Optimal State Feedback Boundary Control of Parabolic PDEs Using SOS Polynomials

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***Abstract*— In this paper we synthesize $\mathcal{L}(L_2)$-optimal full-state feedback controllers for a one dimensional linear PDE with point actuation and distributed disturbances. We use Sum-of-Squares (SOS) polynomials and Semi-Definite Programming (SDP) to parametrize positive operators which define quadratic Lyapunov functions and the controller gains. Additionally, we calculate the upper bound on the system state ensured by the calculated controllers. Moreover, we provide numerical results, but not proofs, for PDEs with additional types of boundary conditions.**

I. INTRODUCTION

Processes in which a physical quantity undergoes diffusion, convection and reaction are modeled by parabolic Partial Differential Equations (PDEs). For e.g., plasma in thermonuclear fusion [21], chemical reactors [2] and heat propagation in media. In this paper we consider a one dimensional inhomogeneous linear parabolic PDE with polynomial, spatially distributed coefficients. Such PDEs are used to model diffusion, convection and reaction in anisotropic media while being subjected to unknown, but bounded, exogenous disturbance due to, e.g. unmodeled nonlinear dynamics [12]. The goal of this article is to construct full-state feedback controllers, with control via point actuation at the boundary which minimizes the effect of disturbances on an output which is some function of the distributed state. The controller is optimized in the metric of the induced $\mathcal{L}(L_2)$ gain from external disturbance to regulated output. The performance analysis and controller design is performed entirely in the infinite-dimensional framework, with no discretization or model reduction at any step.

There is significant existing work on the stability and control of PDE systems, although most of this work ultimately relies on a reducation to a finite-dimensional state space or discretized solution of an operator-valued equation or inequality. There are however, some works which consider the problem in its native infinite-dimensional setting. For example, work on the use of Lyapunov functions for the analysis and control of PDEs can be found in [3], [5], [4]. Additionally, the use of optimization algorithms for boundary control of PDEs using LMIs and a Lyapunov approach may be found in [7]. Prior work on the use of SOS polynomials to PDEs and other infinite dimensional systems may be found in [14], [17] and [20].

Our approach is based on the construction of quadratic Lyapunov functions parameterized by positive operators on L_2. Such positive operators are in turn parametrized by Sum-of-Squares (SOS) polynomials. Moreover, controller gains are parameterized by polynomials as well. This polynomial parametrization of quadratic Lyapunov functions and controller gains allows us to search for the desired polynomials algorithmically. Operator positivity constraints are represented using matrix positivity constraints in an approach inspired the the SOS approximation to the cone of positive polynomials [15]. The proposed algorithm is an infinite-dimensional extension of H_∞-optimal controller synthesis for systems governed by Ordinary Differential Equations (ODEs). For ODEs, the H_∞-norm of a system can be represented as an LMI constraint by means of the Kalman-Yakubovich-Popov (KYP) lemma [6] which says that a system with H_∞ norm of γ implies the existence of a quadratic Lyapunov function a bound on the induced norm of γ. We replicate this approach by searching for a controller and quadratic Lyapunov function which minimizes the induced $\mathcal{L}(L_2)$ of the closed-loop system.

This article extends our work in [8] wherein we designed point-actuation point-observation output feedback controllers for exponential stabilization using the Luenberger framework. In addition, in [9] we showed that such output feedback controllers ensure that the state of the PDE remains bounded in the presence of a bounded, uniformly distributed exogenous input. The contribution of this article is that while we simplify the problem by considering state-feedback, we also improve over existing results by enabling the algorithm to *optimize* the closed-loop upper bound on the system. The ultimate goal of this line of research is to then extend the optimization step to design optimal dynamic output feedback controllers which optimize over the controller and the estimator.

The quadratic Lyapunov functionals we consider are parameterized as $V = \langle \phi, \mathcal{P}\phi \rangle_{L_2}$ where the operator $\mathcal{P}$ is parameterized by polynomial multiplier M and polynomial semi-separable kernels N_1 and N_2 as

$$\begin{aligned}(\mathcal{P}\phi)(x) =& M(x)\phi(x) + \int_0^x N_1(x,\xi)\phi(\xi)d\xi \\ &+ \int_x^1 N_2(x,\xi)\phi(\xi)dx, \quad \phi \in L_2(0,1).\end{aligned}$$

The polynomials M, N_1, N_2 are parameterized by positive matrices using a Sum-of-Squares type condition which ensures positivity of the operator. Likewise, the feedback controller is parameterized as $u = \mathcal{K}w$ where $w \in L_2$ is the distributed state and the operator $\mathcal{K}$ is parameterized by the scalar F_1 and the polynomial F_2 as

$$u(t) = \mathcal{K}w := F_1 w(t,1) + \int_0^1 F_2(x) w(t,x)dx.$$

The polynomial F_2 is parameterized by its vector of monomial coefficients.

This article is organized as follows: Section III gives the problem statement and some background material. Section IV gives an SDP parametrization of positive operators. Section V presents the state-feedback optimization problem as an SDP. Finally, Section VI describes a series of numerical tests, for two example PDEs, numerical justification for the use of the semi-separable kernels and extensions to different types of boundary conditions.

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II. NOTATION

We denote by $\mathbb{S}^n$ the set of n-by-n symmetric real matrices. We denote by I_n the identity matrix of dimensions n-by-n and $I = I_n$ when n is clear from context. We denote by $C^\infty(0,1)$ the set of real valued infinitely differentiable functions on $[0,1]$. $C^{1,2}((0,\infty),[0,1])$ denotes the set of bivariate real valued functions continuously differentiable on $(0,\infty)$ and twice continuously differentiable on $[0,1]$. The shorthand u_x denotes the partial derivative of u with respect to x. $L_2(0,1)$ denotes the Hilbert space of real valued square integrable functions endowed with norm $\|f\|_{L_2} = \sqrt{\int_0^1 \|f(x)\|^2 dx}$ and inner product $\langle f,g\rangle_{L_2} = \int_0^1 f(x)^T g(x)dx$. For convenience, given $f,g \in L_2(0,1)$, we say that $g = f(\cdot)g(\cdot)$ if $g(x) = f(x)g(x)$.

$L_2(0,\infty;L_2(0,1))$ denotes the set of functions $f(t,x)$ such that for any $t \in [0,\infty)$, if $g(x) = f(t,x)$, then $g \in L_2(0,1)$. For $f \in L_2(0,\infty;L_2(0,1))$, we denote by $f(t,\cdot)$ the map from $[0,\infty) \to L_2(0,1)$ defined by $f(t,\cdot)(x) = f(t,x)$. We equip $L_2(0,\infty;L_2(0,1))$ with the norm $\|f\|_{L_2(0,\infty;L_2(0,1))} := \sqrt{\int_0^\infty \|f(t,\cdot)\|_{L_2}^2 dt}$.

$H^n(0,1) := \{f \in L_2 : \frac{\partial^i}{\partial x^i} f \in L_2,\ i = 1,\cdots,n\}$ denotes the Sobolev subspace with norm $\|f\|_{H^n} = \sum_{i=1}^n \|\frac{\partial^i}{\partial x^i} f\|_{L_2}$. $\mathcal{I}$ denotes the identity operator on any Hilbert space. We define $Z_d(x)$ to be the column vector of all monomials in variable x of degree d or less. Similarly, we define $Z_d(x,\xi)$ to be a column vector of all monomials in variables x and ξ of degree d or less. We define $Z_{n,d}(x,\xi) = I_n \otimes Z_d(x,\xi)$.

III. PROBLEM STATEMENT

We consider the following class of possibly unsteady, one-dimensional inhomogeneous linear parabolic PDEs

$$\begin{aligned} w_t(t,x) =& a(x)w_{xx}(t,x) + b(x)w_x(t,x) + c(x)w(t,x) \\ &+ r(x)d(t,x), \end{aligned} \tag{1}$$

for $x \in [0,1]$ and $t \geq 0$, with initial condition $w(x,0) = 0$ and boundary conditions of the form

$$w(t,0) = 0, \quad w_x(t,1) = u(t). \tag{2}$$

The coefficients a, b, c and r are polynomials in x with $a(x) \geq \alpha > 0$, for all $x \in [0,1]$. The function $d \in L_2(0,\infty;L_2(0,1))$ is the spatially distributed *exogenous input* or disturbance. $u(t) \in \mathbb{R}$ is the *control input* which is to be determined by state feedback. We define our *regulated output* as

$$y(t,x) = s(x)w(t,x), \tag{3}$$

which is the quantity to be minimized and where s is a given polynomial. In addition to the boundary conditions as defined in (2), the problem can be generalized to consider Dirichlet, Neumann and Robin boundary conditions, as is discussed in Section VI.

We define our feedback controller to have the form $u(t) = \mathcal{K}w(t)$ where $\mathcal{K}$ is defined as

$$u(t) = \mathcal{K}w(t,\cdot) = F_1 w(t,1) + \int_0^1 F_2(x)w(t,x)dx, \tag{4}$$

for some to be determined scalar F_1 and polynomial $F_2 \in L_\infty(0,1)$. We consider optimal control in the $\mathcal{L}(L_2)$ norm. That is, we seek to minimize over $\mathcal{K}$ and the bound γ such that if $u(t) = \mathcal{K}w(t)$ and $y(t,x) = s(x)w(t,x)$, then for any $d \in L_2(0,\infty : L_2(0,1))$, $y \in L_2(0,\infty : L_2(0,1))$ and

$$\|y\|_{L_2(0,\infty:L_2(0,1))} \leq \gamma \|d\|_{L_2(0,\infty:L_2(0,1))}.$$

For any such pair $\mathcal{K}$ and γ, we term γ the $\mathcal{L}(L_2)$ gain and the associated controller the γ-*optimal controller*. Note that for a given controller, we obtained a bound on γ in [9]. The contribution of this paper is to represent this bound in such a way that it can be *optimized*.

Finally, we note that for the PDE (1)-(2) under feedback of the form in Equation (4) we assume that for any $d \in L_2(0,\infty : L_2(0,1))$, there exists a unique solution $w \in C^{1,2}((0,\infty),[0,1])$. Relevant conditions for existence of solutions may be found in [1, Section 6].

IV. POSITIVE OPERATORS ON $L_2(0,1)$

In this section, we show how positive matrices can be used to parameterize positive operators. Specifically, we consider operators of the form

$$\begin{aligned} (\mathcal{P}f)(x) =& M(x)f(x) + \int_0^x N_1(x,\xi)f(\xi)d\xi \\ &+ \int_x^1 N_2(x,\xi)f(\xi)d\xi, \end{aligned} \tag{5}$$

where $M : [0,1] \to \mathbb{S}^n$ and $N_1, N_2 : [0,1]\times[0,1] \to \mathbb{R}^{n\times n}$ are polynomials. The following theorem shows that any positive operator with a square root belonging to this same class of operators can be represented by a positive matrix.

Theorem 1: Suppose there exists some symmetric $U \geq 0$ such that

$$U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix} \geq \begin{bmatrix} \epsilon I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6}$$

Then if M, N_1 and N_2 are defined as

$$M(x) = Z_1(x)^T U_{11} Z_1(x), \tag{7}$$

$$N_1(x,\xi) = Z_1(x)^T U_{12} Z_2(x,\xi) + Z_2(\xi,x)^T U_{31} Z_1(\xi) \tag{8}$$

$$+ \int_0^\xi Z_2(\eta,x)^T U_{33} Z_2(\eta,\xi)d\eta + \int_\xi^x Z_2(\eta,x)^T U_{32} Z_2(\eta,\xi)d\eta \tag{9}$$

$$+ \int_x^1 Z_2(\eta,x)^T U_{22} Z_2(\eta,\xi)d\eta, \tag{10}$$

where $Z_1(x) = Z_{n,d_1}(x)$, $Z_2(x,y) = Z_{n,d_2}(x,y)$ and $N_2(x,\xi) = N_1(\xi,x)^T$, then the operator $\mathcal{P}$ as defined in Eqn. (5) is bounded, self-adjoint and satisfies

$$\langle \mathcal{P}f, f\rangle_{L_2(0,1)} \geq \epsilon \|f\|_{L_2(0,1)}^2,$$

for all $f \in L_2(0,1)$.

Proof: The proof is based on the result in [16] and is omitted for brevity. ■

For convenience, we define the following set

$$\begin{aligned} \Xi_{n,d_1,d_2,\epsilon} =& \{M, N_1, N_2 : M, N_1, N_2 \text{ satisfy Theorem 1} \\ & \text{for } d_1, d_2, n \in N,\ \epsilon > 0 \text{ and some } U \geq 0\}. \end{aligned}$$

Theorem 1 allows us to use $M, N_1, N_2 \in \Xi_{n,d_1,d_2,\epsilon}$ to enforce the constraint that the operator $\mathcal{P}$ defined by M, N_1, and N_2 as in Eqn.(5), be positive. By expanding Eqns. (7), the coefficients of the polynomials M, K_1 and K_2 are linear combinations of the elements of the matrix variable U. Constructing the matrices which relate the elements of U to the coefficients of the polynomials M, N_1, and N_2 can be automated using Matlab toolboxes for polynomial manipulation such as MULTIPOLY, contained in the package SOSTOOLS [18] and further developed in our package DELAYTOOLS [16].

Finally, we note that, as will be seen in the following Section, given a positive operator $\mathcal{P}$ parameterized by $M, N_1, N_2 \in \Xi_{n,d_1,d_2,\epsilon}$, it will be necessary to construct the inverse of $\mathcal{P}$. Naturally, because the operator is positive, such an inverse exists and, moreover, it turns out that such an inverse is of the same form as $\mathcal{P}$ (Although M, N_1, and N_2 may not be polynomial). Furthermore, this inverse can be constructed as described in [10] and expanded in [8].

V. CONTROLLER SYNTHESIS

In this section, we define an SDP for synthesis of a controller which minimizes a bound on the $\mathcal{L}(L_2)$ norm of the closed loop PDE defined in Eqn. (1). For convenience, we here restate the dynamics.

$$\begin{aligned} w_t(t,x) =& a(x)w_{xx}(t,x) + b(x)w_x(t,x) + c(x)w(t,x) \\ &+ r(x)d(t,x), \end{aligned} \tag{11}$$

with $w(x,0) = 0$ and boundary conditions $w(t,0) = 0$ and $w_x(t,1) = u(t)$ where recall $a(x) \geq \alpha > 0$ and the regulated output is $y(t,x) = s(x)w(t,x)$.

Theorem 2: For given scalar $\epsilon > 0$ and $d_1, d_2 \in \mathbb{N}$, suppose there exist scalars Y_1 and $\gamma > 0$ and polynomials $M, Y_2 : [0,1] \to \mathbb{R}$, $N_1, N_2 : [0,1] \times [0,1] \to \mathbb{R}$ such that

$$\begin{aligned} \{M, N_1, N_2\} &\in \Xi_{1,d_1,d_2,\epsilon}, \\ \{-Q_0, -Q_1, -Q_2\} &\in \Xi_{3,d_1,d_2,0}, \\ N_2(0,x) &= 0, \\ 2a(1)Y_1 - a(1)M_x(1) + (b(1) - a_x(1))M(1) &= 0, \\ Y_2(x) - N_{1,x}(1,x) &= 0, \end{aligned}$$

where

$$\begin{aligned} Q_0(x) &= \begin{bmatrix} T_0(x) & r(x) & s(x)M(x) \\ r(x) & -\gamma & 0 \\ s(x)M(x) & 0 & -\gamma \end{bmatrix}, \\ Q_1(x,\xi) &= \begin{bmatrix} T_1(x,\xi) & 0 & N_1(x,\xi)s(\xi) \\ 0 & 0 & 0 \\ N_1(x,\xi)s(x) & 0 & 0 \end{bmatrix}, \\ Q_2(x,\xi) =& Q_1(\xi,x)^T, \\ T_0(x) =& \left(a_{xx}(x) - b_x(x)\right) M(x) + b(x)M_x(x) \\ &+ a(x)M_{xx}(x) + 2c(x)M(x) - \frac{\pi^2}{2}\alpha\epsilon \\ &+ a(x)\left[2\frac{\partial}{\partial x}\left[N_1(x,\xi) - N_2(x,\xi)\right]\right]_{\xi=x}, \\ T_1(x,\xi) =& a(x)N_{1,xx}(x,\xi) + b(x)N_{1,x}(x,\xi) \\ &+ a(\xi)N_{1,\xi\xi}(x,\xi) + b(\xi)N_{1,\xi}(x,\xi) \\ &+ \left(c(x) + c(\xi)\right)N_1(x,\xi), \\ T_2(x,\xi) =& T_1(\xi,x). \end{aligned}$$

Then if

$$\begin{aligned} u(t) =& Y_1\left(\mathcal{P}^{-1}w\right)(t,1) + \int_0^1 Y_2(x)\left(\mathcal{P}^{-1}w\right)(t,x)dx, \\ =& F_1 w(t,1) + \int_0^1 F_2(x)w(t,x)dx, \end{aligned} \tag{12}$$

where the operator $\mathcal{P}^{-1}$ is the inverse of the operator $\mathcal{P}$ as defined in Eqn. (5), we have that for any $d \in L_2(0,\infty; L_2(0,1))$, Eqn. (11) implies that

$$\|y\|_{L_2(0,\infty;L_2(0,1))} \leq \gamma \|d\|_{L_2(0,\infty;L_2(0,1))}.$$

Proof: Let $\mathcal{P}$ be as defined in Eqn. (5). Then the inverse of $\mathcal{P}$ as defined in [10], [8] is positive, self-adjoint and has the following form.

$$\begin{aligned} \left(\mathcal{P}^{-1}f\right)(x) =& \hat{M}(x)f(x) + \int_0^x \hat{N}_1(x,\xi)f(\xi)d\xi \\ &+ \int_x^1 \hat{N}_2(x,\xi)f(\xi)d\xi, \end{aligned}$$

where $\hat{M}, \hat{N}_1, \hat{N}_2 \in C^\infty[0,1]$. Now let

$$u(t) = Y_1\left(\mathcal{P}^{-1}w\right)(t,1) + \int_0^1 Y_2(x)\left(\mathcal{P}^{-1}w\right)(t,x)dx.$$

Now, since $\mathcal{P}^{-1}$ is self-adjoint, we have that

$$\int_0^1 Y_2(x)\left(\mathcal{P}^{-1}w\right)(t,x)dx = \int_0^1 \left(\mathcal{P}^{-1}Y_2\right)(x)w(t,x)dx.$$

Hence,

$$\begin{aligned} \int_0^1 Y_2(x)\left(\mathcal{P}^{-1}w\right)(t,x)dx =& \int_0^1 \Bigg(\hat{M}(x)Y_2(x) \\ &+ \int_0^x \hat{N}_1(x,\xi)Y_2(\xi)d\xi + \int_x^1 \hat{N}_2(x,\xi)Y_2(\xi)d\xi\Bigg)w(t,x)dx. \end{aligned} \tag{13}$$

Also, since

$$Y_1\left(\mathcal{P}^{-1}w\right)(t,1) = Y_1\left(\hat{M}(1)w(t,1) + \int_0^1 \hat{N}_1(1,x)w(t,x)dx\right),$$

we have that $F_1 = Y_1\hat{M}(1)$ and

$$\begin{aligned} F_2(x) =& Y_1\hat{N}_1(1,x) + \hat{M}(x)Y_2(x) + \int_0^x \hat{N}_1(x,\xi)Y_2(\xi)d\xi \\ &+ \int_x^1 \hat{N}_2(x,\xi)Y_2(\xi)d\xi, \end{aligned}$$

which implies that $F_2 \in C^\infty(0,1)$. Since F_2 is continuous, the closed loop system admits a unique solution $w \in C^{1,2}((0,\infty),[0,1])$.

Now define the Lyapunov function

$$V(w) = \left\langle w, \mathcal{P}^{-1}w\right\rangle. \tag{14}$$

If $w(t,x)$ is a solution of Eqn. (11), then $\dot{V}(w(t,\cdot))$ is given by

$$\frac{dV((t,\cdot))}{dt} = 2\left\langle \mathcal{A}w(t,\cdot), \mathcal{P}^{-1}w(t,\cdot)\right\rangle + 2\left\langle r(\cdot)d(t,\cdot), \mathcal{P}^{-1}w(t,\cdot)\right\rangle, \tag{15}$$

where the differential operator $\mathcal{A} : H^2(0,1) \to L_2(0,1)$ is defined as

$$\mathcal{A} = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x).$$

Let us define $z(t,\cdot) := \mathcal{P}^{-1}w(t,\cdot)$. Then Equation (15) becomes

$$\dot{V}(w(t,\cdot)) = 2\left\langle \mathcal{A}\mathcal{P}z(t,\cdot), z(t,\cdot)\right\rangle + 2\left\langle r(\cdot)d(t,\cdot), z(t,\cdot)\right\rangle.$$

Because of the tedious nature of the calculation, we remove manipulation of the derivative $\dot{V}$ to Lemma 3, which is attached as an Appendix. Applying this lemma, we obtain

$$\begin{aligned} \dot{V}(w(t,\cdot)) \leq& \left\langle \mathcal{T}z(t,\cdot), z(t,\cdot)\right\rangle + 2\left\langle r(\cdot)d(t,\cdot), z(t,\cdot)\right\rangle \\ &+ \left[a(1)M_x(1) + (b(1) - a_x(1))M(1)\right]z(t,1)^2 \\ &+ 2a(1)M(1)z_x(t,1)z(t,1), \end{aligned} \tag{16}$$

where the operator $\mathcal{T} : L_2(0,1) \to L_2(0,1)$ is defined in Lemma 3 as

$$\left(\mathcal{T}f\right)(x) := T_0(x)f(x) + \int_0^x T_1(x,\xi)f(\xi)d\xi + \int_x^1 T_2(x,\xi)f(\xi)d\xi.$$

Since $z = \mathcal{P}^{-1}w$, we have $w = \mathcal{P}z$ which implies

$$w(t,x)=M(x)z(t,x)+\int_0^x N_1(x,\xi)z(t,\xi)d\xi+\int_x^1 N_2(x,\xi)z(t,\xi)dx.$$

Since $N_1(x,\xi) = N_2(\xi,x)$ we have that

$$w_x(t,1)=M_x(1)z(t,1)+M(1)z_x(t,1)+\int_0^1 N_{1,x}(1,x)z(t,x)dx.$$

Rearranging and applying the boundary condition $w_x(t,1) = u(t)$, we have

$$M(1)z_x(t,1) = u(t) - M_x(1)z(t,1) - \int_0^1 N_{1,x}(1,x)z(t,x)dx. \tag{17}$$

Now, from the definition of $u(t)$ in (12), we have that

$$\begin{aligned} u(t) =& F_1 w(t,1) + \int_0^1 F_2(x)w(t,x)dx \\ =& Y_1 \left(\mathcal{P}^{-1}w\right)(t,1) + \int_0^1 Y_2(x)\left(\mathcal{P}^{-1}w\right)(t,x)dx \\ =& Y_1 z(t,1) + \int_0^1 Y_2(x)z(t,x)dx. \end{aligned}$$

Substituting into (17), we obtain

$$\begin{aligned} M(1)z_x(t,1) =& \left(Y_1 - M_x(1)\right)z(t,1) \\ &+ \int_0^1 \left(Y_2(x) - N_{1,x}(1,x)\right)z(t,x)dx. \end{aligned}$$

Substituting for $M(1)z_x(t,1)$ in the last term of Equation (16)

$$\begin{aligned} \dot{V}(w(t,\cdot)) \le& \langle \mathcal{T}z(t,\cdot), z(t,\cdot)\rangle + 2\langle r(\cdot)d(t,\cdot), z(t,\cdot)\rangle \\ &+ \left[2a(1)Y_1 - a(1)M_x(1) + (b(1) - a_x(1))M(1)\right]z(t,1)^2 \\ &+ 2a(1)z(t,1)\int_0^1 \left(Y_2(x) - N_{1,x}(1,x)\right)z(t,x)dx. \end{aligned} \tag{18}$$

Now, from the conditions of the theorem statement, we have

$$\begin{aligned} &2a(1)Y_1 - a(1)M_x(1) + (b(1) - a_x(1))M(1) = 0, \\ &Y_2(x) - N_{1,x}(1,x) = 0, \end{aligned}$$

which implies the last two terms of Eqn. (18) are eliminated, leaving us with

$$\dot{V}(w(t,\cdot)) \le \langle \mathcal{T}z(t,\cdot), z(t,\cdot)\rangle + 2\langle r(\cdot)d(t,\cdot), z(t,\cdot)\rangle. \tag{19}$$

For any $f \in L_2(0,1)$, we define the operators

$$\left(\mathcal{R}f\right)(x) = r(x)f(x), \quad \text{and} \quad \left(\mathcal{S}f\right)(x) = s(x)f(x). \tag{20}$$

Then, by the definition of Q_0, Q_1 and Q_2, we have that for $g \in \mathcal{X} := L_2(0,1)\times L_2(0,1)\times L_2(0,1)$,

$$\begin{aligned} &\left(\begin{bmatrix} \mathcal{T} & \mathcal{R} & (\mathcal{SP})^\star \\ \mathcal{R} & -\gamma & 0 \\ \mathcal{SP} & 0 & -\gamma \end{bmatrix} g\right)(x) \\ &= Q_0(x)g(x) + \int_0^x Q_1(x,\xi)g(\xi)d\xi + \int_x^1 Q_2(x,\xi)g(\xi)d\xi, \end{aligned}$$

and since $\{-Q_0, -Q_1, -Q_2\} \in \Xi_{3,d_1,d_2,0}$, we conclude that

$$\begin{bmatrix} \mathcal{T} & \mathcal{R} & (\mathcal{SP})^\star \\ \mathcal{R} & -\gamma & 0 \\ \mathcal{SP} & 0 & -\gamma \end{bmatrix} \le 0$$

on $\mathcal{X}$. Therefore, using a variation of the Schur complement,

$$\begin{bmatrix} \mathcal{T} & \mathcal{R} \\ \mathcal{R} & -\gamma \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} (\mathcal{SP})^\star \\ 0 \end{bmatrix}\begin{bmatrix} \mathcal{SP} & 0 \end{bmatrix} \le 0,$$

on $\mathcal{Y} := L_2(0,1)\times L_2(0,1)$. Therefore,

$$\left\langle \begin{bmatrix} z(t,\cdot) \\ d(t,\cdot) \end{bmatrix}, \left(\begin{bmatrix} \mathcal{T} & \mathcal{R} \\ \mathcal{R} & -\gamma \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} (\mathcal{SP})^\star \\ 0 \end{bmatrix}\begin{bmatrix} \mathcal{SP} & 0 \end{bmatrix}\right)\begin{bmatrix} z(t,\cdot) \\ d(t,\cdot) \end{bmatrix} \right\rangle_{\mathcal{Y}}$$

is non-positive, and thus

$$\begin{aligned} &\langle \mathcal{T}z(t,\cdot), z(t,\cdot)\rangle + 2\langle r(\cdot)d(t,\cdot), z(t,\cdot)\rangle - \gamma\|d(t,\cdot)\|^2 \\ &\qquad + \frac{1}{\gamma}\langle s(\cdot)\left(\mathcal{P}z\right)(t,\cdot), s(\cdot)\left(\mathcal{P}z\right)(t,\cdot)\rangle \le 0. \end{aligned}$$

Now, since $\mathcal{P}z = w$ and $s(x)w(t,x) = y(t,x)$, we apply Eqn. (19) to get

$$\dot{V}(w(t,\cdot)) + \frac{1}{\gamma}\|y(t,\cdot)\|^2 \le \gamma\|d(t,\cdot)\|^2.$$

Integrating in time, for any $T > 0$, we obtain

$$V(w(T,\cdot)) - V(w(0,\cdot)) + \frac{1}{\gamma}\int_0^T \|y(t,\cdot)\|^2 dt \le \gamma\int_0^T \|d(t,\cdot)\|^2 dt.$$

Finally, since $\mathcal{P}^{-1} \ge 0$, $V(w(T,\cdot)) \ge 0$. Furthermore, since $w(0,x) = 0$, $V(w(0,\cdot)) = 0$. Therefore, we obtain

$$\frac{1}{\gamma}\int_0^T \|y(t,\cdot)\|^2 dt \le \gamma\int_0^T \|d(t,\cdot)\|^2 dt,$$

or

$$\int_0^T \|y(t,\cdot)\|^2 dt \le \gamma^2\int_0^T \|d(t,\cdot)\|^2 dt,$$

Since this holds for any $T > 0$, the proof is complete. ∎

The conditions of Theorem 2 are affine the SDP variables, which consist of the positive matrix U, which defines the polynomials M, N_1, and N_2 and Y_2, as well as the scalar Y_1 and coefficients of polynomial Y_2. Given a feasible solution to Theorem 2, the controller can then be found by constructing the polynomials $\hat{M}$, $\hat{N}_1$ and $\hat{N}_2$ which define the operator $\mathcal{P}^{-1}$ as described in [10], [8]. The controller gains are then recovered as described in the proof as $F_1 = Y_1\hat{M}(1)$ and

$$\begin{aligned} F_2(x) =& Y_1\hat{N}_1(1,x) + \hat{M}(x)Y_2(x) + \int_0^x \hat{N}_1(x,\xi)Y_2(\xi)d\xi \\ &+ \int_x^1 \hat{N}_2(x,\xi)Y_2(\xi)d\xi. \end{aligned}$$

VI. NUMERICAL RESULTS

In this section we test the algorithms defined by Theorem 2 on two arbitrary unstable PDEs. SOSTOOLS [18] was used to assist in the the conversion of the polynomial constraints to an SDP, as described in Section IV. SeDuMi [19] was used to solve the resulting SDP problem.

The first example problem is defined as

$$w_t(t,x) = w_{xx}(t,x) + \left(\frac{\pi^2}{4} + 0.034\right)w(t,x) + (x^3 - 1)d(t,x), \tag{21}$$

$$y(t,x) = (x^2 - 0.5x + 1)w(t,x), \tag{22}$$

The second example problem is defined as

$$\begin{aligned} w_t(t,x) =& (x^3 - x^2 + 2)w_{xx}(t,x) + (3x^2 - 2x)w_x(t,x) \\ &+ (-0.5x^3 + 1.3x^2 - 1.5x + 5.42)w(t,x) + xd(t,x), \end{aligned} \tag{23}$$

$$y(t,x) = (1 + x)w(t,x). \tag{24}$$

Example 1			
degree (d)	3	4	5
minimum $\mathcal{L}(L_2)$ gain	> 50	5.2906	1.3471

TABLE I: Min. achievable closed loop $\mathcal{L}(L_2)$ gain (γ) as a fn. of $d_1 = d_2 = d$ using the algorithm defined by Thm. 2 as applied to Example 1.

Example 2			
degree (d)	4	5	6
minimum $\mathcal{L}(L_2)$ gain	0.42	0.1459	0.1136

TABLE II: Min. achievable closed loop $\mathcal{L}(L_2)$ gain (γ) as a fn. of $d_1 = d_2 = d$ using the algorithm defined by Thm. 2 as applied to Example 2.

The boundary conditions for both examples are given by

$$w(t, 0) = 0, \quad w_x(t, 1) = u(t). \tag{25}$$

The coefficients of these two examples are arbitrary, but are chosen so that when $d(t, x) = 0$ and $u(t) = 0$, the dynamics are unstable. We verified autonomous instability analytically for Example 1 (Eqn. (21)) and by numerical simulation for Example 2 (Eqn. (23)).

For both cases, we seek a controller which minimizes the bound on the $\mathcal{L}(L_2)$ gain, γ, as defined in Thm. 2. We test the algorithm for several degrees $d_1 = d_2 = d$, as defined in Thm. 2. As the degrees increase, the computational complexity of the algorithm increases while the results become increasingly accurate. All numerical experiments for $\epsilon = 0.001$ as defined in Thm. 2. Tables I and II illustrate the minimum bound γ achieved for examples 1 and 2, repsectively, as a function of degree, d.

Tables I and II indicate that increasing d improves the performance of the controller. However, the number of decision variables in the underlying SDP scales as $O(d^2)$ meaning that testing the algorithm for higher values of d requires more computational resources and specifically, more RAM (experiments were performed using 8GB RAM).

We note that the inclusion of semi-separable kernels N_1 and N_2 in Thm. 2 complicates the analysis and increases the computational complexity. To test the significance of the semi-separable kernels, we tested the conditions of Theorem 2 on PDEs (21) and (23) with the additional constraint $N_1 = N_2 = 0$. Tables III and IV present these results. Comparing Tables III-IV with Tables I-II we observe that the inclusion of the kernels N_1 and N_2 leads to synthesis of controllers with significantly improved performance.

A. Numerical Simulation

To verify the performance bound derived in Thm. 2, we synthesized a controller for Example 2 defined above in Eqn. (23) which achieved a performance bound of $\gamma = .1136$ with $d_1 = d_2 = 6$.

Example 1 w/o SS kernels			
degree (d)	4	5	$6 \cdots 10$
minimum $\mathcal{L}(L_2)$ gain	13.777	13.002	13.002

TABLE III: Min. achievable closed loop $\mathcal{L}(L_2)$ gain (γ) as a fn. of $d_1 = d_2 = d$ using the algorithm defined by Thm. 2 with additional constraint with $N_1 = N_2 = 0$ as applied to Example 1.

Example 2 w/o SS kernels			
degree (d)	$d = 4$	5	$6 \cdots 10$
minimum $\mathcal{L}(L_2)$ gain	7.793	6.943	6.855

TABLE IV: Min. achievable closed loop $\mathcal{L}(L_2)$ gain (γ) as a fn. of $d_1 = d_2 = d$ using the algorithm defined by Thm. 2 with additional constraint with $N_1 = N_2 = 0$ as applied to Example 2.

We then used a disturbance defined as

$$d(t, x) = 100 \operatorname{sinc}(t)(1 + x),$$

where

$$\operatorname{sinc}(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}$$

As mentioned, the autonomous system ($u(t) = 0$) is unsteady, as illustrated in Fig. 1. Figure 2 illustrates the same PDE in closed loop using a controller with performance bound $\gamma = .1136$. Finally, Fig. 3 illustrates the associated control signal $u(t)$.

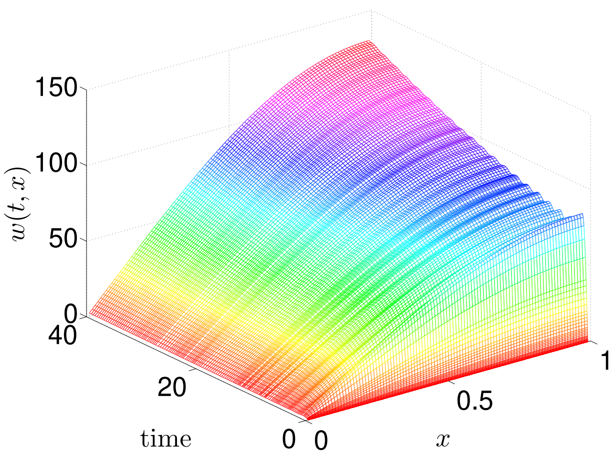

Fig. 1: Unstable dynamics of Eqn. (23) in open loop with disturbance.

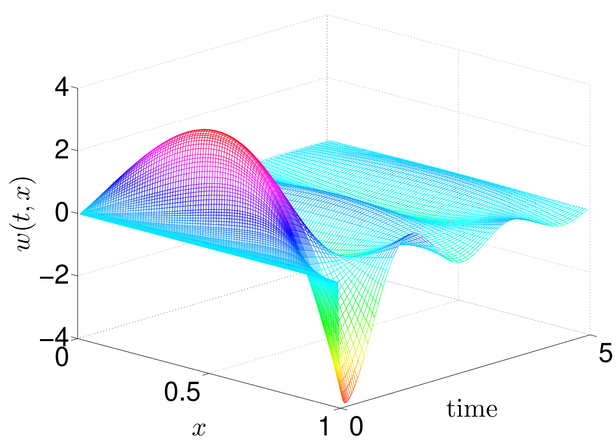

Fig. 2: Closed loop state evolution of Eqn. (23).

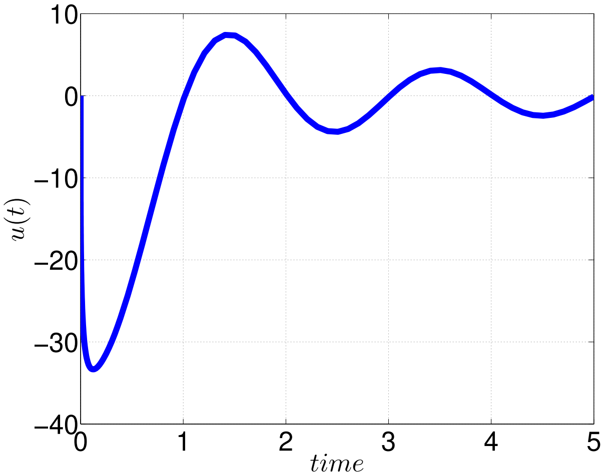

Fig. 3: Control input $u(t)$ for Eqn. (23).

To verify the norm bound, we calculated $L_2(0,\infty;L_2(0,1))$ of both the disturbance and the output of the closed system. The disturbance has norm $\|d\|_{L_2(0,\infty;L_2(0,1))} = 1069.7$ while the output has norm $\|y\|_{L_2(0,\infty;L_2(0,1))} = 28.2761$, yielding a disturbance attenuation of $\frac{\|y\|_{L_2(0,\infty;L_2(0,1))}}{\|d\|_{L_2(0,\infty;L_2(0,1))}} = 0.0264$ which satisfies the predicted bound of $\gamma = 0.1136$. Note that this does not necessarily imply conservatism in the bound or the algorithm, as the norm is the supremum over all possible disturbances.

B. Alternative Boundary Conditions

The algorithms defined in this paper can be readily adapted to alternative boundary conditions. In this subsection, we consider several alternative types of boundary-valued control inputs. For brevity, we do not define the updated conditions explicitly.

We define the first example problem as

$$w_t(t,x) = w_{xx}(t,x) + \lambda w(t,x) + (x^3-1)d(t,x), \tag{26}$$
$$y(t,x) = (x^2-0.5x+1)w(t,x), \tag{27}$$

which is parameterized by the constant λ.

The second example problem is given by

$$\begin{aligned} w_t(t,x) =& (x^3-x^2+2)w_{xx}(t,x) + (3x^2-2x)w_x(t,x) \\ &+ (-0.5x^3+1.3x^2-1.5x+0.7+\lambda)w(t,x) \\ &+ xd(t,x), \end{aligned} \tag{28}$$
$$y(t,x) = (1+x)w(t,x), \tag{29}$$

which is similarly parameterized by λ. For both example problems, we consider the following three types of alternative boundary conditions.

$$\text{Dirichlet:}\quad w(0)=0, \qquad w(1)=u(t), \tag{30}$$
$$\text{Neumann:}\quad w_x(0)=0, \qquad w_x(1)=u(t), \tag{31}$$
$$\text{Robin:}\quad w(0)+w_x(0)=0, \qquad w(1)+w_x(1)=u(t). \tag{32}$$

For each example and set of boundary conditions, the parameter $\lambda \in \mathbb{R}$ is chosen such that the associated autonomous PDE is unstable. These values are listed in Table V.

Tables VI and VII present the minimum achievable closed-loop $\mathcal{L}_{L_2}$ norm bound γ as a function of the polynomial degree$d_1 = d_2 = d$.

	Dirichlet	Neumann	Robin
PDE (26)	$\lambda = \pi^2 + 0.04$	0.033	-0.967
PDE (28)	$\lambda = 19.006$	-0.195	-2.37

TABLE V: Values of parameter λ chosen for Equations (26)-(28) with boundary conditions (30)-(32).

Example 1

	$d=3$	4	5
Dirichlet	$\gamma = 31.62$	9.581	5.266
Neumann	3.525	0.923	0.293
Robin	4.107	1.172	0.313

TABLE VI: Min. achievable closed loop $\mathcal{L}(L_2)$ gain (γ) as a fn. of $d_1 = d_2 = d$ using the algorithm defined by Thm. 2 as applied to Example 1 (Eqn. (26)) with boundary conditions (30)-(32).

VII. CONCLUSIONS

In this paper, we proposed a convex approach to the construction of optimal controllers for parabolic PDE systems with input at the boundary and measurements of the entire distributed state. The algorithms and controllers are formulated in an infinite-dimensional framework and do not require discretization of the dynamics at any stage. Optimality is defined with respect to a bound on the induced $\mathcal{L}(L_2)$ norm of the map from exogenous disturbance to output - similar to the H_∞ framework. Our methodology is based on an SDP parametrization of positive quadratic Lyapunov functions and distributed feedback gains. Numerical tests were used to illustrate the accuracy of the algorithm and the associated $\mathcal{L}(L_2)$ bound. The work presented here can ultimately be used to synthesize optimal output feedback controllers and to determine optimal sensor placement - topics of ongoing work.

APPENDIX

Lemma 1 ([11],[13]): let $w \in H^2(0,1)$ be a scalar function. Then

$$\int_0^1 w(x)^2 dx \le w(0)^2 + \frac{4}{\pi^2}\int_0^1 w_x(x)^2 dx.$$

The following lemma, which we shall use subsequently, is established by dividing the two double integrals in half and applying a change of order of integration.

Lemma 2: For any bivariate polynomials L and N, for any $z \in L_2(0,1)$, the following identity holds

$$\begin{aligned} &\int_0^1\int_0^x z(x)L(x,\xi)z(\xi)d\xi dx + \int_0^1\int_x^1 z(x)N(x,\xi)z(\xi)d\xi dx \\ &= \frac{1}{2}\int_0^1\int_0^x z(x)\left[L(x,\xi)+N(\xi,x)\right]z(\xi)d\xi dx \\ &\quad + \frac{1}{2}\int_0^1\int_x^1 z(x)\left[N(x,\xi)+L(\xi,x)\right]z(\xi)d\xi dx. \end{aligned}$$

Example 2

	$d=4$	5	6
Dirichlet	$\gamma = 31.127$	5.025	0.889
Neumann	0.225	0.102	0.076
Robin	0.135	0.074	0.072

TABLE VII: Min. achievable closed loop $\mathcal{L}(L_2)$ gain (γ) as a fn. of $d_1 = d_2 = d$ using the algorithm defined by Thm. 2 as applied to Example 2 (Eqn. (28)) with boundary conditions (30)-(32).

Lemma 3: Suppose there exist scalars $0 < \epsilon_1 < \epsilon_2 < \infty$ and $d_1, d_2 \in \mathbb{N}$, and polynomials $M : [0,1] \to \mathbb{R}$ and $N_1, N_2 : [0,1] \times [0,1] \to \mathbb{R}$ such that

$$\begin{aligned}\{M, N_1, N_2\} \in& \Xi_{d_1,d_2,\epsilon_1,\epsilon_2},\\ N_2(0,x) =& 0.\end{aligned}$$

Let M, N_1 and N_2 define $\mathcal{P}$ as in Equation (5). Additionally, let $\mathcal{A} : H^2(0,1) \to L_2(0,1)$ be defined as

$$\mathcal{A} = a(x)\frac{d^2}{dx^2} + b\frac{d}{dx} + c(x),$$

where a, b and c are the coefficients of the PDE (11).

Then, for any $z = \mathcal{P}^{-1}w$, $w \in H^2(0,1)$ with $w(0) = 0$, the following identity holds

$$\begin{aligned}2\langle \mathcal{A}\mathcal{P}z, z\rangle \le& \langle \mathcal{T}z, y\rangle + 2a(1)M(1)z_x(1)y(1)\\ &+ \left[a(1)M_x(1) + (b(1) - a_x(1))M(1)\right] z(1)^2.\end{aligned}$$

Here, for any $f \in L_2(0,1)$, we define

$$\begin{aligned}(\mathcal{T}f)(x) =& T_0(x)f(x) + \int_0^x T_1(x,\xi)f(\xi)d\xi\\ &+ \int_x^1 T_2(x,\xi)f(\xi)d\xi,\end{aligned}$$

and

$$\begin{aligned}T_0(x) =& (a_{xx}(x) - b_x(x))M(x) + b(x)M_x(x)\\ &+ a(x)M_{xx}(x) + 2c(x)M(x) - \frac{\pi^2}{2}\alpha\epsilon\\ &+ a(x)\left[2\frac{\partial}{\partial x}\left[N_1(x,\xi) - N_2(x,\xi)\right]\right]_{\xi=x},\\ T_1(x,\xi) =& a(x)N_{1,xx}(x,\xi) + b(x)N_{1,x}(x,\xi)\\ &+ a(\xi)N_{1,\xi\xi}(x,\xi) + b(\xi)N_{1,\xi}(x,\xi)\\ &+ (c(x) + c(\xi))N_1(x,\xi),\\ T_2(x,\xi) =& T_1(\xi,x).\end{aligned}$$

Proof: Using the definitions of operators $\mathcal{P}$ and $\mathcal{A}$ we obtain

$$2\langle \mathcal{A}\mathcal{P}z, z\rangle = \sum_{n=1}^{5}\Gamma_n, \tag{33}$$

where

$$\begin{aligned}\Gamma_1 =& \int_0^1 a(x)\frac{\partial^2}{\partial x^2}(M(x)z(x))z(x)dx,\\ \Gamma_2 =& \int_0^1 b(x)\frac{\partial}{\partial x}(M(x)z(x))z(x)dx,\\ \Gamma_3 =& \int_0^1 a(x)\frac{\partial^2}{\partial x^2}\left(\int_0^x N_1(x,\xi)z(\xi)d\xi\right)z(x)dx\\ &+ \int_0^1 a(x)\frac{\partial^2}{\partial x^2}\left(\int_x^1 N_2(x,\xi)z(\xi)d\xi\right)z(x)dx,\\ \Gamma_4 =& \int_0^1 b(x)\frac{\partial}{\partial x}\left(\int_0^x N_1(x,\xi)z(\xi)d\xi\right)z(x)dx\\ &+ \int_0^1 b(x)\frac{\partial}{\partial x}\left(\int_x^1 N_2(x,\xi)z(\xi)d\xi\right)z(x)dx,\\ \Gamma_5 =& \int_0^1 c(x)M(x)z(x)^2dx\\ &+ \int_0^1\int_0^x z(x)c(x)N_1(x,\xi)z(\xi)d\xi dx\\ &+ \int_0^1\int_x^1 z(x)c(x)N_2(x,\xi)z(\xi)d\xi dx.\end{aligned}$$

The definition $z = \mathcal{P}^{-1}w$ implies

$$w(0) = M(0)z(0) + \int_0^1 N_2(0,x)z(x)dx.$$

Therefore, since $w(0) = 0$ and $N_2(0,x) = 0$, we get $z(0) = 0$.

Since $M(x)a(x) \ge \alpha\epsilon$ and $z(0) = 0$, applying integration by parts twice and using Lemma 1 gives us

$$\begin{aligned}\Gamma_1 =& \frac{1}{2}\int_0^1\left(a_{xx}(x)M(x) + a(x)M_{xx}(x) - \frac{\pi^2}{2}\alpha\epsilon\right)z(x)^2dx\\ &+ \frac{1}{2}\big(a(1)M_x(1) - a_x(1)M(1)\big)z(1)^2\\ &+ a(1)M(1)z_x(1)z(1).\end{aligned}\tag{34}$$

Similarly, applying integration by parts once gives us

$$\begin{aligned}\Gamma_2 =& \frac{1}{2}\int_0^1 (b(x)M_x(x) - b_x(x)M(x))z(x)^2dx\\ &+ \frac{1}{2}b(1)M(1)z(1)^2.\end{aligned}\tag{35}$$

Applying integration by parts twice gives us

$$\begin{aligned}\Gamma_3 =& \int_0^1\left(a(x)\left[\frac{\partial}{\partial x}\left[N_1(x,\xi) - N_2(x,\xi)\right]\right]_{\xi=x}\right)z(x)^2dx\\ &+ \int_0^1\int_0^x z(x)a(x)N_{1,xx}(x,\xi)z(\xi)d\xi dx\\ &+ \int_0^1\int_x^1 z(x)a(x)N_{2,xx}(x,\xi)z(\xi)d\xi dx.\end{aligned}$$

Applying Lemma 2

$$\begin{aligned}\Gamma_3 =& \int_0^1\left(a(x)\left[\frac{\partial}{\partial x}\left[N_1(x,\xi) - N_2(x,\xi)\right]\right]_{\xi=x}\right)z(x)^2dx\\ &+ \frac{1}{2}\int_0^1\int_0^x z(x)a(x)N_{1,xx}(x,\xi)z(\xi)d\xi dx\\ &+ \frac{1}{2}\int_0^1\int_0^x z(x)a(\xi)N_{1,\xi\xi}(x,\xi)z(\xi)d\xi dx\\ &+ \frac{1}{2}\int_0^1\int_x^1 z(x)a(x)N_{2,xx}(x,\xi)z(\xi)d\xi dx\\ &+ \frac{1}{2}\int_0^1\int_x^1 z(x)a(\xi)N_{2,\xi\xi}(x,\xi)z(\xi)d\xi dx.\end{aligned}\tag{36}$$

Similarly, applying integration by parts once followed by Lemma 2 produces

$$\begin{aligned}\Gamma_4 =& \frac{1}{2}\int_0^1\int_0^x z(x)b(x)N_{1,x}(x,\xi)z(\xi)d\xi dx\\ &+ \frac{1}{2}\int_0^1\int_0^x z(x)b(\xi)N_{1,\xi}(x,\xi)z(\xi)d\xi dx\\ &+ \frac{1}{2}\int_0^1\int_x^1 z(x)b(x)N_{2,x}(x,\xi)z(\xi)d\xi dx\\ &+ \frac{1}{2}\int_0^1\int_x^1 z(x)b(\xi)N_{2,\xi}(x,\xi)z(\xi)d\xi dx.\end{aligned}\tag{37}$$

Finally, applying Lemma 2 produces

$$\begin{aligned}\Gamma_5 =& \int_0^1 c(x)M(x)z(x)^2dx\\ &+ \frac{1}{2}\int_0^1\int_0^x z(x)(c(x) + c(\xi))N_1(x,\xi)z(\xi)d\xi dx\\ &+ \frac{1}{2}\int_0^1\int_x^1 z(x)(c(x) + c(\xi))N_2(x,\xi)z(\xi)d\xi dx.\end{aligned}\tag{38}$$

Substituting Equations (34)-(38) into Equation (33) completes the proof.

■

ACKNOWLEDGMENT

This research was supported by the Chateaubriand program and NSF Grant# CMMI-1301851 CAREER.

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