{bmatrix} \geq 0.$$

- Let

$$\begin{aligned} M(x) =& Z_1(x)^T \left(U_{11} + \epsilon\right) Z_1(x), \\ K_1(x,\xi) =& Z_1(x)^T U_{12} Z_2(x,\xi) + Z_2(\xi,x)^T U_{13}^T Z_1(\xi) \\ & + \int_0^\xi Z_2(\eta,x)^T U_{33} Z_2(\eta,\xi) d\eta + \int_\xi^x Z_2(\eta,x)^T U_{23}^T Z_2(\eta,\xi) d\eta \\ & + \int_x^1 Z_2(\eta,x)^T U_{22} Z_2(\eta,\xi) d\eta, \\ K_2(x,\xi) =& K_1(\xi,x)^T. \end{aligned}$$

Parametrization of Choice

- If we define the operator

$$\left(\mathcal{P}z\right)(x) = M(x)z(x) + \int_0^x K_1(x,\xi)z(\xi)d\xi + \int_x^1 K_2(x,\xi)z(\xi)d\xi.$$

- Then

$$\text{then } \langle \mathcal{P}z, z\rangle \geq \epsilon\|z\|^2 \Rightarrow \mathcal{P} - \epsilon\mathcal{I} \geq 0.$$

- Positivity is implied by the existence of an operator $\mathcal{G}$ satisfying

$$\mathcal{P} = \mathcal{G}\mathcal{G}^\star.$$

Implications of the Parametrization

Given a scalar $\epsilon > 0$ and polynomials $M(x)$, $K_1(x,\xi)$ and $K_2(x,\xi)$, we say that

$$\{M, K_1, K_2\} \in \Xi_\epsilon,$$

if

$$\left(\mathcal{P}z\right)(x) = M(x)z(x) + \int_0^x K_1(x,\xi)z(\xi)d\xi + \int_x^1 K_2(x,\xi)z(\xi)d\xi,$$

satisfies $\mathcal{P} \geq \epsilon\mathcal{I}$.

- Given any $U \geq 0$, we can construct $M(x)$, $K_1(x,\xi)$ and $K_2(x,\xi)$ such that

$$\{M, K_1, K_2\} \in \Xi_\epsilon \text{ and thus } \mathcal{P} > 0.$$

- Conversely, given any operator $\mathcal{P}$ as defined above, we can always search for a matrix $U \geq 0$ such that

$$\{M, K_1, K_2\} \in \Xi_\epsilon \text{ and determine if } \mathcal{P} > 0.$$

- Therefore, the construction and the determination of positivity of such operators depend on the existence of positive semidefinite matrices subject to linear constraints.
  - Can use SDP to accomplish these tasks.

Lyapunov Function for the Observer Dynamics

Let us define $\mathcal{P}$ as

$$(\mathcal{P}z)(x) = M(x)z(x) + \int_0^x K_1(x,\xi)z(\xi)d\xi + \int_x^1 K_2(x,\xi)z(\xi)d\xi.$$

We know how to enforce

$$\{M, K_1, K_2\} \in \Xi_\epsilon.$$

Therefore

$$\mathcal{P} \geq \epsilon \mathcal{I}.$$

Consecutively

$$V_o(\hat{w}) = \left\langle \hat{w}(t,x), \mathcal{P}^{-1}\hat{w}(t,x) \right\rangle > 0.$$

Lyapunov Function for the Observer Dynamics

Now let us define

$$u(t) = \mathcal{F}\hat{w}(t,x) = \mathcal{Y}\mathcal{P}^{-1}\hat{w}(t,x),$$

where, we define

$$\mathcal{Y}z(x) = Y_1 z_x(1) + \int_0^1 Y_2(x)z(x)dx,$$

for scalar Y_1 and polynomial $Y_2(x)$.

- The choice of the structure of $\mathcal{Y}$ reveals itself since it is chosen to cancel out all terms at boundary $x = 1$.

Applying integration by parts and using integral inequalities

$$\frac{d}{dt}V_o(\hat{w}) \leq \left\langle \left(\mathcal{P}^{-1}\hat{w}(t,\cdot)\right), \mathcal{T}\left(\mathcal{P}^{-1}\hat{w}(t,\cdot)\right)\right\rangle + \text{boundary terms},$$

where

$$\left(\mathcal{T}z\right)(x) = T_0(x)z(x) + \int_0^x T_1(x,\xi)z(\xi)d\xi + \int_x^1 T_2(x,\xi)z(\xi)d\xi,$$

where T_0, T_1 and T_2 are linear functions in M, K_1, K_2, Y_1 and Y_2.

- Observation: Both $\mathcal{P}$ and $\mathcal{T}$ have the similar structure.
  - We can enforce positivity/negativity.

Lyapunov Function for the Observer Dynamics

The problem of constructing a controller and a Lyapunov function for the observer is reduced to:

Given a scalar $\epsilon > 0$, find scalar Y_1 and polynomials $M(x)$, $K_1(x, \xi)$, $K_2(x, \xi)$ and $Y_2(x)$ such that

$$\{M, K_1, K_2\} \in \Xi_\epsilon \quad \text{Can be solved by SDP!},$$
$$\{-T_0, -T_1, -T_2\} \in \Xi_\epsilon \quad \text{Can be solved by SDP!},$$

where T_0, T_1 and T_2 are linear functions in M, K_1, K_2, Y_1 and Y_2.

Lyapunov Function for the Error Dynamics

Let us define $\mathcal{S}$ as

$$\left(\mathcal{S}z\right)(x) = N(x)z(x) + \int_0^x P_1(x,\xi)z(\xi)d\xi + \int_x^1 P_2(x,\xi)z(\xi)d\xi.$$

We know how to enforce

$$\{N, P_1, P_2\} \in \Xi_\epsilon.$$

Therefore

$$\mathcal{S} \geq \epsilon\mathcal{I}.$$

Consecutively

$$V_e(e) = \langle e(t,\cdot), \mathcal{S}e(t,\cdot)\rangle > 0.$$

Lyapunov Function for the Error Dynamics

Now let us define the observer gain $O_1(x)$ as

$$O_1(x) = \left(\mathcal{S}^{-1}R\right)(x),$$

for some polynomial R.

Applying integration by parts and using integral inequalities

$$\frac{d}{dt}V_e(e) \leq \langle e(t,\cdot), \mathcal{Q}e(t,\cdot)\rangle - 2\left\langle f(t,\cdot), \mathcal{S}e(t,\cdot)\right\rangle + \text{boundary terms},$$

where

$$\left(\mathcal{Q}z\right)(x) = Q_0(x)z(x) + \int_0^x Q_1(x,\xi)z(\xi)d\xi + \int_x^1 Q_2(x,\xi)z(\xi)d\xi,$$

where Q_0, Q_1 and Q_2 are linear functions in N, P_1, P_2, R and O_2.

- Observation: Both $\mathcal{S}$ and $\mathcal{Q}$ have the similar structure.
  - We can enforce positivity/negativity.

Lyapunov Function for the Error Dynamics

The problem of constructing the observer gains and a Lyapunov function for the error dynamics is reduced to:

Given a scalar $\epsilon > 0$, find scalar O_2 and polynomials $N(x)$, $P_1(x,\xi)$, $P_2(x,\xi)$ and $R(x)$ such that

$$\{N, P_1, P_2\} \in \Xi_\epsilon \quad \text{Can be solved by SDP!},$$
$$\{-Q_0, -Q_1, -Q_2\} \in \Xi_\epsilon \quad \text{Can be solved by SDP!},$$

where Q_0, Q_1 and Q_2 are linear functions in N, P_1, P_2, R and O_2.

Piecing it Together

We are performing the following search which can be cast as a SDP feasibility problem:

Given a scalar $\epsilon > 0$, find scalars Y_1 and O_2 and polynomials $M(x)$, $N(x)$, $P_1(x,\xi)$, $P_2(x,\xi)$, $K_1(x,\xi)$, $K_2(x,\xi)$, $Y_2(x)$ and $R(x)$ such that

$$\begin{aligned}
\{M, K_1, K_2\} \in \Xi_\epsilon &\Rightarrow \mathcal{P} > 0, \\
\{N, P_1, P_2\} \in \Xi_\epsilon &\Rightarrow \mathcal{S} > 0, \\
\{-T_0, -T_1, -T_2\} \in \Xi_\epsilon &\Rightarrow \mathcal{T} < 0, \\
\{-Q_0, -Q_1, -Q_2\} \in \Xi_\epsilon &\Rightarrow \mathcal{Q} < 0.
\end{aligned}$$

- Since $\mathcal{P} > 0$ and $\mathcal{S} > 0$, we have that

$$V_o(\hat{w}) + V_e(e) = \left\langle \hat{w}(t,\cdot), \mathcal{P}^{-1}\hat{w}(t,\cdot)\right\rangle + \langle e(t,\cdot), \mathcal{S}e(t,\cdot)\rangle > 0.$$

Piecing it Together

- Additionally, we have

$$\frac{d}{dt}V_o(\hat{w}) + \frac{d}{dt}V_e(e)$$
$$\leq \left\langle \left(\mathcal{P}^{-1}\hat{w}(t,\cdot)\right), \mathcal{T}\left(\mathcal{P}^{-1}\hat{w}(t,\cdot)\right)\right\rangle + \langle e(t,\cdot), \mathcal{Q}e(t,\cdot)\rangle - 2\langle f(t,\cdot), \mathcal{S}e(t,\cdot)\rangle$$
$$+ \text{boundary terms.}$$

- Since $\mathcal{T} < 0$ and $\mathcal{Q} < 0$, using Schur complements we can show that there exists a bounded $\gamma > 0$ such that

$$\frac{d}{dt}V_o(\hat{w}) + \frac{d}{dt}V_e(e) + \|\hat{w}(t,\cdot)\|^2 + \|e(t,\cdot)\|^2 \leq \frac{\gamma^2}{4}\|f(t,\cdot)\|^2.$$

- Integrating in time, using the positivity of $V_o + V_e$ and the fact that $\|w\| \leq \|\hat{w}\| + \|e\|$ we obtain that

$$\|w\|_{L_2(0,\infty;L_2(0,1))} \leq \gamma\|f\|_{L_2(0,\infty;L_2(0,1))}.$$

Feasibility Problem

Conclusion: Controller synthesis is reduced to the following search, which can be cast as an SDP feasibility problem.

Given a scalar $\epsilon > 0$, find scalars Y_1 and O_2 and polynomials $M(x)$, $N(x)$, $P_1(x,\xi)$, $P_2(x,\xi)$, $K_1(x,\xi)$, $K_2(x,\xi)$, $Y_2(x)$ and $R(x)$ such that

$$\begin{aligned}
\{M, K_1, K_2\} &\in \Xi_\epsilon, \\
\{N, P_1, P_2\} &\in \Xi_\epsilon, \\
\{-T_0, -T_1, -T_2\} &\in \Xi_\epsilon, \\
\{-Q_0, -Q_1, -Q_2\} &\in \Xi_\epsilon.
\end{aligned}$$

- M, K_1, K_2, Y_1 and Y_2 define the controller gains.
- N, P_1, P_2, O_2 and R define the observer gains.

Numerical Results

We consider the following two systems

$$
\begin{aligned}
w_t(t,x) =& w_{xx}(t,x) + \lambda w(t,x) + f(t,x), \text{ and} \quad (1)\\
w_t(t,x) =& \left(x^3 - x^2 + 2\right) w_{xx}(t,x) + \left(3x^2 - 2x\right) w_x(t,x) \\
&+ \left(-0.5x^3 + 1.3x^2 - 1.5x + 0.7 + \lambda\right) w(t,x) + f(t,x), \quad (2)
\end{aligned}
$$

where λ is a scalar which may be chosen freely.

With the following boundary conditions

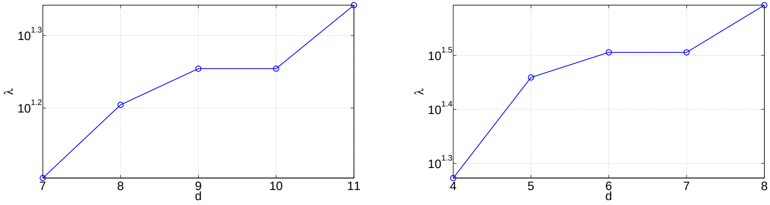
$$w_x(t,0) = 0, \quad w(1,t) = u(t).$$

Increasing λ shifts the eigenvalues of the differential operator to the right in the complex plane. Therefore the systems become unstable for larger values of λ.

TASK: Determine the maximum $\lambda > 0$ such that we can construct output feedback controllers.

- Ideal scenario would be an ability to synthesize boundary controllers for any arbitrary $\lambda \in \mathbb{R}$.

Numerical Results



(a) PDE 1 (b) PDE 2

Maximum λ as a function of polynomial degree d for which we can construct a controller.

- Increasing the degree d means that we are searching over the larger set of polynomials of a higher degree.
- Searching over a set of positive semidefinite matrices of larger dimensions.

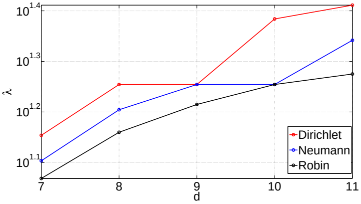
Numerical Results: Alternative Boundary Conditions

The methodology presented can be changed to accommodate other types of boundary conditions.

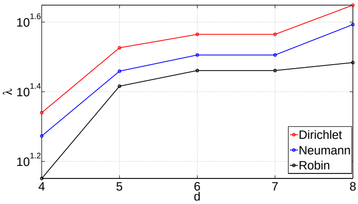
We now present the numerical results for the example PDEs with the following boundary conditions and outputs:

	Boundary Condition	Output $y(t)$
Dirichlet	$w(0,t)=0$ $w(1,t)=u(t)$	$w_x(1,t)$
Neumann	$w_x(0,t)=0$ $w_x(1,t)=u(t)$	$w(1,t)$
Robin	$w(0,t)+w_x(0,t)=0$ $w(1,t)+w_x(1,t)=u(t)$	$w(1,t)$

Numerical Results



(a) PDE 1



(b) PDE 2

Maximum λ as a function of polynomial degree d for which we can construct a controller.

Numerical Results

Conclusions from these numerical results:

- Increasing the degree d leads to synthesis of boundary controllers for higher values of parameter λ.
- These numerical results indicate that the method is asymptotically accurate.
  - Given any controllable system, we can synthesize boundary controllers for a large enough d.
- Size of the underlying SDP problem scales as $O(d^2)$.
  - Increasing the degree d leads to increased memory requirements.

Numerical Simulation

We provide numerical simulation for the following system

$$\begin{aligned} w_t(t,x) = {} & \left(x^3 - x^2 + 2\right) w_{xx}(t,x) + \left(3x^2 - 2x\right) w_x(t,x) \\ & + \left(-0.5x^3 + 1.3x^2 - 1.5x + 0.7 + 39\right) w(t,x), \end{aligned}$$

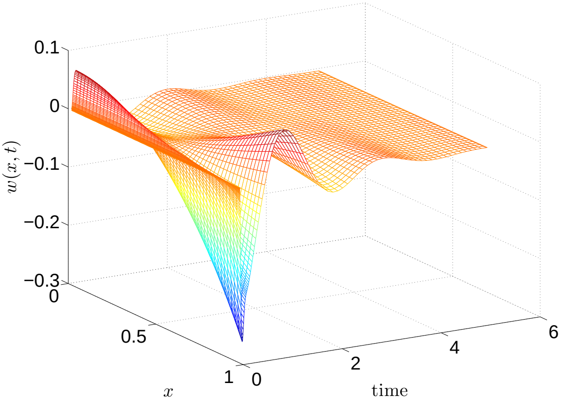
coupled to the following boundary conditions and output

$$w_x(t,0) = 0, \quad w(t,1) = u(t), \quad y(t) = w_x(t,1).$$

The system is initiated by a zero initial condition and we choose

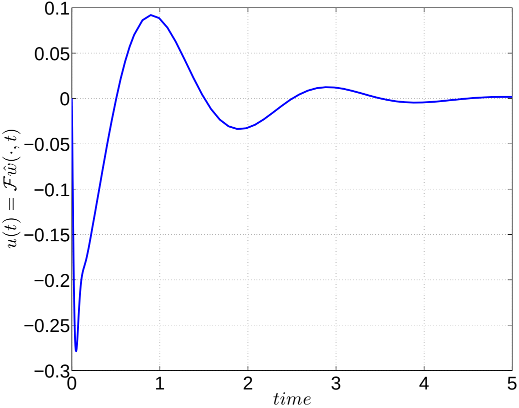
$$f(t,x) = e^{-t}\cos(\pi t)\left(1 + \sin(0.1\pi x)\right).$$

Numerical Simulation



Controlled state evolution.

Numerical Simulation



Control effort.

Conclusions

- We presented a computationally tractable numerical method, using operators parametrized by PSD matrices and convex optimization, for boundary controller synthesis using boundary measurement.
- The method is asymptotically accurate.
- Algorithm performance depends on the degree of polynomial representation.
- Memory requirements enforce a constraint on the method.
- Method can be tweaked to consider Mixed, Neumann, Dirichlet or Robin boundary conditions.
- Numerical experiments performed using SOSTOOLS and SeDuMi.

Future Work

Final Goal

- Extend current work to a numerical method for synthesis of optimal output feedback controller synthesis for parabolic PDEs.
  - Analogous to the Kalman-Yakubovich-Popov (KYP) lemma based approach for ODEs.
- This would involve the following steps:
  - Development of optimal state-feedback based controller synthesis. (Under review for ACC 2016)
  - Development of optimal output-feedback based controller synthesis using a Luenberger observer. (Ongoing)
  - Removal of the Luenberger structure constraint on the observer.

Other Directions

- Application to uncertain (non) linear systems.
- Boundary controller synthesis for hyperbolic PDEs (wave and beam equations).
- Observer synthesis for poloidal magnetix flux in Tokamaks.