A Dual to Lyapanov's Second Method for Linear Systems with Multiple Delays and Implementation using SOS

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***Abstract*—We present a dual form of Lyapunov-Krasovskii functional which allows the problem of controller synthesis of multi-delay systems to be formulated and solved in a convex manner. First, we give a generalized version of the dual stability condition formulated in terms of Lyapunov operators which are positive, self-adjoint and preserve the structure of the state-space. Second, we provide a class of such operators and express the stability conditions as positivity and negativity of quadratic Lyapunov-Krasovskii functional forms. Next, we adapt the SOS methodology to express positivity and negativity of these forms as LMIs, describing a new set of polynomial manipulation tools designed for this purpose. We apply the resulting LMIs to a battery of numerical examples and demonstrate that the stability conditions are not significantly conservative. Finally, we formulate a test for controller synthesis for systems with multiple delays, apply the test to a numerical example, and simulate the resulting closed-loop system.**

***Index Terms*—Delay Systems, LMIs, Controller Synthesis.**

I. Introduction

Systems with delay have been well-studied for some time [1], [2], [3]. In recent years, however, there has been an increased emphasis on the use of optimization and semidefinite programming for stability analysis of linear and nonlinear time-delay systems. Although the computational question of stability of a linear state-delayed system is believed to be NP-hard, several techniques have been developed which use LMI methods [4] to construct asymptotically exact algorithms. An asymptotically exact algorithm is a sequence of polynomial-time algorithms wherein each instance in the sequence provides sufficient conditions for stability, the computational complexity of the instances is increasing, the accuracy of the test is increasing, and the sequence converges to what appears to be a necessary and sufficient condition. Examples of such sequential algorithms include the piecewise-linear approach [2], the delay-partitioning approach [5], the Wirtinger-based method of [6] and the SOS approach [7]. In addition, there are also frequency-domain approaches such as [8], [9]. These asymptotic algorithms are sufficiently reliable so that for the purposes of this paper, we may consider the problem of stability analysis of linear discrete-delay systems to be solved.

The purpose of this paper is to explore methods by which we may extend the success in the use of asymptotic algorithms for stability analysis of time-delay systems to the field of robust and optimal controller synthesis - an area which is relatively underdeveloped. Although there have been a number of results on controller synthesis for time-delay systems [10], none of these results has been able to resolve the fundamental bilinearity of the synthesis problem. Bilinearity here means that for a given feedback controller, the search for a Lyapunov functional is linear in the decision variables which define the functional and relatively tractable. Furthermore, given a predefined Lyapunov functional, the search for a controller ensuring negativity of the time-derivative of that functional is linear in the decision variables which define the feedback gains. However, if we are looking for both a controller and a Lyapunov functional which establishes stability of that controller, then the resulting stability condition is nonlinear and non-convex in the combined set of decision variables.

Without a convex formulation of the controller synthesis problem, we cannot search over the set of provably stabilizing controllers without significant conservatism, much less address the problems of robust and quadratic stability. To resolve this difficulty, some papers use iterative methods to alternately optimize the Lyapunov functional and then the controller as in [11] or [12] (via a "tuning parameter"). However, this iterative approach is not guaranteed to converge. Meanwhile, approaches based on frequency-domain methods, discrete approximation, or Smith predictors result in controllers which are not provably stable or are sensitive to variations in system parameters or in delay. Finally, we mention that delays often occur in both state and input and to date most methods do not provide a unifying formulation of the controller synthesis problem with both state and input delay.

In this paper, we propose a dual Lypunov-type stability criterion, wherein the decision variables do not parameterize a Lyapunov functional per se, but where the feasibility of this criterion *implies* the existence of such a functional. The advantage of such an approach for controller synthesis is that it allows for an invertible variable substitution which eliminates all bilinear terms in the criterion for controller synthesis.

To motivate this dual stability criterion, consider the LMI framework for synthesis of controllers for linear finite-dimensional state-space systems of the form $\dot{x} = Ax + Bu$. Specifically, if $u = 0$, the LMI condition for the existence of a quadratic Lyapunov function $V(x) = x^T Px$ is the existence of a $P > 0$ such that $A^T P + PA < 0$. The feasibility of this LMI implies that $V(x) = x^T Px > 0$ and $\dot{V}(x) = x^T (A^T P + PA)x < 0$. This LMI is in primal form because the decision variable P defines the Lyapunov function directly. However, when we add a controller $u = Kx$, we get $\dot{x} = (A + BK)x$ and the synthesis condition becomes $A^T P + PA + K^T B^T P + PBK < 0$ which is bilinear in decision variables P and K and hence intractable. Bilinearity can be eliminated, however, if we use the *implied* Lyapunov function $V(x) = x^T P^{-1}x$. Using the implied Lyapunov function $V(x) = x^T P^{-1}x$, the time-derivative becomes $\dot{V}(x) = x^T (A^T P^{-1} + P^{-1}A)x = (P^{-1}x)^T (PA^T + AP)(P^{-1}x) = z^T (PA^T + AP)z$, where $z = P^{-1}x$. This

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implies that stability of $\dot{x} = Ax$ is equivalent to the existence of $P > 0$ such that $AP + PA^T < 0$. If we now add a controller $u = Kx$, the controller synthesis condition becomes $(AP + BKP) + (AP + BKP)^T < 0$, which is still bilinear. However, if we consider the variable substitution $Z = KP$, then stabilizability is equivalent to the existence of a $P > 0$ and Z such that $(AP+BZ)+(AP+BZ)^T < 0$ - which is an LMI. The stabilizing controller gains can then be reconstructed as $K = ZP^{-1}$. If $A(\delta)$ and $B(\delta)$ are uncertain, $\delta \in \Delta$, then we search for $P(\delta)$ and $Z(\delta)$ (or a fixed P for quadratic stability) and the LMI must hold for all $\delta \in \Delta$ - a problem which is more difficult, but still convex in the variables P and Z. LMIs of this form were introduced in [13] and are the basis for a majority of LMI methods for controller synthesis (See Chapter 5 Notes in [4] for a discussion). The first contribution of this paper, then, is an operator-valued equivalent of the dual Lyapunov inequality $P > 0$, $AP + PA^T < 0$ which implies stability of a general class of infinite-dimensional systems. The second contribution of the paper is a computational framework for verifying this dual inequality using LMIs.

The standard approach to state-space representation of infinite-dimensional systems is to define the state as evolving on a Hilbert space Z and satisfying the derivative condition $\dot{x}(t) = \mathcal{A}x(t)$. The state is constrained to a subspace X of Z and the operator $\mathcal{A}$ is typically unbounded. It is known that if $\mathcal{A}$ generates a strongly continuous semigroup, then exponential stability of this system is equivalent to the existence of an operator $\mathcal{P}$ such that $\langle x, \mathcal{P}x\rangle \geq \|x\|^2$ and $\langle x, \mathcal{P}\mathcal{A}x\rangle + \langle \mathcal{P}\mathcal{A}x, x\rangle \leq -\epsilon \|x\|^2$ [14]. In Section IV, we show that under mild additional conditions on $\mathcal{P}$, the dual version of this result also holds. Namely, existence of an operator $\mathcal{P}$ such that $\langle x, \mathcal{P}x\rangle \geq \|x\|^2$ and $\langle x, \mathcal{A}\mathcal{P}x\rangle + \langle \mathcal{A}\mathcal{P}x, x\rangle \leq -\epsilon \|x\|^2$ implies exponential stability of $\dot{x} = \mathcal{A}x$. Specifically, these additional conditions on $\mathcal{P}$ are that $\mathcal{P}$ be self-adjoint and preserve specified properties of the solution. This result applies to any well-posed infinite-dimensional system, and is not conservative if X is a closed subspace of Z.

The second contribution of the paper considers the special case of systems with multiple delays and is based on a parametrization of a class of operators which satisfy these additional constraints, and which are defined by the combination of multiplier and integral operators with constraints on the associated multipliers and kernels. This second result allows us to represent the dual stability criterion in a manner similar to classical Lyapunov-Krasovskii stability conditions, but with an additional tri-diagonal structure which may prove useful for solving these Lyapunov equations. Finally, the third contribution of this paper is an LMI/SOS method for enforcing positivity and negativity of the operators under the assumption that all multipliers and kernels are polynomial. Finally, we discuss how these results can be used to solve the controller synthesis problem and give a numerical example using the methods defined in [15] and [16].

Having stated the main contributions of the paper, we note that while we show how to enforce the operator inequalities using a slight generalization of existing SOS-based results, the duality results are presented in such a way as to encourage the reader to use other methods of enforcing these inequalities, methods including those contained in [17], [5], or [6]. Indeed, we emphasize that the first two contributions of the paper (Theorems 1 and 5) are formulated independent of whichever numerical method is used for enforcing the inequalities. In this way, our goal is to simply establish a new class of Lyapunov stability conditions which are well-suited to the problem of controller synthesis, leaving the method of enforcement of these conditions to the reader.

Finally, we note that there have been a number of results on dual and adjoint systems [18]. Unfortunately, however, these dual systems are not delay-type systems and there is no clear relationship between stability of these adjoint and dual systems and stability of the original delayed system.

This paper is organized as follows. In Sections II and III we develop a mathematical framework for expressing Lyapunov-based stability conditions as operator inequalities. In Section IV we show that given additional constraints on the Lyapunov operator, satisfaction of the dual Lyapunov inequality $\langle x, \mathcal{A}\mathcal{P}x\rangle + \langle \mathcal{A}\mathcal{P}x, x\rangle \leq -\epsilon \|x\|^2$ proves stability of $\dot{x}(t) = \mathcal{A}x(t)$. In Sections VI and V we define a restricted class of Lyapunov functionals and operators which are valid for the dual stability condition in both the single-delay and mutliple-delay cases, applying these classes of operators in Subsections VI-B and V-B to obtain dual stability conditions. These dual stability conditions are formulated as positivity and negativity of Lyapunov functionals. In Section VII, we show how SOS-based methods can be used to parameterize positive Lyapunov functionals and thereby enforce the inequality conditions in Sections VI-B and V-B, results which are summarized in Corollary 13. Finally, in Section VIII, we summarize our results with a set of LMI conditions for dual stability in both the single and multiple-delay cases. Section IX describes our Matlab toolbox, available online, which facilitates construction and solution of the LMIs. Section X applies the results to a variety of stability problems and verifies that the dual stability test is not conservative. Finally Section XI discusses the problem of full-state feedback controller synthesis and gives a numerical illustration in the case of a single delay.

A. Technical Summary of Results

Before proceeding, we give a brief summary of the main results of Section VI-B using as little mathematical formalism as possible in order to illustrate how these results differ from the classical Lyapunov-Krasovskii stability conditions. These results are stated for systems with a single delay in order to avoid much of the notation and mathematical progression needed for the multiple delay case. That is, we consider the system:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau).$$

Classical Lyapunov-Krasovskii Stability Conditions: The standard necessary and sufficient conditions for stability in the single delay case are the existence of a

$$V(\phi) = \int_{-\tau}^{0} \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix}^T \begin{bmatrix} M_{11} & \tau M_{12}(s)\\ \tau M_{21}(s) & \tau M_{22}(s)\end{bmatrix} \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix} ds + \tau \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s)^T N(s,\theta)\phi(\theta) d\theta ds$$

such that $V(\phi) \geq \|\phi(0)\|^2$ and

$$\dot{V}(\phi) =$$

$$\int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(s) \end{bmatrix}^T \begin{bmatrix} D_{11}+D_{11}^T & D_{12} & \tau D_{13}(s) \\ D_{12}^T & -M_{22}(-\tau) & \tau D_{23}(s) \\ \tau D_{13}(s)^T & \tau D_{23}(s)^T & -\tau \dot{M}_{22}(s) \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(s) \end{bmatrix} ds$$

$$-\tau \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s)^T \left(\frac{d}{ds}N(s,\theta) + \frac{d}{d\theta}N(s,\theta)\right)\phi(\theta) d\theta ds \leq -\epsilon \|\phi\|^2$$

$$D_{11} = M_{11}A_0 + M_{12}(0) + \frac{1}{2}M_{22}(0),$$

$$D_{12} = M_{11}A_1 - M_{12}(-\tau), \quad D_{23} = A_1^T M_{12}(s) - N(-\tau, s)$$

$$D_{13} = A_0^T M_{12}(s) - \dot{M}_{12}(s) + N(0,s).$$

New Dual Lyapunov-Krasovskii Stability Conditions: As per Corollary 7, the single-delay system is stable if there exists a

$$V(\phi) = \int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T \begin{bmatrix} \tau(R(0,0)+S(0)) & \tau R(0,s) \\ \tau R(s,0) & \tau S(s) \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds$$
$$+ \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s)^T R(s,\theta)\phi(\theta) d\theta ds$$

such that $V(\phi) \geq \left\| \begin{bmatrix} \phi(0) \\ \phi \end{bmatrix} \right\|^2$ and

$$V_D(\phi) = \langle \phi, \mathcal{D}\phi \rangle$$
$$= \int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(s) \end{bmatrix}^T \begin{bmatrix} S_{11}+S_{11}^T & S_{12} & \tau S_{13}(s) \\ S_{12}^T & S_{22} & 0_n \\ \tau S_{13}(s)^T & 0_n & \tau \dot{S}(s) \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(s) \end{bmatrix} ds$$
$$+ \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s)^T \left(\frac{d}{ds}R(s,\theta) + \frac{d}{d\theta}R(s,\theta)\right)\phi(\theta) d\theta ds$$
$$\leq -\epsilon \left\| \begin{bmatrix} \phi(0) \\ \phi \end{bmatrix} \right\|.$$

where

$$S_{11} := \tau A_0(R(0,0)+S(0)) + \tau A_1 R(-\tau,0) + \frac{1}{2}S(0),$$
$$S_{12} := \tau A_1 S(-\tau), \qquad S_{22} := -S(-\tau),$$
$$S_{13}(s) := A_0 R(0,s) + A_1 R(-\tau,s) + \dot{R}(s,0)^T.$$

Although this subsection only considers the single-delay case, one can see the two primary differences between the primal and dual stability conditions. First, as was the case for delay-free systems, the A_0, A_1 system matrices appear on the left as opposed to the right hand side of the Lyapunov variables. This allows for controller synthesis via variable substitution as we will demonstrate in Section XI. The second difference is that in the dual stability conditions, the (2,3) and (3,2) blocks of the derivative condition are zero. This unexpected structure extends to the multiple-delay case, wherein *ALL* (i,j) blocks are zero for $i,j \neq 1$, $i \neq j$. This tri-diagonal structure can be exploited by numerical optimization algorithms. Alternatively, it may be possible to adapt these results to the algebraic approach of [19].

B. Notation

Shorthand notation used throughout this paper includes the Hilbert spaces $L_2^m[X] := L_2(X;\mathbb{R}^m)$ of square integrable functions from X to $\mathbb{R}^m$ and $W_2^m[X] := W^{1,2}(X;\mathbb{R}^m) = H^1(X;\mathbb{R}^m) = \{x : x, \dot{x} \in L_2^m[X]\}$. We use L_2^m and W_2^m when domains are clear from context. We also use the extensions $L_2^{n\times m}[X] := L_2(X;\mathbb{R}^{n\times m})$ and $W_2^{n\times m}[X] := W^{1,2}(X;\mathbb{R}^{n\times m})$ for matrix-valued functions. $\mathcal{C}[X] \supset W_2[X]$ denotes the continuous functions on X. $S^n \subset \mathbb{R}^{n\times n}$ denotes the symmetric matrices. We say an operator $\mathcal{P} : Z \to Z$ is positive on a subset X of Hilbert space Z if $\langle x, \mathcal{P}x\rangle_Z \geq 0$ for all $x \in X$. $\mathcal{P}$ is coercive on X if $\langle x, \mathcal{P}x\rangle_Z \geq \epsilon \|x\|_Z^2$ for some $\epsilon > 0$ and for all $x \in X$. Given an operator $\mathcal{P} : Z \to Z$ and a set $X \subset Z$, we use the shorthand $\mathcal{P}(X)$ to denote the image of $\mathcal{P}$ on subset X. $I_n \in \mathbb{S}^n$ denotes the identity matrix. $0_{n\times m} \in \mathbb{R}^{n\times m}$ is the matrix of zeros with shorthand $0_n := 0_{n\times n}$. We will occasionally denote the intervals $T_i^j := [-\tau_i, -\tau_j]$ and $T_i^0 := [-\tau_i, 0]$. For a natural number, $K \in \mathbb{N}$, we adopt the index shorthand notation which denotes $[K] = \{1, \cdots, K\}$.

II. Standard Results on Lyapunov Stability of Linear Time-Delay Systems

In this paper, we consider stability of linear discrete-delay systems of the form

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{K} A_i x(t-\tau_i) \quad \text{for all} \quad t \geq 0,$$
$$x(t) = \phi(t) \qquad \text{for all} \quad t \in [-\tau_K, 0], \quad (1)$$

where $A_i \in \mathbb{R}^{n\times n}$, $\phi \in \mathcal{C}[-\tau_K, 0]$, $K \in \mathbb{N}$ and for convenience $\tau_1 < \tau_2 < \cdots < \tau_K$. We associate with any solution x and any time $t \geq 0$, the 'state' of System (1), $x_t \in \mathcal{C}[-\tau_K, 0]$, where $x_t(s) = x(t+s)$. For linear discrete-delay systems of Form (1), the system has a unique solution for any $\phi \in \mathcal{C}[-\tau_K, 0]$ and global, local, asymptotic and exponential stability are all equivalent.

Stability of Equations (1) may be certified through the use of Lyapunov-Krasovskii functionals - an extension of Lyapunov theory to systems with infinite-dimensional state-space. In particular, it is known [2] that the linear time-delay system (1) is stable if and only if there exist functions M and N, continuous in their respective arguments everywhere except possibly at points $H := \{-\tau_1, \cdots, -\tau_{K-1}\}$, such that the quadratic Lyapunov-Krasovskii functional

$$V(\phi) = \int_{-\tau_K}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds$$
$$+ \int_{-\tau_K}^{0}\int_{-\tau_K}^{0} \phi(s)^T N(s,\theta)\phi(\theta)\, ds\, d\theta \qquad (2)$$

satisfies $V(\phi) \geq \epsilon \|\phi(0)\|^2$ and the Lie (upper-Dini) derivative of the functional is negative along any solution x of (1). That is,

$$\dot{V}(x_t) = \lim_{h\to 0} \frac{V(x_{t+h}) - V(x_t)}{h} \leq -\epsilon \|x_t(0)\|^2$$

for all $t \geq 0$ and some $\epsilon > 0$.

For the dual stability conditions we propose in this paper, discontinuities in the unknown functions M and N pose challenges which make this form of Lyapunov-Krasovskii functional poorly suited to controller synthesis. For this reason, we use an alternative formulation of the necessary Lyapunov-Krasovskii functional. Specifically, it has been shown in [20], Theorem 3, that exponential stability is also equivalent to the existence of a Lyapunov-Krasovskii functional of the form

$$V(\phi) = \tau_K \phi(0)^T P \phi(0) + \tau_K \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(0)^T Q_i(s)\phi(s)ds$$
$$+ \tau_K \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(s)^T Q_i(s)^T \phi(0)ds + \tau_K \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(s)^T S_i(s)\phi(s)$$
$$+ \sum_{i,j=1}^{K} \int_{-\tau_i}^{0} \int_{-\tau_j}^{0} \phi(s)^T R_{ij}(s,\theta)\phi(\theta)d\theta \geq \epsilon \left\|\phi(0)\right\|^2, \quad (3)$$

where $\dot{V}(x_t) \leq -\epsilon \left\|x_t(0)\right\|^2$ for some $\epsilon > 0$ and the functions Q_i, S_i and R_{ij} may be assumed continuous on their respective domains of definition.

III. Reformulating the Lyapunov Stability Conditions using Positive Operators

In this section, we introduce the mathematical formalism which will be used to express both the primal and dual stability conditions. We begin by reviewing the well-established semi-group framework - a generalization of the concept of differential equations. Sometimes known as the 'flow map', a 'strongly continuous semigroup' is an operator, $S(t) : Z \to Z$, defined by the Hilbert space Z, which represents the evolution of the state of the system so that for any solution x, $x_{t+s} = S(s)x_t$. Associated with a semigroup on Z is an operator $\mathcal{A}$, called the 'infinitesimal generator' which satisfies $\frac{d}{dt}S(t)\phi = \mathcal{A}S(t)\phi$ for any $\phi \in X$. The space $X \subset Z$ is often referred to as the domain of the generator $\mathcal{A}$, and is the space on which the generator is defined and need not be a closed subspace of Z. In this paper we will refer to X as the 'state-space'.

For the multi-delay System (1), we define $Z_{m,n,K} := \{\mathbb{R}^m \times L_2^n[-\tau_1, 0] \times \cdots \times L_2^n[-\tau_K, 0]\}$ and for $\{x, \phi_1, \cdots, \phi_K\} \in Z_{m,n,K}$, we define the following shorthand notation

$$\begin{bmatrix} x \\ \phi_i \end{bmatrix} := \{x, \phi_1, \cdots, \phi_K\},$$

which allows us to simplify expression of the inner product on $Z_{m,n,K}$, which we define to be

$$\left\langle \begin{bmatrix} y \\ \psi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{m,n,K}} = \tau_K y^T x + \sum_{i=1}^{K} \int_{-\tau_i}^{0} \psi_i(s)^T \phi_i(s)ds.$$

When $m = n$, we simplify the notation using $Z_{n,K} := Z_{n,n,K}$. We may now conveniently write the state-space for System 1 as

$$X := \left\{ \begin{bmatrix} x \\ \phi_i \end{bmatrix} \in Z_{n,K} \;:\; \begin{smallmatrix} \phi_i \in W_2^n[-\tau_i, 0] \text{ and} \\ \phi_i(0) = x \text{ for all } i \in [K] \end{smallmatrix} \right\}.$$

Note that X is a subspace of $Z_{n,K}$, inherits the norm of $Z_{n,K}$, but is not closed in $Z_{n,K}$. We furthermore extend this notation to say

$$\begin{bmatrix} x \\ \phi_i \end{bmatrix}(s) = \begin{bmatrix} y \\ f(s,i) \end{bmatrix}$$

if $x = y$ and $\phi_i(s) = f(s,i)$ for $s \in [-\tau_i, 0]$ and $i \in [K]$. This also allows us to compactly represent the infinitesimal generator, $\mathcal{A}$, of Eqn. (1) as

$$\mathcal{A} \begin{bmatrix} x \\ \phi_i \end{bmatrix}(s) := \begin{bmatrix} A_0 x + \sum_{i=1}^{K} A_i \phi_i(-\tau_i) \\ \partial_s \phi_i(s) \end{bmatrix}.$$

Using these definitions of $\mathcal{A}$, Z and X, for matrix P and functions Q_i, S_i, R_{ij}, we define an operator $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ of the "complete-quadratic" type as

$$\left(\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \begin{bmatrix} x \\ \phi_i \end{bmatrix}\right)(s) :=$$
$$\begin{bmatrix} Px + \sum_{i=1}^{K} \int_{-\tau_i}^{0} Q_i(s)\phi_i(s)ds \\ \tau_K Q_i(s)^T x + \tau_K S_i(s)\phi_i(s) + \sum_{j=1}^{K} \int_{-\tau_j}^{0} R_{ij}(s,\theta)\phi_j(\theta)\, d\theta. \end{bmatrix}$$

This notation will be used throughout the paper and allows us to associate P, Q_i, S_i and R_{ij} with the corresponding complete-quadratic functional in Eqn. (3) as

$$V(\phi) = \left\langle \begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}}.$$

That is, the Lyapunov functional is defined by the operator $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ which is a variation of a classical combined multiplier and integral operator whose multipliers and kernel functions are defined by P, Q_i, S_i, R_{ij}.

The time-derivative of the complete-quadratic functional can similarly be represented using complete quadratic operators as

$$\dot{V}(\phi) = \left\langle \begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \mathcal{A} \begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}}$$
$$+ \left\langle \mathcal{A} \begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}}$$
$$= \left\langle \begin{bmatrix} \phi(0) \\ \vdots \\ \phi(-\tau_K) \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}} \begin{bmatrix} \phi(0) \\ \vdots \\ \phi(-\tau_K) \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n(K+1),n,K}}$$

where [21]

$$D_1 = \begin{bmatrix} \Delta_0 & \Delta_1 & \cdots & \Delta_K \\ \Delta_1^T & S_1(-\tau_1) & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \Delta_K^T & 0 & 0 & S_K(-\tau_K) \end{bmatrix},$$

$$\Delta_0 = PA_0 + A_0^T P + \sum_{k=1}^{K} Q_k(0) + Q_k(0)^T + S_k(0),$$
$$\Delta_j = PA_j - Q_j(-\tau_j), \quad V_i(s) = \begin{bmatrix} \Pi_{0,i}(s)^T & \ldots & \Pi_{K,i}(s)^T \end{bmatrix}^T,$$
$$\Pi_{0j}(s) = A_0^T Q_j(s) + \sum_{k=1}^{K} R_{jk}^T(s,0) - \dot{Q}_j(s)$$
$$\Pi_{ij}(s) = A_i^T Q_j(s) - R_{ji}^T(s, -\tau_i),$$
$$G_{ij}(s,\theta) = -\frac{\partial}{\partial s} R_{ij}(s,\theta) - \frac{\partial}{\partial \theta} R_{ij}(s,\theta).$$

In this subsection, we have reformulated $\mathcal{A}^*\mathcal{P} + \mathcal{P}\mathcal{A} < 0$ as negativity of a multiplier/integral operator on a lifted space. The classical Lyapunov-Krasovskii stability condition, then, states that the delay-differential Equation (1) is stable if there exists an $\epsilon > 0$, matrix P and functions Q_i, S_i, R_{ij} such that $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \geq \epsilon \hat{I}_1$ and $\mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}} \leq -\epsilon \hat{I}_2$ for suitably defined $\hat{I}_1 = diag(I_n, 0, \cdots)$ and $\hat{I}_2 = diag(I_n, 0, \cdots)$.

Note that there are many possible ways of defining Z and X which lead to Lyapunov functional in Eqn. (3) and the associated primal stability conditions. These choices are not arbitrary. As will be seen: they are critical in ensuring that $\mathcal{P}\mathcal{A}^* + \mathcal{A}\mathcal{P} < 0$ can be reformulated as a multiplier operator as we did here for the primal stability criterion - a requirement that precludes the use of the standard Hilbert space $\mathbb{R}^n \times L_2[-\tau_K, 0]$ (or the Banach space $\mathbb{R}^n \times \mathcal{C}[-\tau_K, 0]$). A third option would be the Sobolev space $\mathbb{R}^n \times W^2[-\tau_K, 0]$. In this third case, however, the resulting inequality would imply a slightly stronger stability property of stability in the Sobolev norm rather than the standard L_2-norm.

IV. A Dual Stability Condition for Infinite-Dimensional Systems

Using the notation we have introduced in the preceding section, we may compactly represent the dual stability condition which forms the main theoretical contribution of the paper. Note that the results of this section apply to infinite-dimensional systems in general and are not specific to systems with delay. The following sections will consider application of these results to systems with delay.

Theorem 1: Suppose that $\mathcal{A}$ generates a strongly continuous semigroup on Hilbert space Z with domain X. Further suppose there exists an $\epsilon > 0$ and a bounded, coercive linear operator $\mathcal{P} : X \to X$ with $\mathcal{P}(X) = X$ and which is self-adjoint with respect to the Z inner product and satisfies

$$\langle \mathcal{A}\mathcal{P}z, z\rangle_Z + \langle z, \mathcal{A}\mathcal{P}z\rangle_Z \le -\epsilon \|z\|_Z^2$$

for all $z \in X$. Then a dynamical system which satisfies $\dot{x}(t) = \mathcal{A}x(t)$ generates an exponentially stable semigroup.

Proof: Because $\mathcal{P}$ is coercive and bounded there exist $\gamma, \delta > 0$ such that $\langle x, \mathcal{P}x\rangle_Z \ge \gamma \|x\|_Z^2$ and $\|Px\| \le \delta \|x\|_Z$. By the Lax-Milgren theorem [22], $\mathcal{P}^{-1}$ exists and is bounded and $\mathcal{P}(X) = X$ implies $\mathcal{P}^{-1} : X \to X$. The inverse is self-adjoint since $\mathcal{P}$ is self-adjoint and hence $\left\langle \mathcal{P}^{-1}x, y\right\rangle_Z = \left\langle \mathcal{P}^{-1}x, \mathcal{P}\mathcal{P}^{-1}y\right\rangle_Z = \left\langle x, \mathcal{P}^{-1}y\right\rangle_Z$. Since $\sup_z \frac{\|\mathcal{P}z\|}{\|z\|} = \delta < \infty$, $\inf_y \frac{\|\mathcal{P}^{-1}y\|}{\|y\|} = \inf_x \frac{\|x\|}{\|\mathcal{P}x\|} = \frac{1}{\delta} > 0$ and hence $\left\langle y, \mathcal{P}^{-1}y\right\rangle_Z = \left\langle \mathcal{P}\mathcal{P}^{-1}y, \mathcal{P}^{-1}y\right\rangle_Z \ge \gamma \left\|\mathcal{P}^{-1}y\right\|_Z^2 \ge \frac{\gamma}{\delta^2}\|y\|_Z^2$. Hence $\mathcal{P}^{-1}$ is coercive.

Define the Lyapunov functional $V(y) = \left\langle y, \mathcal{P}^{-1}y\right\rangle_Z \ge \frac{\gamma}{\delta^2}\|y\|_Z^2$ which holds for all $y \in X$. If $y(t)$ satisfies $\dot{y}(t) = \mathcal{A}y(t)$, then V has time derivative

$$\begin{aligned}\frac{d}{dt}V(y(t)) &= \left\langle \dot{y}(t), \mathcal{P}^{-1}y(t)\right\rangle_Z + \left\langle y(t), \mathcal{P}^{-1}\dot{y}(t)\right\rangle_Z \\ &= \left\langle \mathcal{A}y(t), \mathcal{P}^{-1}y(t)\right\rangle_Z + \left\langle y(t), \mathcal{P}^{-1}\mathcal{A}y(t)\right\rangle_Z \\ &= \left\langle \mathcal{A}y(t), \mathcal{P}^{-1}y(t)\right\rangle_Z + \left\langle \mathcal{P}^{-1}y(t), \mathcal{A}y(t)\right\rangle_Z.\end{aligned}$$

Now define $z(t) = \mathcal{P}^{-1}y(t) \in X$ for all $t \ge 0$. Then $y(t) = \mathcal{P}z(t)$ and since $\mathcal{P}$ is bounded and $\mathcal{P}^{-1}$ is coercive,

$$\begin{aligned}\dot{V}(y(t)) &= \left\langle \mathcal{A}y(t), \mathcal{P}^{-1}y(t)\right\rangle_Z + \left\langle \mathcal{P}^{-1}y(t), \mathcal{A}y(t)\right\rangle_Z \\ &= \langle \mathcal{A}\mathcal{P}z(t), z(t)\rangle_Z + \langle z(t), \mathcal{A}\mathcal{P}z(t)\rangle_Z \\ &\le -\epsilon \|z(t)\|_Z^2 \le -\frac{\epsilon}{\delta}\langle z(t), \mathcal{P}z(t)\rangle_Z \\ &= -\frac{\epsilon}{\delta}\left\langle y(t), \mathcal{P}^{-1}y(t)\right\rangle_Z \le -\frac{\epsilon\gamma}{\delta^3}\|y(t)\|_Z^2.\end{aligned}$$

Negativity of the derivative of the Lyapunov function implies exponential stability in the square norm of the state by, e.g. [14] or by the invariance principle. ■

The constraint $\mathcal{P}(X) = X$ ensures $\mathcal{P}^{-1} : X \to X$ and is satisfied if X is a closed subspace of Z or if X is itself a Hilbert space contained in Z and $\mathcal{P}$ is coercive on the space X with respect to the inner product in which X is closed. For the case of time-delay systems, X is not a closed subspace and we do not wish to constrain $\mathcal{P}$ to be coercive on X, since this space requires the Sobolev inner product in order to be closed. For these reasons, in Lemma 4, we will directly show that for our class of operators (to be defined), $\mathcal{P}(X) = X$.

The advantage of the dual stability condition is that we replace $\mathcal{P}\mathcal{A}$, which appears in the classical stability conditions with $\mathcal{A}\mathcal{P}$. Although relatively subtle, this distinction allows convexification of the controller synthesis problem. In the following sections, we discuss how to parameterize operators which satisfy the conditions of Theorem 1, first in the case of multiple delays, and then for the special case of a single delay. We start with the constraints $\mathcal{P} = \mathcal{P}^*$ and $\mathcal{P} : X \to X$. Note that without additional restrictions on P, Q_i, S_i, R_{ij}, the operator $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ satisfies neither constraint.

Before moving to the next section, a natural question is whether the dual stability condition is significantly conservatism. That is, does stability of the system imply that the conditions of Theorem 1 are feasible. We refer to the following theorem from which is from Theorem 5.1.3 in [14].

Theorem 2: Suppose that $\mathcal{A}$ is the infinitesimal generator of the C_0-semigroup $S(t)$ on the Hilbert space Z with domain $D(\mathcal{A})$. Then $S(t)$ is exponentially stable if and only if there exists a positive, self-adjoint operator $\mathcal{P} \in \mathcal{L}(Z)$ such that

$$\langle \mathcal{P}\mathcal{A}z, z\rangle_Z + \langle z, \mathcal{P}\mathcal{A}z\rangle_Z = -\langle z, z\rangle_Z \qquad \text{for all} \quad z \in D(\mathcal{A})$$

Absent from the conditions of the theorem is the restriction $\mathcal{P} : D(\mathcal{A}) \to D(\mathcal{A})$ and indeed the unique operator $\mathcal{P}$ which satisfies the theorem instead maps $D(A) \to D(A^*)$, with $D(A^*)$ the domain defined by $\mathcal{A}^*$ and which has a structure significantly different than that of $D(A)$. Alternatively, if $X = D(\mathcal{A})$ is a closed subspace of L_2, then we can apply a projection operator to construct a $\mathcal{P}$ such that $\mathcal{P}(X) = X$. However, for X defined in the previous section, while X is compactly embedded in L_2, it is not closed in that space.

V. Dual Conditions for Multiple-Delay Systems

In this section, we translate the results of Section IV into positivity and negativity of Lyapunov-Krasovskii-like functionals for systems with multiple delays. First, we give a class of operators $\mathcal{P}$ which satisfy the conditions of Theorem 1. Specifically, we give a parametrization of operators which are self-adjoint with respect to the Hilbert space $Z_{n,K}$, map $X \to X$ and satisfy $\mathcal{P}(X) = X$. Next, we show how the conditions of Theorem 1 can be applied to this class of operators to obtain stability conditions similar to the primal Lyapunov-Krasovskii conditions presented in Section II. Note that in Section VI, we will apply these results specifically to systems with a single delay and the exposition in that section is significantly reduced.

A. A Parametrization of $\mathcal{P}$ which Satisfies Theorem 1 on $Z_{n,K}$

In this subsection, we parameterize a class of operators which are self-adjoint and map $X \to X$, where recall we have defined the state-space as

$$X := \left\{ \begin{bmatrix} x \\ \phi_i \end{bmatrix} \in Z_{n,K} \; : \; \begin{smallmatrix} \phi_i \in W_2^n[-\tau_i, 0] \text{ and} \\ \phi_i(0) = x \text{ for all } i \in [K] \end{smallmatrix} \right\}.$$

Likewise, recall the inner product on $Z_{m,n,K}$ as

$$\left\langle \begin{bmatrix} y \\ \psi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{m,n,K}} = \tau_K y^T x + \sum_{i=1}^K \int_{-\tau_i}^0 \psi_i(s)^T \phi_i(s) ds.$$

Lastly, recall we consider operators of the form

$$\left(\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}\begin{bmatrix}x\\ \phi_i\end{bmatrix}\right)(s) := \tag{4}$$
$$\begin{bmatrix} Px+\sum_{i=1}^K\int_{-\tau_i}^0 Q_i(s)\phi_i(s)ds \\ \tau_K Q_i(s)^T x+\tau_K S_i(s)\phi_i(s)+\sum_{j=1}^K\int_{-\tau_j}^0 R_{ij}(s,\theta)\phi_j(\theta)\,d\theta.\end{bmatrix}$$

The following lemma gives constraints on the matrix P and functions Q_i, S_i, and R_{ij} for which $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ is self-adjoint and maps $X \to X$.

Lemma 3: Suppose that $S_i \in W_2^{n\times n}[-\tau_i,0]$, $R_{ij} \in W_2^{n\times n}\left[[-\tau_i,0]\times[-\tau_j,0]\right]$ and $S_i(s)=S_i(s)^T$, $R_{ij}(s,\theta)=R_{ji}(\theta,s)^T$, $P=\tau_K Q_i(0)^T+\tau_K S_i(0)$ and $Q_j(s)=R_{ij}(0,s)$ for all $i,j\in[K]$. Then $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ is a bounded linear operator, maps $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}: X\to X$, and is self-adjoint with respect to the inner product defined on $Z_{n,K}$.

Proof: To simplify the presentation, let $\mathcal{P} := \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$. We first establish that $\mathcal{P}: X\to X$. If $\begin{bmatrix}x\\ \phi_i\end{bmatrix}\in X$, then $\phi_i\in\mathcal{C}[-\tau_i,0]$ and $\phi_i(0)=x$. Now if $\begin{bmatrix}y\\ \psi_i(s)\end{bmatrix} = \left(\mathcal{P}\begin{bmatrix}x\\ \phi_i\end{bmatrix}\right)(s)$ then since $P=\tau_K Q_i(0)^T+\tau_K S_i(0)$ and $Q_j(s)=R_{ij}(0,s)$, we have that

$$\begin{aligned}\psi_i(0) &= \tau_K Q_i(0)^T x+\tau_K S_i(0)\phi_i(0)+\sum_{j=1}^K\int_{-\tau_j}^0 R_{ij}(0,\theta)\phi_j(\theta)d\theta\\ &=\Big(\tau_K Q_i(0)^T+\tau_K S_i(0)\Big)x+\sum_{j=1}^K\int_{-\tau_j}^0 R_{ij}(0,\theta)\phi_j(\theta)d\theta\\ &= Px+\sum_{j=1}^K\int_{-\tau_j}^0 Q_j(s)\phi_j(s)ds = y.\end{aligned}$$

Since $S_i \in W_2^{n\times n}[-\tau_i,0]$ and $R_{ij} \in W_2^{n\times n}\left[[-\tau_i,0]\times[-\tau_j,0]\right]$, clearly $\phi_i\in W_2^n[-\tau_i,0]$, and hence we have $\begin{bmatrix}y\\ \psi_i\end{bmatrix}\in X$. This proves that $\mathcal{P}: X\to X$. Furthermore, boundedness of the functions Q_i, S_i and R_{ij} implies boundedness of the linear operator $\mathcal{P}$.

Now, to prove that $\mathcal{P}$ is self-adjoint with respect to the inner product $\langle\cdot,\cdot\rangle_{Z_{n,K}}$, we show $\langle y,\mathcal{P}x\rangle_{Z_{n,K}}=\langle\mathcal{P}y,x\rangle_{Z_{n,K}}$ for any $x,y\in Z_{n,K}$. Using the properties $S_i(s)=S_i(s)^T$ and $R_{ij}(s,\theta)=R_{ji}(\theta,s)^T$, we have the following.

$$\begin{aligned}\left\langle\begin{bmatrix}y\\ \psi_i\end{bmatrix},\mathcal{P}\begin{bmatrix}x\\ \phi_i\end{bmatrix}\right\rangle_{Z_{n,K}} &= \tau_K y^T\left(Px+\sum_{i=1}^K\int_{-\tau_i}^0 Q_i(\theta)\phi_i(\theta)d\theta\right)\\ &\quad+\sum_{i=1}^K\int_{-\tau_i}^0\psi_i(s)^T\Big(\tau_K Q_i(s)^T x+\tau_K S_i(s)\phi_i(s)\\ &\qquad\qquad+\sum_{j=1}^K\int_{-\tau_j}^0 R_{ij}(s,\theta)\phi_j(\theta)d\theta\Big)\\ &=\tau_K\left(Py+\sum_{j=1}^K\int_{-\tau_j}^0 Q_i(s)\psi_j(s)ds\right)^T x\\ &\quad+\sum_{i=1}^K\int_{-\tau_i}^0\Big(\tau_K Q_i(s)^T y+\tau_K S_i(s)^T\psi_i(s)\\ &\qquad\qquad+\sum_{j=1}^K\int_{-\tau_j}^0 R_{ji}(\theta,s)^T\psi_j(\theta)d\theta\Big)^T\phi_i(s)\,ds\\ &=\left\langle\mathcal{P}\begin{bmatrix}y\\ \psi_i\end{bmatrix},\begin{bmatrix}x\\ \phi_i\end{bmatrix}\right\rangle_{Z_{n,K}}\end{aligned}$$

■

Finally, we show that for this class of operators, if $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ is coercive with respect to the L_2 norm, then $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}(X)=X$.

Lemma 4: Suppose that there exist P, Q_i, S_i and R_{ij} which satisfy the conditions of Lemma 3. If $\left\langle x,\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}x\right\rangle_{Z_{n,K}}\ge\epsilon\|x\|^2_{Z_{n,K}}$ for all $x\in X$ and some $\epsilon>0$, then $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}(X)=X$.

Proof: By Lemma 3, $\mathcal{P}$ is self-adjoint and maps $X\to X$. Since $\mathcal{P}$ is coercive, bounded and self-adjoint, $\mathcal{P}^{-1}$ is coercive, bounded and self adjoint. To show $\mathcal{P}(X)=X$, we need only show that $\mathbf{y}=\mathcal{P}\mathbf{x}\in X$ implies that $\mathbf{x}\in X$. First, we show that if $\mathbf{y}=\begin{bmatrix}y\\ \psi_i(\theta)\end{bmatrix}\in X$, then $\mathbf{x}=\begin{bmatrix}x\\ \phi_i(\theta)\end{bmatrix}=\mathcal{P}^{-1}\mathbf{y}$ satisfies $x=\phi(0)$. We proceed by contradiction. Suppose $x-\phi_i(0)\neq 0$ for some i. Then we have

$$y=P(\phi_i(0)+x-\phi_i(0))+\sum_{i=1}^K\int_{-\tau_i}^0 Q_i(s)\phi_i(s)ds.$$

Now, since $\mathbf{y}\in X$, $y=\psi_i(0)$ and hence

$$y=P\phi_i(0)+\sum_{j=1}^K\int_{-\tau_j}^0 R_{ij}(0,\theta)\phi_j(\theta)\,d\theta,$$

which implies $P(x-\phi_i(0))=0$. Now, $\langle\mathbf{x},\mathcal{P}\mathbf{x}\rangle_{Z_{n,K}}\ge\epsilon\|\mathbf{x}\|^2_{Z_{n,K}}$ implies $P\ge\epsilon I$. Hence $x-\phi(0)\neq 0$ implies $P(x-\phi(0))\neq 0$, which is a contradiction. We conclude that $x=\phi_i(0)$. Next, we establish $\phi_i\in W_2$ for any i by showing $\left\|\dot\phi_i\right\|_{L_2}<\infty$. For this, we differentiate ψ_i to obtain

$$\begin{aligned}\dot\psi_i(s)&=\tau_K\dot Q_i(s)^T x+\tau_K\dot S_i(s)\phi_i(s)+\tau_K S_i(s)\dot\phi_i(s)\\ &\quad+\sum_{j=1}^K\int_{-\tau_j}^0\partial_s R_{ij}(s,\theta)\phi_j(\theta)\,d\theta,\end{aligned}$$

which we reverse to obtain

$$\begin{aligned}\tau_K S_i(s)\dot\phi_i(s)&=\dot\psi_i(s)-\tau_K\dot Q_i(s)^T x-\tau_K\dot S_i(s)\phi_i(s)\\ &\quad-\sum_{j=1}^K\int_{-\tau_j}^0\partial_s R_{ij}(s,\theta)\phi_j(\theta)\,d\theta,\end{aligned}$$

which is L_2-bounded since $\dot\psi_i,\phi_i,\dot Q_i\in L_2$, and $\dot S_i$ and $\partial_s R_{i.j}$ are continuous and thus bounded on $[-\tau_i,0]$. Now, for $x=0$ and $\phi_j=0$ for $j\neq i$, the constraint $\langle\mathbf{x},\mathcal{P}\mathbf{x}\rangle_{Z_{n,K}}\ge\epsilon\|\mathbf{x}\|^2_{Z_{n,K}}$, implies

$$\tau_K S_i(s)\phi_i(s)+\int_{-\tau_i}^0 R_{ii}(s,\theta)\phi_i(\theta)\,d\theta$$

is coercive. Thus, since integral operators cannot be coercive for L_2-bounded kernels R_{ii}, we have that $S_i(s)\ge\eta I$ for some $\eta>0$. Therefore, for each i, we conclude $\left\|\dot\phi_i\right\|_{L_2}\le\frac{1}{\eta}\left\|S_i(s)\dot\phi_i(s)\right\|_{L_2}<\infty$. Hence $\mathbf{x}\in X$. We conclude that $\mathcal{P}(X)=X$. ■

B. The Duality Conditions for Multiple Delays

For the multiple-delay case, we apply the operator $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$, with P,Q_i,S_i,R_{ij} satisfying the conditions of Lemma 4 to the dual stability condition in Theorem 1 and eliminate differential operators from the result. This subsection provides additional justification for the unique choice of state space X and Hilbert space $Z_{m,n,K}$ used in this paper. Specifically, elimination of differential operators and reformulation as negativity of a multiplier/integral operator on $Z_{n(K+1),n,K}$ would not be possible using the more classical state and inner product spaces which allow for discontinuities in the state.

Recall the generator, $\mathcal{A}$, is defined as

$$\left(\mathcal{A}\begin{bmatrix} x \\ \phi_i \end{bmatrix}\right)(s) = \begin{bmatrix} A_0 x + \sum_{i=1}^k A_i \phi_i(-\tau_i) \\ \frac{d}{ds}\phi_i(s) \end{bmatrix}.$$

Theorem 5: Suppose that there exist P, Q_i, S_i and R_{ij} which satisfy the conditions of Lemma 3. If $\left\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} x \right\rangle_{Z_{n,K}} \geq \epsilon \|x\|^2$ for all $x \in Z_{n,K}$ and

$$\left\langle \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}} \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n(K+1),n,K}} \leq -\epsilon \left\| \begin{bmatrix} y_1 \\ \phi_i \end{bmatrix} \right\|^2_{Z_{n,K}}$$

for all $y_1 \in \mathbb{R}^n$ and $\begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix} \in Z_{n(K+1),n,K}$ where

$$D_1 := \begin{bmatrix} C_0 + C_0^T & C_1 & \cdots & C_k \\ C_1^T & -S_1(-\tau_1) & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ C_{k^T} & 0 & 0 & -S_k(-\tau_K) \end{bmatrix},$$

$$\begin{aligned}
&C_0 := A_0 P + \tau_K \sum_{i=1}^K (A_i Q_i(-\tau_i)^T + \frac{1}{2} S_i(0)), \\
&C_i := \tau_K A_i S_i(-\tau_i), \quad i \in [K] \\
&V_i(s) := \begin{bmatrix} B_i(s)^T & 0 & \cdots & 0 \end{bmatrix}^T, \quad i \in [K] \\
&B_i(s) := A_0 Q_i(s) + \dot{Q}_i(s) + \sum_{j=1}^K A_j R_{ji}(-\tau_j, s), \quad i \in [K] \\
&G_{ij}(s,\theta) := \frac{\partial}{\partial s} R_{ij}(s,\theta) + \frac{\partial}{\partial \theta} R_{ji}(s,\theta)^T, \quad i,j \in [K],
\end{aligned}$$

then the system defined by Eqn. (1) is exponentially stable.

Proof: Define the operators $\mathcal{A}$ and $\mathcal{P} = \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ as above. By Lemma 3, $\mathcal{P}$ is self-adjoint and maps $X \to X$. Since $\mathcal{P}$ is coercive by assumption, this implies by Theorem 1 and Lemma 4 that the system is exponentially stable if

$$\left\langle \mathcal{A}\mathcal{P} \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} + \left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{A}\mathcal{P} \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} \leq -\epsilon \left\| \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\|^2_{Z_{n,K}}$$

for all $\begin{bmatrix} x \\ \phi_i \end{bmatrix} \in X$. We begin by constructing $\mathcal{A}\mathcal{P}\begin{bmatrix} x \\ \phi_i \end{bmatrix} := \begin{bmatrix} y \\ \psi_i(s) \end{bmatrix}$, where

$$\begin{aligned}
y = A_0 P x &+ \sum_{i=1}^K \int_{-\tau_i}^0 A_0 Q_i(s) \phi_i(s) ds \\
&+ \sum_{i=1}^K A_i \Bigg(\tau_K Q_i(-\tau_i)^T x + \tau_K S_i(-\tau_i) \phi_i(-\tau_i) \\
&\qquad + \sum_{j=1}^K \int_{-\tau_j}^0 R_{ij}(-\tau_i, \theta) \phi_j(\theta) d\theta \Bigg),
\end{aligned}$$

$$\begin{aligned}
\psi_i(s) = \tau_K \dot{Q}_i(s)^T x + \tau_K \dot{S}_i(s) \phi_i(s) + \tau_K S_i(s) \dot{\phi}_i(s) \\
+ \sum_{j=1}^K \int_{-\tau_j}^0 \frac{d}{ds} R_{ij}(s,\theta) \phi_j(\theta) d\theta.
\end{aligned}$$

Now divide the expression into terms as

$$\left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{A}\mathcal{P} \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} := \tau_K x^T y + \sum_{i=1}^K \int_{-\tau_i}^0 \phi_i(s)^T \psi_i(s) ds.$$

Examining the first term and using $x = \phi_i(0)$, we have

$$\begin{aligned}
x^T y = x^T A_0 P x &+ \sum_{i=1}^K \int_{-\tau_i}^0 x^T A_0 Q_i(s) \phi_i(s) ds \\
&+ \sum_{i=1}^K \tau_K x^T A_i Q_i(-\tau_i)^T x + \sum_{i=1}^K \tau_K x^T A_i S_i(-\tau_i) \phi_i(-\tau_i) \\
&+ \sum_{i=1}^K \int_{-\tau_i}^0 \sum_{j=1}^K x^T A_j R_{ji}(-\tau_j, \theta) \phi_i(\theta) d\theta
\end{aligned}$$

Next, we examine the second term and use integration by parts to eliminate $\dot{\phi}$.

$$\begin{aligned}
&\sum_{i=1}^K \int_{-\tau_i}^0 \phi_i(s)^T \psi_i(s) ds = \sum_{i=1}^K \tau_K \int_{-\tau_i}^0 \phi_i(s)^T \dot{Q}_i(s)^T x \, ds \\
&+ \sum_{i=1}^K \tau_K \int_{-\tau_i}^0 \phi_i(s)^T \dot{S}_i(s) \phi_i(s) ds + \sum_{i=1}^K \tau_K \int_{-\tau_i}^0 \phi_i(s)^T S_i(s) \dot{\phi}_i(s) ds \\
&\quad + \sum_{i,j} \int_{-\tau_i}^0 \int_{-\tau_j}^0 \phi_i(s)^T \frac{\partial}{\partial s} R_{ij}(s,\theta) \phi_j(\theta) \, ds \, d\theta \\
&= \sum_{i=1}^K \tau_K \int_{-\tau_i}^0 \phi_i(s)^T \dot{Q}_i(s)^T x \, ds + \frac{\tau_K}{2} \sum_{i=1}^K \int_{-\tau_i}^0 \phi_i(s)^T \dot{S}_i(s) \phi_i(s) ds \\
&\quad + \frac{\tau_K}{2} x^T \sum_{i=1}^K S_i(0) x - \frac{\tau_K}{2} \sum_{i=1}^K \phi_i(-\tau_i)^T S_i(-\tau_i) \phi_i(-\tau_i) \\
&\quad + \sum_{i,j} \int_{-\tau_i}^0 \int_{-\tau_j}^0 \phi_i(s)^T \frac{\partial}{\partial s} R_{ij}(s,\theta) \phi_j(\theta) \, ds \, d\theta.
\end{aligned}$$

Combining both terms, we obtain

$$\begin{aligned}
&\left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{A}\mathcal{P} \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} = \tau_K x^T y + \sum_{i=1}^K \int_{-\tau_i}^0 \phi_i(s)^T \psi_i(s) ds \\
&= x^T \left(\tau_K A_0 P + \sum_{i=1}^K \tau_K^2 A_i Q_i(-\tau_i)^T + \frac{\tau_K}{2} \sum_{i=1}^K S_i(0) \right) x \\
&+ \tau_K^2 \sum_{i=1}^K x^T A_i S_i(-\tau_i) \phi_i(-\tau_i) - \frac{\tau_K}{2} \sum_{i=1}^K \phi_i(-\tau_i)^T S_i(-\tau_i) \phi_i(-\tau_i) \\
&+ \tau_K \sum_{i=1}^K \int_{-\tau_i}^0 x^T \Big(A_0 Q_i(s) + \dot{Q}_i(s) + \sum_{j=1}^K A_j R_{ji}(-\tau_j, s) \Big) \phi_i(s) ds \\
&+ \frac{\tau_K}{2} \sum_{i=1}^K \int_{-\tau_i}^0 \phi_i(s)^T \dot{S}_i(s) \phi_i(s) ds \\
&+ \sum_{i,j} \int_{-\tau_i}^0 \int_{-\tau_j}^0 \phi_i(s)^T \frac{\partial}{\partial s} R_{ij}(s,\theta) \phi_j(\theta) \, ds \, d\theta.
\end{aligned}$$

Combining the expression with its adjoint, we recover

$$\left\langle \mathcal{A}\mathcal{P} \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} + \left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{A}\mathcal{P} \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}}$$

$$= \left\langle \begin{bmatrix} \begin{bmatrix} x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_k(-\tau_K) \end{bmatrix} \\ \phi_i \end{bmatrix}, \mathcal{D} \begin{bmatrix} \begin{bmatrix} x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_k(-\tau_K) \end{bmatrix} \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n(K+1),n,K}} \leq -\epsilon \left\| \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\|^2_{Z_{n,K}},$$

where $\mathcal{D} := \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}}$. We conclude that all conditions of Theorem 1 are satisfied and hence System (1) is stable. ■

Theorem 5 provides stability conditions expressed as positivity of $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ and negativity of the multiplier/integral operator $\mathcal{D} = \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}}$. Note that positivity is defined with respect to the inner product $Z_{m,n,K}$. In Section VII, we will show how to reformulate positivity on $Z_{m,n,K}$ as an equivalent positivity condition using the L_2 inner product. Positive operators with respect to the L_2 inner product can then be parameterized using LMIs as also described in Section VII. Before moving to the next section, we note that the derivative operator $\mathcal{D} = \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}}$ is sparse in the sense that no terms of the form $\phi(-\tau_i)^T\phi_j(-\tau_j)$ for $i \neq j$ or $\phi_i(-\tau_i)^T\phi_i(s)$ for any i appear in $\left\langle \phi, \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}}\phi \right\rangle$. This is extraordinary, as all such terms do appear in the similar formulation of the primal stability conditions (i.e. the $\left\langle \phi, \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}}\phi \right\rangle$ from Section III). To emphasize this difference, we fully expand both versions of the form $\left\langle \phi, \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}}\phi \right\rangle$ to obtain the following.

Dual Lyapunov-Krasovskii Form: Theorem 5 implies that System 1 is stable if there exists a

$$\begin{aligned}V(\phi) &= \tau_K\phi(0)^TP\phi(0) + \tau_K\sum_{i=1}^K\int_{-\tau_i}^0 \phi(0)^TQ_i(s)\phi(s)ds\\ &+ \tau_K\sum_{i=1}^K\int_{-\tau_i}^0\phi(s)^TQ_i(s)^T\phi(0)ds + \tau_K\sum_{i=1}^K\int_{-\tau_i}^0\phi(s)^TS_i(s)\phi(s)ds\\ &+ \sum_{i,j=1}^K\int_{-\tau_i}^0\int_{-\tau_j}^0 \phi(s)^TR_{ij}(s,\theta)\phi(\theta)d\theta,\end{aligned}$$

such that $V(\phi) \geq \epsilon \left\| \begin{bmatrix}\phi(0)\\ \phi_i\end{bmatrix}\right\|^2_{Z_{n,K}}$ and

$$\begin{aligned}V_D(\phi) &= \tau_K\phi(0)^T(C_0+C_0^T)\phi(0)\\ &- \tau_K\sum_{i=1}^K\phi_i(-\tau_i)^TS_i(-\tau_i)\phi_i(-\tau_i) + 2\tau_K\sum_{i=1}^K\phi(0)^TC_i\phi_i(-\tau_i)\\ &+ 2\tau_K\sum_{i=1}^K\int_{-\tau_i}^0 \phi(0)^TB_i(s)\phi_i(s)ds\\ &+ \tau_K\sum_{i=1}^K\int_{-\tau_i}^0 \phi_i(s)^T\dot{S}_i(s)\phi_i(s)ds\\ &+ \sum_{i,j=1}^K\int_{-\tau_i}^0\int_{-\tau_j}^0 \phi_i(s)^TG_{ij}(s,\theta)\phi_i(\theta)\,ds\,d\theta \leq -\epsilon\left\|\begin{bmatrix}\phi(0)\\ \phi_i\end{bmatrix}\right\|^2_{Z_{n,K}}.\end{aligned}$$

Primal Lyapunov-Krasovskii Form: Now, compare with the associated primal classical Lyapunov-Krasovskii derivative condition [21] which states that System 1 is stable if there exists a

$$\begin{aligned}V(\phi) &= \phi(0)^TP\phi(0) + \sum_{i=1}^K\int_{-\tau_i}^0 \phi(0)^TQ_i(s)\phi(s)ds\\ &+ \sum_{i=1}^K\int_{-\tau_i}^0\phi(s)^TQ_i(s)^T\phi(0)ds + \sum_{i=1}^K\int_{-\tau_i}^0\phi(s)^TS_i(s)\phi(s)ds\\ &+ \sum_{i,j=1}^K\int_{-\tau_i}^0\int_{-\tau_j}^0 \phi(s)^TR_{ij}(s,\theta)\phi(\theta)d\theta,\end{aligned}$$

such that $V(\phi) \geq \epsilon\left\|\phi(0)\right\|^2$ and

$$\begin{aligned}\dot{V}(\phi) &= \phi(0)^T\Delta_0\phi(0) + \sum_{i=1}^K\phi_i(-\tau_i)^TS_i(-\tau_i)\phi_i(-\tau_i)\\ &+ 2\sum_{i=1}^K\phi(0)^T\Delta_i\phi_i(-\tau_i) + 2\sum_{i=1}^K\int_{-\tau_i}^0 \phi(0)^T\Pi_{0i}(s)\phi_i(s)ds\\ &+ \sum_{i=1}^K\int_{-\tau_i}^0 \phi_i(s)^T\dot{S}_i(s)\phi_i(s)ds\\ &+ 2\sum_{i,j=1}^K\int_{-\tau_i}^0 \phi_i(-\tau_i)^T\Pi_{ij}(s)\phi_j(s)ds\\ &- \sum_{i,j=1}^K\int_{-\tau_i}^0\int_{-\tau_j}^0 \phi_i(s)^TG_{ij}(s,\theta)\phi_i(\theta)\,ds\,d\theta \leq -\epsilon\left\|\phi(0)\right\|^2.\end{aligned}$$

From this comparison, we see that the structure of the dual stability condition is very similar to the structure of the primal except for the fourth line of the derivative, which is absent from the dual. Roughly speaking, it is as if all the Π_{ij} terms in the primal form have been combined in Π_{0i}. This sparsity pattern yields a multiplier of the form

$$\begin{bmatrix}\cdot & \cdots\\ \vdots & \ddots\end{bmatrix}$$

consisting of a single row, single column, and diagonal. For an example of how to exploit such sparsity, positivity of such a multiplier would be equivalent to positivity of the diagonal and positivity of the scalar $[\cdot] - \cdots \begin{bmatrix}\ddots\end{bmatrix}^{-1} \vdots$

VI. Duality Conditions for Single Delay Systems

In this section, we simplify the results of Section VIII-A for systems with a single delay. We find that in the case of single-delay the parametrization of the operator $\mathcal{P}$ is direct (it does not rely on equality constraints to enforce the mapping conditions of Theorem 1) - which allows us to arrive at the explicit forms described in Subsection I-A.

A. A Parametrization of $\mathcal{P}$ which Satisfies Theorem 1 on $Z_{n,1}$

First, we consider a class of operators which are self-adjoint with respect to Z and map $X \to X$. This is simplified in the case of a single-delay case partially due to the fact that $Z = Z_{n,1} = \mathbb{R}^n \times L_2^n$ equipped with the L_2^{2n} inner product and subspace $X := \{\{x,\phi\} \in \mathbb{R}^n \times W_2^n[-\tau,0] \;:\; \phi(0) = x\}$. Specifically, given functions $S, R \in W_2^{n\times n}[-\tau,0]$, in this section we will define $\mathcal{P}$ as follows.

$$\left(\mathcal{P}\begin{bmatrix}x\\ \phi\end{bmatrix}\right)(s) := \begin{bmatrix}\tau(R(0,0)+S(0))x + \int_{-\tau}^0 R(0,s)\phi(s)ds\\ \tau R(s,0)\phi(0) + \tau S(s)\phi(s) + \int_{-\tau}^0 R(s,\theta)\phi(\theta)d\theta\end{bmatrix} \quad (5)$$

Clearly, we have that $\mathcal{P}$ is a bounded linear operator and since S, R are continuous, it is trivial to show that $\mathcal{P} : X \to X$. Furthermore, $\mathcal{P}$ is self-adjoint with respect to the L_2^{2n} inner product, as indicated in the following lemma.

Lemma 6: Suppose $S \in W_2^{n\times n}[-\tau,0]$, $R \in W_2^{n\times n}[[-\tau,0]\times[-\tau,0]]$, $R(s,\theta) = R(\theta,s)^T$ and $S(s) \in \mathbb{S}^n$.

Then the operator $\mathcal{P}$, as defined in Equation (5), is self-adjoint with respect to the L_2^{2n} inner product. Furthermore, if there exists $\epsilon > 0$ such that $\langle x, \mathcal{P}x\rangle_{L_2^{2n}} \geq \epsilon \|x\|^2$ for all $x \in X$, then $\mathcal{P}(X) = X$.

Proof: The proof is a direct application of Lemma 3. First, we note that $\mathcal{P} = \mathcal{P}_{\{P,Q,S,R\}}$ where $P = \tau(R(0,0) + S(0))$ and $Q(s) = R(0,s)$. Noting that $P = \tau(R(0,0) + S(0)) = \tau Q(0)^T + \tau S(0)$, we see that $\mathcal{P}_{\{P,Q,S,R\}}$ satisfies the conditions of Lemma 3 and hence the proof is complete ■

Note that the constraints $\mathcal{P} : X \to X$ and $\mathcal{P} = \mathcal{P}^*$ significantly reduce the number of free variables. In the single delay case, we could make this explicit by replacing P and Q with $P = \tau(R(0,0) + S(0))$ and $Q(s) = R(0,s)$.

Having introduced a parametrization of $\mathcal{P}$ and established properties of this operator, we now apply this structured operator to Theorem 1 to obtain Lyapunov-like conditions on S and R for which stability holds.

B. Dual Stability Conditions: Single Delay

In this subsection, we specialize the results of Theorem 5 to single-delay systems. First, recall that the dynamics of the single-delay system are represented by the infinitesimal generator, $\mathcal{A}$, defined as

$$\left(\mathcal{A}\begin{bmatrix} x \\ \phi \end{bmatrix}\right)(s) = \begin{bmatrix} A_0 x + A_1\phi(-\tau) \\ \frac{d}{ds}\phi(s) \end{bmatrix}.$$

Then we have the following.

Corollary 7: Suppose S and R satisfy the conditions of Lemma 6 and there exists $\epsilon > 0$ such that

$$\langle x, \mathcal{P}_{\{P,Q,S,R\}}x\rangle_{L_2^{2n}} \geq \epsilon \|x\|^2_{L_2^{2n}}$$

for all $x \in \mathbb{R}^n \times L_2^n[-\tau, 0]$ where $P = \tau(R(0,0) + S(0))$ and $Q(s) = R(0,s)$. Furthermore, suppose

$$\left\langle \begin{bmatrix} x \\ y \\ \phi \end{bmatrix}, \mathcal{D}\begin{bmatrix} x \\ y \\ \phi \end{bmatrix}\right\rangle_{L_2^{3n}} \leq -\epsilon \left\| \begin{bmatrix} x \\ \phi \end{bmatrix}\right\|^2_{L_2^{2n}}$$

for all $\begin{bmatrix} x \\ y \\ \phi \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}^n \times L_2^n[-\tau, 0]$ where $\mathcal{D} = \mathcal{P}_{\{D_1,V,\dot{S},G\}}$ and

$$D_1 := \begin{bmatrix} C_0 + C_0^T & C_1 \\ C_1^T & -S(-\tau) \end{bmatrix}, \quad V(s) = \begin{bmatrix} B(s) \\ 0 \end{bmatrix},$$

$$C_0 := \tau A_0(R(0,0) + S(0)) + \tau A_1 R(-\tau, 0) + \frac{1}{2}S(0),$$

$$C_1 := \tau A_1 S(-\tau), \qquad ,$$

$$B(s) := A_0 R(0,s) + A_1 R(-\tau, s) + \dot{R}(s,0)^T,$$

$$G(s,\theta) := \frac{d}{ds}R(s,\theta) + \frac{d}{d\theta}R(s,\theta).$$

Then the system defined by Equation (1) in the case $K = 1$ with $\tau_1 = \tau$ is exponentially stable.

Proof: The proof is a direct application of Lemma 6 and Theorem 5. ■

Note that expanding the term

$$\left\langle \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi \end{bmatrix}, \mathcal{D}\begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi \end{bmatrix}\right\rangle_{L_2^{3n}}$$

from Corollary 7 yields the new dual stability conditions previously described in Subsection I-A.

VII. Using LMIs to solve LOIs on $Z_{m,n,K}$

In previous sections, we have formulated dual stability conditions, with decision variables parameterized by the matrix P and functions Q_i, S_i, and R_{ij}. The dual stability conditions were reformulated as positivity of

$$\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}x\rangle_{Z_{n,K}} \geq \epsilon \|x\|^2_{Z_{n,K}}$$

for all $x \in Z_{n,K}$ and negativity of

$$\left\langle \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}}\begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix}\right\rangle_{Z_{n(K+1),n,K}} \leq -\epsilon \left\| \begin{bmatrix} y_1 \\ \phi_i \end{bmatrix}\right\|^2_{Z_{n,K}}$$

$y_1 \in \mathbb{R}^n$ and $\begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix} \in Z_{n(K+1,n,K)}$ where $D_1, V_i, \dot{S}_i, G_{ij}$ are as defined in Theorem 5. Operator feasibility conditions of this form are termed Linear Operator Inequalities (LOIs) and in this section we will show how LMIs can be used to solve LOIs under the presumption that the functions Q_i, S_i, and R_{ij} are polynomial (which implies $D_1, V_i, \dot{S}_i, G_{ij}$ are polynomial). Specifically, the variables in this case become the coefficients of the polynomials Q_i, S_i, and R_{ij} and the goal of the section is to find LMI constraints on P and these polynomial coefficients which ensure that

$$\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}x\rangle_{Z_{m,n,K}} \geq 0.$$

Our approach to solving LOIs on $Z_{m,n,K}$ is to construct an equivalent feasibility condition using operators on L_2^{m+n}. For this reason, we now define a new parametrization of operators which map $L_2 \to L_2$. Specifically, for given integrable functions, M and N, we define $\mathcal{P}_{\{M,N\}} : L_2 \to L_2$ as:

$$\left(\mathcal{P}_{\{M,N\}}x\right)(s) := M(s)x(s) + \int_{-\tau_K}^{0} N(s,\theta)x(\theta)d\theta.$$

The strategy, then is a) to find LMI constraints of the coefficients on polynomials M and N which ensure that $\mathcal{P}_{\{M,N\}} \geq 0$; b) Given P, Q_i, S_i, R_{ij}, define a linear map from the coefficients of P, Q_i, S_i, R_{ij} to coefficients of some M and N such that $\mathcal{P}_{\{M,N\}} \geq 0$ if and only if $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \geq 0$.

This strategy is accomplished in 3 parts. First, in Subsection VII-A, we construct polynomials M and N such that $\mathcal{P}_{\{M,N\}}$ is coercive on $\mathbb{R}^m \times L_2^n[-\tau_K, 0]$ if and only if $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ is coercive on $Z_{m,n,K}$. Second, in Subsection VII-B, we convert the problem of positivity on $\mathbb{R}^m \times L_2^n[-\tau_K, 0]$ to positivity on L_2^{n+m} through the use of auxiliary variables called spacing functions. Specifically, we show that $\mathcal{P}_{\{M,N\}}$ is positive on $\mathbb{R}^m \times L_2^n[-\tau_K, 0]$ if and only if $\mathcal{P}_{\{M,N\}} = \mathcal{P}_{\{\hat{M},\hat{N}\}} + \mathcal{T}$ for some operator $\mathcal{P}_{\{\hat{M},\hat{N}\}}$ which is positive on $L_2^{m+n}[-\tau_K, 0]$ and some $\mathcal{T}$ for which $\langle x, \mathcal{T}x\rangle_{L_2} = 0$ for all $x \in \mathbb{R}^m \times L_2^n[-\tau_K, 0]$.

Finally, for the third step, in Subsection VII-C, we impose LMI constraints on the coefficients of these polynomials $\hat{M}$ and $\hat{N}$, constraints which are denoted $\{\hat{M}, \hat{N}\} \in \Xi_{d,n,K}$ and which ensure that $\mathcal{P}_{\{\hat{M},\hat{N}\}} \geq 0$ on $L_2^{m+n}[-\tau_K, 0]$. In addition, in Subsection VII-D, we give linear equality constraints on the coefficients of the polynomials F, G (denoted $\{F, H\} \in \Theta_{m,n,K}$) which ensure $\langle x, \mathcal{T}x\rangle_{L_2} = 0$ for all $x \in \mathbb{R}^m \times L_2^n[-\tau_K, 0]$ where $\mathcal{T} = \mathcal{P}_{\{F,G\}}$.

All steps are combined into a single summarizing statment in Corollary 13.

A. Equivalence between $Z_{m,n,K}$ and $\mathbb{R}^m \times L_2^n[-\tau_K, 0]$

In this subsection, we address positivity of $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ on $Z_{m,n,K}$ by constructing a linear map from the matrix P and coefficients of Q_i, S_i, R_{ij} to the coefficients of new polynomial variables M and N, where the positivity of $\mathcal{P}_{\{M,N\}}$ on $\mathbb{R}^m \times L_2^n[-\tau_K, 0]$ is equivalent to positivity of $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ on $Z_{m,n,K}$.

Given matrix P and polynomials Q_i, S_i, R_{ij}, define the linear map $\mathcal{L}_1$ by

$$\{M, N\} := \mathcal{L}_1(P, Q_i, S_i, R_{ij}) \tag{6}$$

if

$$\begin{aligned} M(s) &= \Big\{ M_i(s) \quad s \in T_i^{i-1} \\ N(s,\theta) &= \Big\{ N_{ij}(s,\theta) \quad s \in T_i^{i-1}, \theta \in T_j^{j-1} \\ M_i(s) &:= \begin{bmatrix} P & \frac{\tau_K}{\sqrt{a_i}} Q_i(\frac{s+\tau_{i-1}}{a_i}) \\ \frac{\tau_K}{\sqrt{a_i}} Q_i(\frac{s+\tau_{i-1}}{a_i})^T & \tau_K S_i(\frac{s+\tau_{i-1}}{a_i}) \end{bmatrix} \\ N_{ij}(s,\theta) &:= \frac{1}{\sqrt{a_i a_j}} R_{ij}\left(\frac{s+\tau_{i-1}}{a_i}, \frac{\theta+\tau_{j-1}}{a_j} \right), \end{aligned}$$

where $a_i = \frac{\tau_i - \tau_{i-1}}{\tau_i}$. Then we have the following result.

Lemma 8: Let $\{M, N\} := \mathcal{L}_1(P, Q_i, S_i, R_{ij})$ and

$$\left(\mathcal{P}_{\{M,N\}}x\right)(s) := M(s)x(s) + \int_{-\tau_K}^{0} N(s,\theta)x(\theta)d\theta.$$

$\langle x, \mathcal{P}_{\{M,N\}}x\rangle_{L_2^{m+n}} \geq \alpha \|x\|^2_{L_2^{m+n}}$ for all $x \in \mathbb{R}^m \times L_2^n[-\tau_K, 0]$ if and only if $\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}x\rangle_{Z_{m,n,K}} \geq \alpha \|x\|^2_{Z_{m,n,K}}$ for all $x \in Z_{m,n,K}$.

Proof: The proof is straightforward. For sufficiency, we have, through a simple change of integration variables,

$$\begin{aligned} &\left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{m,n,K}} \\ &= \tau_K x^T P x + 2\tau_K \sum_{i=1}^{K} \int_{-\tau_i}^{0} x^T Q_i(s)\phi_i(s)ds \\ &\quad + \tau_K \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi_i(s)^T S_i(s)\phi_i(s)ds \\ &\quad + \sum_{i,j=1}^{K} \int_{-\tau_i}^{0} \int_{-\tau_j}^{0} \phi_i(s)^T R_{ij}(s,\theta)\phi_j(\theta)d\theta \\ &= \int_{\tau_K}^{0} x^T P x ds + 2\tau_K \sum_{i=1}^{K} \frac{1}{\sqrt{a_i}} \int_{-\tau_i}^{-\tau_{i-1}} x^T Q_i(s)\hat{\phi}_i(s)ds \\ &\quad + \tau_K \sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \hat{\phi}_i(s)^T S_i(s)\hat{\phi}_i(s)ds \\ &\quad + \sum_{i,j=1}^{K} \frac{1}{\sqrt{a_i a_j}} \int_{-\tau_i}^{-\tau_{i-1}} \int_{-\tau_j}^{-\tau_{j-1}} \hat{\phi}_i(s)^T R_{ij}(s,\theta)\hat{\phi}_j(\theta)d\theta \end{aligned}$$

We conclude that

$$\begin{aligned} &\left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{m,n,K}} \\ &= \int_{-\tau_K}^{0} \begin{bmatrix} x \\ \hat{\phi}(s) \end{bmatrix}^T M(s) \begin{bmatrix} x \\ \hat{\phi}(s) \end{bmatrix} ds \\ &\qquad + \int_{-\tau_K}^{0} \int_{-\tau_K}^{0} \hat{\phi}(s)^T N(s,\theta)\hat{\phi}(s)ds\, d\theta \\ &= \left\langle \begin{bmatrix} x \\ \hat{\phi} \end{bmatrix}, \mathcal{P}_{\{M,N\}} \begin{bmatrix} x \\ \hat{\phi} \end{bmatrix} \right\rangle^2_{L_2^{m+n}} \geq \alpha \left\| \begin{bmatrix} x \\ \hat{\phi} \end{bmatrix} \right\|^2_{L_2^{m+n}} = \alpha \left\| \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\|^2_{Z_{m,n,K}} \end{aligned}$$

where

$$\hat{\phi}(s) = \Big\{ \hat{\phi}_i(s) := \tfrac{1}{\sqrt{a_i}} \phi_i(\tfrac{s+\tau_{i-1}}{a_i}) \quad s \in T_i^{i-1}.$$

Necessity can be shown similarly by reversing the steps and using

$$\phi_i(s) = \sqrt{a_i}\hat{\phi}(a_i s - \tau_{i-1}) \qquad s \in T_i^0$$

∎

Note that if Q_i, S_i and R_{ij} are polynomials whose coefficients are variables in the optimization problem, then the constraint $\{M, N\} = \mathcal{L}_1(P, Q_i, S_i, R_{ij})$ defines a linear equality constraint between the coefficients of Q_i, S_i and R_{ij} and the coefficients of the piecewise-polynomials which define M and N. In the following subsection, we will discuss how to enforce positivity of operators on $\mathbb{R}^m \times L_2^n[-\tau_K, 0]$ defined by piecewise-polynomial multipliers and kernels.

B. Positive Operators on $\mathbb{R}^m \times L_2^n[-\tau_K, 0]$

To parameterize operators which are positive on $\mathbb{R}^m \times L_2^n[-\tau_K, 0]$, we show that any such operator can be decomposed into an operator which is positive on $L_2^{m+n}[-\tau_K, 0]$ and an operator, $\mathcal{T}$, for which $\langle x, \mathcal{T}x\rangle = 0$ for all $x \in \mathbb{R}^m \times L_2^n[-\tau_K, 0]$. In the following two subsections, we will then parameterize the set of operators positive on $L_2^{m+n}[-\tau_K, 0]$ (Theorem 10) and the set of operators for which $\langle x, \mathcal{T}x\rangle = 0$ for all $x \in \mathbb{R}^m \times L_2^n[-\tau_K, 0]$ (Theorem 12).

Theorem 9: Suppose X is a closed subspace of a Hilbert space Z. Then $\langle x, \mathcal{P}x\rangle \geq 0$ for all $x \in X$ if and only if there exist operators $\mathcal{M}$ and $\mathcal{T}$ such that $\mathcal{P} = \mathcal{M} + \mathcal{T}$ and $\langle x, \mathcal{M}x\rangle \geq 0$ for all $x \in Z$ and $\langle x, \mathcal{T}x\rangle = 0$ for all $x \in X$.
The proof of Theorem 9 is trivial and can be found in [23].

Since $\mathbb{R}^m \times L_2^n[-\tau_K, 0]$ is a closed subspace of $L_2^{m+n}[-\tau_K, 0]$, this proposition implies that the class of operators which are positive on $\mathbb{R}^m \times L_2^n[-\tau_K, 0]$ is the direct sum of the cone of operators, $\mathcal{M}$ which are positive on $L_2^{m+n}[-\tau_K, 0]$ and the space of operators, $\mathcal{T}$, which are null on $\mathbb{R}^m \times L_2^n[-\tau_K, 0]$. In the following subsection, we address the problem of parameterizing operators which are positive on $L_2^{m+n}[-\tau_K, 0]$. Then, in Subsection VII-D, we will parameterize operators which are null on $\mathbb{R}^m \times L_2^n[-\tau_K, 0]$.

C. LMI conditions for Positivity of Multiplier and Integral Operators on $L_2^{m+n}[-\tau_K, 0]$

In this subsection, we define LMI-based conditions for positivity of operators of the form

$$\left(\mathcal{P}_{\{M,N\}}x\right)(s) := M(s)x(s) + \int_{-\tau_K}^{0} N(s,\theta)x(\theta)d\theta. \tag{7}$$

where $x \in L_2^{m+n}[-\tau_K, 0]$ and M and N are continuous except possibly for $s, \theta \in \{-\tau_1, \cdots, -\tau_{K-1}\}$.

Our approach to positivity is based on the observation that a positive operator will always have a square root. If we assume that this square root is also of the form of Operator (7) with functions M and N piecewise-polynomial of bounded degree, then the results of this subsection give necessary and sufficient conditions for the positivity of (7). Note that although this assumption is restrictive, it is unclear whether it implies conservatism. For example, while not all positive polynomials are Sum-of-Squares, any positive polynomial can be approximated arbitrarily well in the sup norm on a bounded domain by a polynomial with a polynomial "root". Specifically, the following theorem assumes a square root of the form

$$\left(\mathcal{P}^{\frac{1}{2}} x\right)(s) := Q^{\frac{1}{2}} \begin{bmatrix} \sqrt{g(s)} Y_1(s) \\ \sqrt{g(s)} \int_{-\tau_K}^{0} Y_2(s,\theta) x(\theta) \end{bmatrix}$$

where here $Q^{\frac{1}{2}}$ is a matrix, Y_1 and Y_2 are functions, and g is a function which is positive on the interval $[-\tau_K, 0]$ and is often taken as $g(s) = 1$. The choice of $g(s)$ will be discussed following the theorem statement.

Theorem 10: For any functions $Y_1 : [-\tau_K, 0] \to \mathbb{R}^{m_1 \times n}$ and $Y_2 : [-\tau_K, 0] \times [-\tau_K, 0] \to \mathbb{R}^{m_2 \times n}$, square integrable on $[-\tau_K, 0]$ with $g(s) \geq 0$ for $s \in [-\tau_K, 0]$, suppose that

$$M(s) = g(s) Y_1(s)^T Q_{11} Y_1(s) \tag{8}$$

$$\begin{aligned} N(s,\theta) &= g(s) Y_1(s) Q_{12} Y_2(s,\theta) + g(\theta) Y_2(\theta,s)^T Q_{12}^T Y_1(\theta) \\ &\quad + \int_{-\tau_K}^{0} g(\omega) Y_2(\omega,s)^T Q_{22} Y_2(\omega,\theta)\, d\omega \end{aligned} \tag{9}$$

where $Q_{ij} \in \mathbb{R}^{m_i \times m_j}$ and

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \geq 0.$$

Then for $\mathcal{P}_{\{M,N\}}$ as defined in Equation (7), $\left\langle x, \mathcal{P}_{\{M,N\}} x \right\rangle_{L_2^n} \geq 0$ for all $x \in L_2^n[-\tau_K, 0]$.

Proof: Using the definition of $\mathcal{P}^{\frac{1}{2}}$ introduced above, it is straightforward to show that $\left\langle x, \mathcal{P}_{\{M,N\}} x \right\rangle_{L_2^n} = \left\langle \mathcal{P}^{\frac{1}{2}} x, \mathcal{P}^{\frac{1}{2}} x \right\rangle_{L_2^n} \geq 0$. An extended version of this proof can be found in [23]. ■

Theorem 10 gives a linear parametrization of a cone of positive operators using positive semidefinite matrices. Note that there are few constraints on the vector-valued functions Y_1 and Y_2, functions whose elements are a basis for the multiplier and kernel functions found in the square root of $\mathcal{P}_{\{M,N\}}$. Therefore, if we wish to construct multiplier and kernel functions which are piecewise-continuous, we must choose Y_1 and Y_2 to be a basis for the piecewise-continuous functions. This choice of basis will be addressed shortly.

Inclusion of g is inspired by the Positivstellensatz approach to local positivity of polynomials, as can be found in, e.g. [24], [25], [26]. For example, under mild conditions, Putinar's P-Satz states that a polynomial, $p(x)$, is positive for all $x \in \{x : g(x) \geq 0\}$ if and only if it can be represented as $p(x) = s_1(x) + g(x) s_2(x)$ for some sum-of-squares polynomials s_1, s_2. In this way, Theorem 10 can be seen as an operator-valued version of this classical result. Note, however, in our case g is a function of the variable of integration and not the state and so the analogy is somewhat specious. Furthermore, for this paper, we restrict ourselves to linear maps of the state space. A partial discussion of parametrization of positive nonlinear operators for stability of nonlinear time-delay systems can be found in [27], [23].

We now turn our attention to the choices of Y_1 and Y_2 which yield piecewise-polynomial functions M and N.

a) Piecewise-Polynomials Multipliers and Kernels: To define multipliers and kernels with discontinuities at known points, we divide the region of integration $T_K^0 = [-\tau_K, 0]$ into almost disjoint subregions $T_i^{i-1} = [-\tau_i, -\tau_{i-1}]$, $i \in [K]$ on which continuity holds and assume the functions are polynomial on these subregions. To do this, we introduce the indicator functions (not to be confused with the identity matrix)

$$I_i(t) = \begin{cases} 1 & t \in T_i^{i-1} \\ 0 & \text{otherwise,} \end{cases} \qquad i \in [K]$$

and the vector of indicator functions $J = \begin{bmatrix} I_1 & \cdots & I_K \end{bmatrix}^T$. We can now define the vectors-valued functions Y_1 and Y_2 which are used to define M and N in Theorem 10.

$$Y_{1pc}(s) = Y_{1p}(s) \otimes J(s), \;\; Y_{2pc}(s,\theta) = Y_{2p}(s,\theta) \otimes J(s) \otimes J(\theta)$$

where

$$Y_{1p}(s) = Y_d(s) \otimes I_n, \qquad Y_{2p}(s,\theta) = Y_d(s,\theta) \otimes I_n,$$

where $Y_d(s)$ is a vector whose elements form a basis for the scalar polynomials in variables s of degree d or less. e.g. The vector of monomials. Note that $Y_d(s) \in \mathbb{R}^{d+1}$, hence $Y_{1p}(s) \in \mathbb{R}^{n(d+1) \times n}$, and $Y_{1pc}(s) \in \mathbb{R}^{nK(d+1) \times n}$. Similarly, $Y_d(s,\theta) \in \mathbb{R}^q$ where $q = (d+1)(d+2)/2$, $Y_{2p}(s,\theta) \in \mathbb{R}^{nq \times n}$, and $Y_{2pc}(s,\theta) \in \mathbb{R}^{nKq \times n}$.

Theorem 11: If $Y_1(s) = Y_{1pc}(s)$ and $Y_2(s,\theta) = Y_{2pc}(s,\theta)$ and M and N satisfy the conditions of Theorem 10, then M and N are piecewise-polynomial matrices ($\mathbb{R}^{n \times n}$) of degree $2d$ with possible discontinuities at $s, \theta \in \{-\tau_i\}_i$. In this case, if $g_i(s) \geq 0$ for $s \in T_i^{i-1}$, the functions M and N can be defined piecewise as

$$M(s) = \Big\{ M_i(s) \quad s \in T_i^{i-1}$$

where

$$M_i(s) = g_i(s) Y_d(s)^T Q_{11,ii} Y_d(s)$$

where $Q_{11,i,j} \in \mathbb{R}^{n(d+1) \times n(d+1)}$ is the i,jth block of $Q_{11} \in \mathbb{S}^{n(d+1)K}$. Likewise,

$$N(s,\theta) = \Big\{ N_{ij}(s,\theta) \quad s \in T_i^{i-1},\ \theta \in T_j^{j-1}$$

where

$$\begin{aligned} N_{ij} &= g_i(s) Y_{1p}(s) Q_{12,\substack{i \\ (i-1)K+j}} Y_{1p}(s,\theta) \\ &\quad + g_j(\theta) Y_{2p}(\theta,s)^T Q^T_{12,\substack{(j-1)K+i \\ j}} Y_{1p}(\theta) \\ &+ \sum_{l=1}^{K} \int_{-\tau_l}^{-\tau_{l-1}} g_l(\omega) Y_{2p}(\omega_l, s)^T Q_{22,\substack{i+(l-1)K \\ j+(l-1)K}} Y_{2p}(\omega_l,\theta)\, d\omega_l \end{aligned}$$

where $Q_{12,\substack{i \\ j}} \in \mathbb{R}^{n(d+1) \times nq}$ is the i,jth block of $Q_{12} \in \mathbb{R}^{n(d+1)K \times nqK}$ and $Q_{22,\substack{i \\ j}} \in \mathbb{R}^{nq \times nq}$ is the i,jth block of $Q_{22} \in \mathbb{S}^{nqK}$.

The proof is a straightforward application of Theorem 10, but involves the use of indicator functions and cumbersome notation and hence we simply refer to [23], which is devoted entirely to parametrization of positive operators using positive matrices.

Before moving on to the next subsection, we note that, for the intervals $s \in T_i^{i-1} = [-\tau_i, -\tau_{i-1}]$, we will be using two choices for g_i, both $g_i(s) = 1$ and $g_i = -(s+\tau_i)(s+\tau_{i-1})$. This yields two sets of positive operators which we combine using a direct sum as $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$, where $\mathcal{P}_1$ is defined using the first choice of $g_i = 1$ and $\mathcal{P}_2$ is defined using the second choice $g_i = -(s+\tau_i)(s+\tau_{i-1})$. To simplify notation, throughout the remainder of the paper, we will use the notation $\{M,N\} \in \Xi_{d,n,K}$ to denote the LMI constraints on the coefficients of the polynomials M, N implied by the conditions of Theorem 11 using both $g_i(s) = 1$ and $g_i = -(s+\tau_i)(s+\tau_{i-1})$ as

$$\Xi_{d,n,K} := \left\{ \{M,N\} : \begin{array}{c} M{=}M_1{+}M_2,\ N{=}N_1{+}N_2, \text{ where } \{M_1,N_1\} \\ \text{and } \{M_2,N_2\} \text{ satisfy Thm. 11 with } g_i = 1 \text{ and} \\ g_i = -(s+\tau_i)(s+\tau_{i-1}), \text{ respectively.} \end{array} \right\}$$

D. A Parametrization of Operators where $\langle x, \mathcal{T}x\rangle = 0$ for all $x \in \mathbb{R}^m \times L_2^n[-\tau_K, 0]$

Since we now know how to parameterize operators which are positive on L_2^{n+m}, our last step is to parameterize a set of multiplier/integral operators, $\mathcal{T}$, which satisfy the constraint $\langle x, \mathcal{T}x\rangle = 0$ for all $x \in \mathbb{R}^m \times L_2^n[-\tau_K, 0]$. In prior work, we have referred to the multipliers and kernels which define such operators as "spacing functions".

Theorem 12: Suppose that F and H are defined as

$$F(s) = \begin{bmatrix} L_0(s) + \int\limits_{-\tau_K}^{0}\int\limits_{-\tau_K}^{0} \frac{L_{11}(\omega,t)}{\tau_K} d\omega dt & \int\limits_{-\tau_K}^{0} L_{12}(\omega,s) d\omega \\ \int\limits_{-\tau_K}^{0} L_{21}(s,\omega) d\omega & 0 \end{bmatrix}$$

$$H(s,\theta) = -\begin{bmatrix} L_{11}(s,\theta) & L_{12}(s,\theta) \\ L_{21}(s,\theta) & 0 \end{bmatrix}$$

for some square-integrable functions L_0 and L_{ij} where $L_0(s) \in \mathbb{R}^{m\times m}$, $L_{11}(s,\theta) \in \mathbb{R}^{m\times m}$, and $L_{12}(s,\theta) \in \mathbb{R}^{m\times n}$ such that $\int_{-\tau_K}^0 K(s)ds = 0$. Then if

$$\mathcal{T} := \mathcal{P}_{\{F,H\}}$$

then for any $z \in \mathbb{R}^m \times L_2^n$,

$$\langle z, \mathcal{T}z\rangle_{L_2^{m+n}} = 0$$

Proof: The proof is straightforward. For $z(s) = \begin{bmatrix} c & y(s)\end{bmatrix}^T$ with $c \in \mathbb{R}^m$ and $y \in L_2^n[-\tau_K, 0]$, we have

$$\begin{aligned} \langle z, \mathcal{T}z\rangle_{L_2^{m+n}} &= \int_{-\tau_K}^0 \begin{bmatrix} c \\ y(s)\end{bmatrix}^T F(s) \begin{bmatrix} c \\ y(s)\end{bmatrix} ds \\ &\quad + \int_{-\tau_K}^0\int_{-\tau_K}^0 \begin{bmatrix} c \\ y(s)\end{bmatrix}^T H(s,\theta) \begin{bmatrix} c \\ y(\theta)\end{bmatrix} d\theta ds \\ &= \int_{-\tau_K}^0 \begin{bmatrix} c \\ y(s)\end{bmatrix}^T \begin{bmatrix} \frac{1}{\tau_K}\int_{-\tau_K}^0 L_0(\omega)d\omega & 0 \\ 0 & 0\end{bmatrix} \begin{bmatrix} c \\ y(s)\end{bmatrix} ds \\ &\quad + \int_{-\tau_K}^0\int_{-\tau_K}^0 \begin{bmatrix} c \\ y(s)\end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix} \begin{bmatrix} c \\ y(\theta)\end{bmatrix} d\theta\, ds = 0. \end{aligned}$$

■

For simplicity, we use $\{F,H\} \in \Theta_{m,n,K}$ to denote the conditions of Theorem 12 where K and L_{ij} are piecewise-polynomial matrices.

$$\Theta_{m,n,K} := \{\{F,H\} : F, H \text{ satisfy the conditions of Thm. 12.}\}$$

E. A Summary of Conditions for Positivity on $Z_{m,n,K}$

The following corollary summarizes the main result of this section, combining all subsections.

Corollary 13: Suppose there exist $d \in \mathbb{N}$, constant $\epsilon > 0$, matrix $P \in \mathbb{R}^{m\times m}$, polynomials Q_i, S_i, R_{ij} for $i,j \in [K]$, and $\{F,H\} \in \Theta_{m,n,K}$ such that

$$\mathcal{L}_1(P, Q_i, S_i, R_{ij}) + \{F,H\} \in \Xi_{d,m+n,K}.$$

Then $\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}x\rangle_{Z_{m,n,K}} \geq 0$ for all $x \in Z_{m,n,K}$.

Proof: Define $\{M,N\} = \mathcal{L}_1(P, Q_i, S_i, R_{ij})$, $\{M_0,N_0\} = \{M,N\} + \{F,H\}$, and $\mathcal{T} := \mathcal{P}_{\{F,H\}}$. Since $\{M_0,N_0\} \in \Xi_{d,m+n,K}$, by Theorem 11 $\langle x, \mathcal{P}_{\{M_0,N_0\}}x\rangle_{L_2^{m+n}} \geq 0$ for all $x \in L_2^{m+n}$. Since $\{F,H\} \in \Theta_{m,n,K}$, by Theorem 12, $\langle x, \mathcal{T}x\rangle = 0$ for all $x \in \mathbb{R}^m \times L_2^n[-\tau_K, 0]$. Therefore, since $\mathcal{P}_{\{M,N\}} = \mathcal{P}_{\{M_0,N_0\}} - \mathcal{T}$, by Theorem 9, $\langle x, \mathcal{P}_{\{M,N\}}x\rangle_{L_2^{m+n}} \geq 0$ for all $x \in \mathbb{R}^m \times L_2^n[-\tau_K, 0]$. Finally, since $\{M,N\} = \mathcal{L}_1(P, Q_i, S_i, R_{ij})$, by Lemma 8, $\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}x\rangle_{Z_{m,n,K}} \geq 0$ for all $x \in Z_{m,n,K}$. ■

To simplify presentation, the main results of the following section will reference Corollary 13 instead of the individual theorem statements which it combines.

VIII. An LMI Formulation of the Dual Stability Test

In this section, we apply the positivity conditions developed in Section VII to the operators parameterized in Section V-B, yielding a computational method for verification of the dual stability conditions of Theorem 5 and Corollary 7.

A. An LMI Test for Dual Stability with Multiple Delays

We first consider the case of systems with multiple delays. The variables in the LMI are the matrix P, the coefficients of the polynomial functions Q_i S_i, R_{ij}, and the coefficients of each polynomial term in the piecewise-polynomial functions F_i, H_i, D and E. Polynomial equality constraints (including: $\in \Theta_{n,n,K}$; $\in \Theta_{n(K+1),n,K}$ as per Theorem 10; and the definition of $\mathcal{L}_1$) are equivalent to equality constraints on the coefficients of the respective polynomials. The polynomial constraints $\in \Xi_{d,2n,K}$ and $\in \Xi_{d,n(K+1),K}$ represent LMI constraints on the coefficients of the polynomials as per Theorem 11.

Theorem 14: Suppose there exist $d \in \mathbb{N}$, constant $\epsilon > 0$, matrix $P \in \mathbb{R}^{n\times n}$, polynomials $S_i, Q_i \in W_2^{n\times n}[T_i^0]$, $R_{ij} \in W_2^{n\times n}\left[T_i^0 \times T_j^0\right]$ for $i,j \in [K]$, $\{F_1,H_1\} \in \Theta_{n,n,K}$, and $\{F_2,H_2\} \in \Theta_{n(K+1),n,K}$ such that

$$\{M,N\} + \{F_1,H_1\} \in \Xi_{d,2n,K}$$
$$\{-D,-E\} + \{F_2,H_2\} \in \Xi_{d,n(K+2),K}$$

where

$$\{M,N\} = \mathcal{L}_1(P-\epsilon I_n, Q_i, S_i-\epsilon I_n, R_{ij})$$
$$\{D,E\} = \mathcal{L}_1(D_1+\epsilon \hat{I}, V_i, \dot{S}_i+\epsilon I_n, G_{ij})$$

where $\hat{I} = \mathrm{diag}(I_n, 0_{nK})$, $\mathcal{L}_1$ is as defined in Eqn. (6), and where

$$D_1 := \begin{bmatrix} C_0+C_0^T & C_1 & \cdots & C_k \\ C_1^T & -S_1(-\tau_1) & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ C_k^T & 0 & 0 & -S_k(-\tau_K) \end{bmatrix},$$

$$\begin{aligned} &C_0 := A_0P + \tau_K \sum_{i=1}^{K} (A_iQ_i(-\tau_i)^T + \frac{1}{2}S_i(0)), \\ &C_i := \tau_K A_i S_i(-\tau_i) \qquad i\in[K], \\ &V_i(s) := \begin{bmatrix} B_i(s)^T & 0 & \cdots & 0 \end{bmatrix}^T \qquad i \in [K], \\ &B_i(s) := A_0Q_i(s) + \dot{Q}_i(s) + \sum_{j=1}^{K} R_{ji}(-\tau_j, s) \qquad i\in[K], \\ &G_{ij}(s,\theta) := \frac{\partial}{\partial s}R_{ij}(s,\theta) + \frac{\partial}{\partial \theta}R_{ji}(s,\theta)^T, \quad i,j\in[K]. \end{aligned}$$

Furthermore, suppose

$$\begin{aligned} &P = \tau_K Q_i(0)^T + \tau_K S_i(0) \quad \text{ for } i\in[K], \\ &S_i(s) = S_i(s)^T, \qquad R_{ij}(s,\theta) = R_{ji}(\theta,s)^T \quad \text{ for } i,j\in[K], \\ &Q_j(s) = R_{ij}(0,s) \quad \text{ for } i,j\in[K]. \end{aligned}$$

Then the system defined by Equation (1) is exponentially stable.

Proof: Clearly, $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ satisfies the conditions of Lemma 3. By Corollary 13, we have

$$\begin{aligned} &\left\langle x, \mathcal{P}_{\{P-\epsilon I_n, Q_i, S_i-\epsilon I_n, R_{ij}\}} x\right\rangle_{Z_{n,K}} \\ &= \left\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} x\right\rangle_{Z_{n,K}} - \epsilon \|x\|^2_{Z_{n,K}} \ge 0 \end{aligned}$$

for all $x\in Z_{n,K}$. Similarly, we have

$$\begin{aligned} &\left\langle \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_1+\epsilon\hat{I}, V_i, \dot{S}_i+\epsilon I_n, G_{ij}\}} \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n(K+1),n,K}} \\ &= \left\langle \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_1, V_i, \dot{S}_i, G_{ij}\}} \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n(K+1),n,K}} + \epsilon \left\| \begin{bmatrix} y_1 \\ \phi_i \end{bmatrix} \right\|^2_{Z_{n,K}} \\ &\le 0. \end{aligned}$$

Hence Theorem 5 establishes exponential stability of Equation (1). ∎

B. An LMI for Dual Stability of Single Delay Systems

We now state an LMI representation of the dual stability condition for a single delay ($\tau_1 = \tau_K = \tau$). This is a simplified version of Theorem 14, where we have eliminated the variables P and Q.

Theorem 15: Suppose there exist $d\in\mathbb{N}$, constant $\epsilon>0$, polynomials $S\in W_2^{n\times n}[-\tau,0]$, $R\in W_2^{n\times n}[[-\tau,0]\times[-\tau,0]]$, with $R(s,\theta) = R(\theta,s)^T$ and $S(s)\in\mathbb{S}^n$ such that

$$\begin{aligned} &\{M,N\} + \{F_1,H_1\} \in \Xi_{d,2n,1}, &&\{F_1,H_1\}\in\Theta_{n,n,1}, \\ &\{-D,-E\} + \{F_2,H_2\} \in \Xi_{d,3n,1}, &&\{F_2,H_2\}\in\Theta_{2n,n,1}, \end{aligned}$$

where

$$\begin{aligned} &M(s) = \begin{bmatrix} \tau R(0,0)+\tau S(0) & \tau R(0,s) \\ \tau R(s,0) & \tau S(s) \end{bmatrix} - \epsilon I_{2n}, \\ &N(s,\theta) = \begin{bmatrix} 0_n & 0_n \\ 0_n & R(s,\theta) \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} &D(s) := \begin{bmatrix} D_1 & \tau V(s) \\ \tau V(s)^T & \tau \dot{S}(s) \end{bmatrix} + \epsilon \begin{bmatrix} I_n & 0_n & 0_n \\ 0_n & 0_n & 0_n \\ 0_n & 0_n & I_n \end{bmatrix}, \\ &E(s,\theta) := \begin{bmatrix} 0_{2n} & 0_{2n,n} \\ 0_{n,2n} & G(s,\theta) \end{bmatrix} \\ &D_1 := \begin{bmatrix} C_0+C_0^T & C_1 \\ C_1^T & -S(-\tau) \end{bmatrix}, \quad V(s) = \begin{bmatrix} B(s) \\ 0 \end{bmatrix}, \\ &C_0 := \tau A_0(R(0,0)+S(0)) + \tau A_1 R(-\tau,0) + \frac{1}{2}S(0), \\ &C_1 := \tau A_1 S(-\tau), \\ &B(s) := A_0 R(0,s) + A_1 R(-\tau,s) + \dot{R}(s,0)^T, \\ &G(s,\theta) := \frac{d}{ds}R(s,\theta) + \frac{d}{d\theta}R(s,\theta). \end{aligned}$$

Then the system defined by Equation (1) in the case $K=1$ with $\tau_1=\tau$ is exponentially stable.

Proof: By the definition of M, N, D, and E, we have

$$\{M,N\} = \mathcal{L}_1(\tau R(0,0)+\tau S(0)-\epsilon I_n, R(0,s), S-\epsilon I_n, R)$$
$$\{D,E\} = \mathcal{L}_1(D_1+\epsilon \hat{I}_n, V, \dot{S}+\epsilon I_n, G)$$

where $\hat{I}_n := \begin{bmatrix} I_n & 0_n \\ 0_n & 0_n \end{bmatrix}$. By Corollary 13, therefore,

$$\begin{aligned} &\left\langle x, \mathcal{P}_{\{P-\epsilon I_n, Q, S-\epsilon I_n, R\}} x\right\rangle_{Z_{n,1}} = \left\langle x, \mathcal{P}_{\{P,Q,S,R\}} x\right\rangle_{Z_{n,1}} - \epsilon \|x\|^2_{Z_{n,1}} \\ &= \left\langle x, \mathcal{P}_{\{P,Q,S,R\}} x\right\rangle_{L_2^{2n}} - \epsilon\|x\|^2_{L_2^{2n}} \ge 0 \end{aligned}$$

for all $x\in Z_{n,1} = \mathbb{R}^n \times L_2^n[-\tau,0]$. Similarly,

$$\left\langle \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi \end{bmatrix}, \mathcal{P}_{\{D_1,V,\dot{S},G\}} \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi \end{bmatrix} \right\rangle_{L_2^{3n}} + \epsilon \left\| \begin{bmatrix} y_1 \\ \phi \end{bmatrix} \right\|^2_{L_2^{2n}} \le 0$$

Therefore, by Corollary 7, the system defined by Equation (1) in the case $K=1$ with $\tau_1=\tau$ is exponentially stable. ∎

IX. A Matlab Toolbox Implementation

To assist with the application of these results, we have created a library of functions for verifying the stability conditions described in this paper. These libraries make use of modified versions of the SOSTOOLS [28] and MULTIPOLY toolboxes coupled with either SeDuMi [29] or Mosek. A complete package can be downloaded from [30] or [31]. Key examples of functions included are:

1) `[M,N]=sosjointpos_mat_ker_ndelay.m`
   - Declares a positive piecewise-polynomial multiplier, kernel pair which satisfies $[M,N]\in\Xi_{d,n,K}$.

2) `sosmateq.m`
   - Declare a matrix-valued equality constraint.
3) `[F,H]=sosspacing_mat_ker_ndelay.m`
   - Declare a matrix-valued equality constraint which satisfies $\{F, H\} \in \Theta_{n,n,K}$.

The functions are implemented within the pvar framework of SOSTOOLS and the user must have some familiarity with this relatively intuitive language to utilize these functions. Note also that the entire toolbox and supporting modified implementations of SOSTOOLS and MULTIPOLY must be added to the path for these functions to execute.

b) Pseudocode: To illustrate how these conditions can be efficiently coded using the Matlab toolbox, we give a pseudocode implmentation of the conditions of Theorem 14.

1) `[M,N]=sosjointpos_mat_ker_ndelay`
2) `[F1,H1]=sosspacing_mat_ker_ndelay`
3) `[D,E]=L(M+F1, N+H1)`
4) `[Q,R]=sosjointpos_mat_ker_ndelay`
5) `[F2,H2]=sosspacing_mat_ker_ndelay`
6) `sosmateq(D+F2+Q)`
7) `sosmateq(E+H2+R)`

Here we use the function L to represent the map $\mathcal{L}_1$. An optimized version of the code is contained in `solver_ndelay_nd_dual_joint.m`.

X. Numerical Validation

In the preceding Sections, we proposed a sufficient condition for stability. However, as discussed, this condition is not necessary and there are several potential sources of conservatism, including the constraint $\mathcal{P}(X) = X$ and the assumption of a SOS representation of the positive operator. In this section, we apply the dual stability condition to a battery of numerical examples in order to determine whether this potential conservatism is significant.

In each case, a table is given which lists the maximum provably stable value of a specified parameter for each degree d. This maximum value is found using bisection on the parameter. In each case d is increased until the maximum parameter value converges to several decimal places. The true maximum is also provided as either the “limit” or “analytic” value, depending on whether this limiting value is known analytically or is a best estimate based on simulation. The computation time is also listed in CPU seconds on an Intel i7-5960X 3.0GHz processor. This time corresponds to the interior-point (IPM) iteration in SeDuMi and does not account for preprocessing, postprocessing, or for the time spent on polynomial manipulations formulating the SDP using SOSTOOLS. Such polynomial manipulations can significantly exceed SDP computation time.

c) Example A: First, we consider a simple example which is known to be stable for $\tau \le \frac{\pi}{2}$.

$$\dot{x}(t) = -x(t-\tau)$$

d	1	2	3	4	analytic
$\tau_{\max}$	1.408	1.5707	1.5707	1.5707	1.5707
CPU sec	.18	.21	.25	.47	

d) Example B: Next, we consider a well-studied 2-dimensional, single delay system.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & .1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau)$$

d	1	2	3	4	limit
$\tau_{\max}$	1.6581	1.716	1.7178	1.7178	1.7178
$\tau_{\min}$	.10019	.10018	.10017	.10017	.10017
CPU sec	.25	.344	.678	1.725	

e) Example C: We consider a scalar, two-delay system.

$$\dot{x}(t) = ax(t) + bx(t-1) + cx(t-2)$$

In this case, we fix $a = -2, c = -1$ and search for the maximum b, which is 3 [32], [33], [34].

d	1	2	3	4	analytic
$b_{\max}$	.7071	2.5895	2.9981	2.9982	3
CPU sec	.3	.976	2.77	12.96	

f) Example D: We consider a 2-D, 2-delay system where $\tau_1 = \tau_2/2$ and search for the maximum stable τ_2.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & .1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x(t-\tau/2) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau)$$

d	1	2	3	4	limit
$\tau_{\max}$	1.33	1.371	1.3717	1.3718	1.372
CPU sec	2.13	6.29	24.45	79.0	

g) Example E: Finally, to illustrate computational scaling with state, we consider a 4-dimensional, one-delay delayed static output feedback system considered in [35]. This example considers the static feedback system

$$\dot{x}(t) = (A - BKC)x(t) + (A + BKC)x(t-\tau),$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & 10 & 0 & 0 \\ 5 & -15 & 0 & -.25 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

In this case, we take $K = 1$. This problem is known to be particularly challenging for SOS techniques and it has been reported that it requires a degree 10 polynomial even in the primal case to prove stability of $h = 3$. Using the dual stability condition, we find a stability proof for degree $d = 6$, perhaps due to the use of the improved parametrization of positive operators from [23]. The computation times for increasing degrees are listed in the following table.

d	1	2	3	4	
CPU sec	1.45	5.99	24.78	63.21	

These numerical examples indicate little, if any conservatism in the LMI implementation of the dual stability conditions and moreover, the method is accurate for relatively low degree. Beyond these examples, to illustrate computational scaling and show that the results hold for non-trivial examples, the algorithm was applied to a 10-state problem, terminating successfully in 22s, and a 20-state problem, terminating successfully in 951s.

XI. An LMI Controllability Test

Establishment of dual stability conditions is the first step in developing full-state feedback controller synthesis conditions. To obtain the stabilizing controller requires two more steps. Specifically, consider the system $\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t)$, where $u(t) \in \mathbb{R}^m$. First, we define the controllability test.

Theorem 16: Suppose there exist $d \in \mathbb{N}$, constant $\epsilon > 0$, matrix $P \in \mathbb{R}^{n\times n}$, polynomials $S_i, Q_i \in W_2^{n\times n}[T_i^0]$, $R_{ij} \in W_2^{n\times n}\left[T_i^0 \times T_j^0\right]$ for $i,j \in [K]$, $\{F_1, H_1\} \in \Theta_{n,n,K}$, $\{F_2, H_2\} \in \Theta_{n(K+1),n,K}$, $W_i \in \mathbb{R}^{m\times n}$ and polynomials $Y_i \in W_2^{m\times n}$ for $i \in [K]$ such that

$$\{M,N\} + \{F_1, H_1\} \in \Xi_{d,2n,K}$$
$$\{-D,-E\} + \{F_2, H_2\} \in \Xi_{d,n(K+2),K}$$

where

$$\{M,N\} = \mathcal{L}_1(P - \epsilon I_n, Q_i, S_i - \epsilon I_n, R_{ij})$$
$$\{D,E\} = \mathcal{L}_1(D_1 + W + \epsilon \hat{I}, V_i + BY_i, \dot{S}_i + \epsilon I_n, G_{ij})$$

where $\hat{I}$, D_1, V_i, G_{ij} are as defined in Theorem 14, $\mathcal{L}_1$ is as defined in Eqn. (6), and

$$W = \begin{bmatrix} BW_0 + W_0^T B^T & BW_1 & \dots & BW_K \\ W_1^T B^T & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 \\ W_K B^T & 0 & 0 & 0 \end{bmatrix}$$

Furthermore, suppose P, Q_i, S_i, R_{ij} satisfy the conditions of Lemma 3. Then the system $\dot{x}(t) = A_0 x(t) = \sum_i A_i x(t-\tau_i) + Bu(t)$ is exponentially stabilizable and $u(t) = \mathcal{Z}\mathcal{P}^{-1}x(t)$ is an exponentially stabilizing controller where

$$\left(\mathcal{Z}\begin{bmatrix} x \\ \phi_i \end{bmatrix}\right)(s) := W_0 x + \sum_{i=1}^K W_i \phi_i(-\tau_i) + \sum_{i=1}^K \int_{-\tau_i}^0 Y_i(s)\phi_i(s)ds.$$

Proof: If $u(t) = \mathcal{Z}\mathcal{P}^{-1}x(t)$, then $\dot{x}(t) = \left(\mathcal{A} + \mathcal{B}\mathcal{Z}\mathcal{P}^{-1}\right)x(t)$ where $(\mathcal{B}u)(s) = \begin{bmatrix} Bu(t) \\ 0 \end{bmatrix}$. Hence as in Theorem 5, the closed loop system is stable if

$$\begin{aligned}
&\left\langle \left(\mathcal{A} + \mathcal{B}\mathcal{Z}\mathcal{P}^{-1}\right)\mathcal{P}\begin{bmatrix} x \\ \phi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix}\right\rangle_{Z_{n,K}} \\
&+ \left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \left(\mathcal{A} + \mathcal{B}\mathcal{Z}\mathcal{P}^{-1}\right)\mathcal{P}\begin{bmatrix} x \\ \phi_i \end{bmatrix}\right\rangle_{Z_{n,K}} \\
&= \left\langle \mathcal{A}\mathcal{P}\begin{bmatrix} x \\ \phi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix}\right\rangle_{Z_{n,K}} + \left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{A}\mathcal{P}\begin{bmatrix} x \\ \phi_i \end{bmatrix}\right\rangle_{Z_{n,K}} \\
&+ \left\langle \mathcal{B}\mathcal{Z}\begin{bmatrix} x \\ \phi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix}\right\rangle_{Z_{n,K}} + \left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{B}\mathcal{Z}\begin{bmatrix} x \\ \phi_i \end{bmatrix}\right\rangle_{Z_{n,K}} \\
&= \left\langle \begin{bmatrix} \begin{bmatrix} x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_k(-\tau_K) \end{bmatrix} \\ \phi_i \end{bmatrix}, \mathcal{D} + \mathcal{D}_Z \begin{bmatrix} \begin{bmatrix} x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_k(-\tau_K) \end{bmatrix} \\ \phi_i \end{bmatrix}\right\rangle_{Z_{n(K+1),n,K}} \\
&\le -\epsilon \left\| \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\|_{Z_{n,K}}^2
\end{aligned}$$

for all $\begin{bmatrix} x \\ \phi_i \end{bmatrix} \in X$ where

$$\mathcal{D}_Z := \mathcal{P}_{\{W, BY_i, 0, 0\}} \quad \text{and} \quad \mathcal{D} := \mathcal{P}_{\{D_1, V_i, \dot{S}_i, G_{ij}\}}.$$

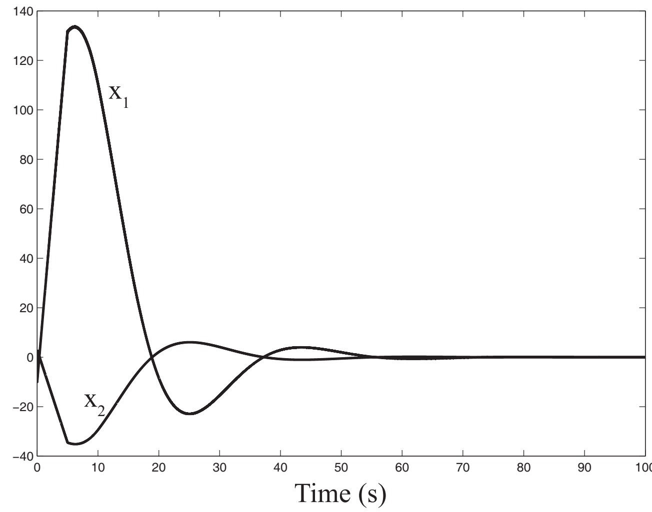

Fig. 1. A Matlab DDE23 simulation of System (10) with Controller (11) and delay $\tau = 5s$.

Now, from Corollary 13, we have

$$\mathcal{P}_{\{D_1 + W + \epsilon\hat{I}, V_i + BY_i, \dot{S}_i + \epsilon I_n, G_{ij}\}} \le 0$$

and hence

$$\begin{aligned}
&\left\langle \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_1 + W + \epsilon\hat{I}, V_i + BY_i, \dot{S}_i + \epsilon I_n, G_{ij}\}} \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix}\right\rangle_{Z_{n(K+1),n,K}} \\
&= \left\langle \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix}, (\mathcal{D} + \mathcal{D}_Z) \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix}\right\rangle_{Z_{n(K+1),n,K}} + \epsilon \left\| \begin{bmatrix} y_1 \\ \phi_i \end{bmatrix} \right\|_{Z_{n,K}}^2 \le 0.
\end{aligned}$$

Therefore, by Theorem 5, the closed-loop system is exponentially stable. ∎

The second step in controller synthesis is construction of the stabilizing controller $u(t) = \mathcal{Z}\mathcal{P}^{-1}_{\{P,Q_i,S_i,R_{ij}\}}$, which requires inversion of the operator $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ - a topic we do not address in this paper. For the single-delay system, an analytic expression for this inverse can be found in [16]. In the multiple-delay case, iterative methods can be used, as were introduced in [15]. We illustrate these results in the single delay case using the well-studied system

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -2 & -.5 \\ 0 & -1 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t). \quad (10)$$

For $\tau = 5$ using simple degree 2 polynomials, we obtained the following exponentially stabilizing controller.

$$\begin{aligned}
u(t) &= \begin{bmatrix} -3601 \\ -944 \end{bmatrix}^T x(t) + \begin{bmatrix} -.00891 \\ .872 \end{bmatrix}^T x(t-\tau) \qquad (11) \\
&+ \int_{-5}^0 \begin{bmatrix} 52.1 + 6.98s + .00839s^2 - .0710s^3 \\ 12.7 + 1.50s - .0407s^2 - .0190s^3 \end{bmatrix}^T x(t+s)ds
\end{aligned}$$

Simulations for fixed initial conditions were performed and can be seen in Figure 1.

XII. Conclusion

We have proposed a new form of dual Lyapunov stability condition which allows convexification of the controller synthesis problem for delayed and other infinite-dimensional systems. This dual principle requires a Lyapunov operator which is positive, invertible, self-adjoint and preserves the structure of the state-space. We have proposed such a class of operators and used them to create stability conditions which can be expressed as positivity and negativity of quadratic Lyapunov functions. These dual stability conditions have a tri-diagonal structure which is distinct from standard Lyapunov-Krasovskii forms and may be exploited to increase performance when studying systems with large numbers of delays. The dual stability condition is presented in a format which can be adapted to many existing computational methods for Lyapunov stability analysis. We have applied the Sum-of-Squares approach to enforce positivity of the quadratic forms and tested the stability condition in both the single and multiple-delay cases. Numerical testing on several examples indicates the method is not likely to be conservative. The contribution of the present paper is not in the efficiency of the stability test, however, as these are likely less efficient when compared to e.g., previous SOS results, due to the structural constraints imposed upon the operator. Rather the contribution is in the convexification of the synthesis problem which opens the door for dynamic output-feedback H_∞ synthesis for infinite-dimensional systems. This potential is demonstrated in the numerical example of controller synthesis for a single-delay system.

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