A Duality-Based Convex Framework for the Control of Coupled Differential-Difference Equations

Matthew M. Peet
Arizona State University
Tempe, AZ USA

joint work with Guoying Miao and Keqin Gu
International Federation of Automatic Control 2017
Toulouse, France

July 11, 2017

Control of Coupled Differential Difference Equations

Consider a MIMO Linear Differential-Difference Equation.

$$\dot{x}(t) = Ax(t) + By(t-r) + Fu, \qquad (1)$$

$$y(t) = Cx(t) + Dy(t-r), \qquad (2)$$

Stability Analysis of linear discrete-delay systems is a CLOSED PROBLEM.

- Lets move on to optimal control.

We would like to use *asymptotic algorithms* to design controllers for this system.
Recall:

- LMIs optimize positive matrices
- SOS optimizes positive polynomials

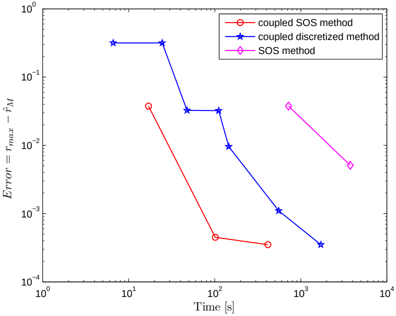

Figure: Comparison of asymptotic algorithms for maximum stable delay

Full-State Feedback Control of **ODE** Systems

Our Template is the LMI Framework

The goal is to find $K \in \mathbb{R}^{m \times n}$ such that

$$\dot{x} = Ax + Bu, \qquad u = Kx \qquad\qquad \text{is Stable}$$

Step 1: DUALITY says the following are equivalent for fixed A, B, K:

1. $\exists P > 0$ such that $P(A + BK) + (A + BK)^T P < 0$.
2. $\exists Q > 0$ such that $(A + BK)Q + Q(A + BK)^T < 0$.

Step 2: Variable Substitution - Define variable $Z = KQ$. The Synthesis condition becomes

$$AQ + BZ + QA^T + Z^T B^T < 0 \qquad Q > 0, \quad Z \in \mathbb{R}^{m \times n}$$

Step 3: Controller Reconstruction. Given solution Q, Z, the controller is

$$K = ZQ^{-1}$$

In this Paper:

A Linear Operator Inequality (LOI) Framework for Synthesis

MAIN IDEA: Replace the Word MATRIX with **OPERATOR**.

An Operator Differential Equation:

$$\dot{x} = \mathcal{A}x + \mathcal{B}u, \qquad u = \mathcal{K}x,$$

- $\mathcal{A} : \underbrace{W^2}_{X} \to \underbrace{L_2}_{Z}$ and $\mathcal{B} : \underbrace{\mathbb{R}^m}_{U} \to \underbrace{L_2}_{Z}$, $\mathcal{K} : \underbrace{W^2}_{X} \to \underbrace{\mathbb{R}^m}_{U}$ are **OPERATORS**.
- We CAN Optimize Operators - Linear Operator Inequalities (LOIs).

Primal Stability (No Feedback): $\mathcal{A}$ is exp. stable iff [Curtain, Zwart] there exists a $\mathcal{P} > 0$

$$\langle x, \mathcal{P}\mathcal{A}x \rangle_Z + \langle \mathcal{P}\mathcal{A}x, x \rangle_Z < 0 \qquad \forall x \in X$$

The First Main Result is Duality: $\mathcal{A}$ is exp. stable if there exists a $\mathcal{Q} : X \to X$ such that $\mathcal{Q} > 0$ and

$$\langle x, \mathcal{A}\mathcal{Q}x \rangle + \langle \mathcal{A}\mathcal{Q}x, x \rangle < 0$$

Other Main Results:

- Solving LOIs with SDP
- Reconstructing $\mathcal{K}$ (Inverting the Controller).

The Duality Theorem

Formal Statement. Applies to any Strongly Continuous Semigroup

Theorem 1.

Suppose that $\mathcal{A}$ generates a strongly continuous semigroup on Hilbert space Z with domain X. Further suppose there exists an $\epsilon > 0$ and a bounded, coercive linear operator $\mathcal{P} : X \to X$ with $\mathcal{P}(X) = X$ and which is self-adjoint with respect to the Z inner product and satisfies

$$\langle \mathcal{A}\mathcal{P}z, z\rangle_Z + \langle z, \mathcal{A}\mathcal{P}z\rangle_Z \le -\epsilon \|z\|_Z^2$$

for all $z \in X$. Then $\dot{x}(t) = \mathcal{A}x(t)$ generates an exponentially stable semigroup.

Key Restriction: $\mathcal{P} : X \to X$. **Not Conservative?**

- When X is a Hilbert Subspace of Z.
- But this is not true for Delay systems.

So now we have An LOI for Controller Synthesis!!!
Find Q, Z such that $\mathcal{Q} : X \to X$,

$$\langle (\mathcal{A}\mathcal{Q} + \mathcal{B}\mathcal{Z})x, x\rangle + \langle x, (\mathcal{A}\mathcal{Q} + \mathcal{B}\mathcal{Z})x\rangle_Z < 0 \qquad \mathcal{Q} > 0, \quad \mathcal{Z} \in \mathcal{L}(X, U)$$

Question: How to Solve LOIs????

What is an LOI

And How do I solve one?

First Rule of LOIs: NO DISCRETIZATION

Formal Definition:
An LOI is a TUPLE $(Z, X, \mathbb{P}, \mathcal{H}, \mathcal{G})$ which defines the feasibility problem: Find $\mathcal{P} \in \mathbb{P}$ such that

$$\mathcal{H}\mathcal{P}\mathcal{G} + (\mathcal{H}\mathcal{P}\mathcal{G})^* > 0, \qquad \mathcal{P} \in \mathbb{P}$$

where the inequality is shorthand for

$$\langle \mathcal{H}\mathcal{P}\mathcal{G}x, x\rangle_Z + \langle x, \mathcal{H}\mathcal{P}\mathcal{G}x\rangle_Z \geq 0 \qquad \text{for all } x \in X \subset Z$$

The key features of an LOI are

1. Inner Product Space Z is an inner-product space (the meaning of ≥ 0).
2. State Space $X \subset Z$ quantifies "for all $x \in X$".
3. Variables The operator $\mathcal{P}$ is constrained to lie in set $\mathbb{P}$.
4. Data $\mathcal{H}$ and $\mathcal{G}$ are operators and may be unbounded.
5. Well-posed Given $\mathcal{H}$ and $\mathcal{G}$, the inner product $\langle x, \mathcal{H}\mathcal{P}\mathcal{G}x\rangle_Z$ must be well-defined for all $\mathcal{P} \in \mathbb{P}$ and $x \in X$.

Applying the LOI Framework to Delay Systems

Represent

$$\begin{aligned}\dot{x} &= Ax + By(t-r) + Fu(t),\\ y(t) &= Cx(t) + Dy(t-r).\end{aligned}$$

as **An Operator Differential Equation:**

$$\dot{x} = \mathcal{A}x + \mathcal{B}u, \qquad u = \mathcal{K}x,$$

In this case

$$\mathcal{A}\begin{bmatrix} x \\ y_t \end{bmatrix} := \begin{bmatrix} Ax + By(t-r) \\ \frac{d}{ds}y_t(s) \end{bmatrix}, \qquad (\mathcal{B}u)(s) := \begin{bmatrix} Fu \\ 0 \end{bmatrix}.$$

Furthermore, we define the inner product and state spaces as

$$Z := L_2^{m+n}[-r,0]$$

$$X := \left\{ \begin{bmatrix} \psi \\ \phi \end{bmatrix} \in \mathbb{R}^m \times L_2^n[-r,0] \,\middle|\, \dot{\phi}(s) \in L_2, \phi(0) = C\psi + D\phi(-r) \right\},$$

Find $\mathcal{Q}, \mathcal{Z} \in \mathbb{P}$ such that $\mathcal{Q} > 0$ and

$$\langle (\mathcal{A}\mathcal{Q} + \mathcal{B}\mathcal{Z})x, x\rangle_{L_2} + \langle x, (\mathcal{A}\mathcal{Q} + \mathcal{B}\mathcal{Z})x\rangle_{L_2} < 0 \qquad \forall x \in X$$

Solving LOIs

The Operators $\mathbb{P}$: The Variables. Our Dual LOI uses operators $\mathbb{P}$ of the form

$$\left(\mathcal{P}_{\{P,Q,S,R\}}\begin{bmatrix}\psi\\ \phi\end{bmatrix}\right)(s)=\begin{bmatrix}P\psi+\int_{-r}^{0}Q(s)\phi(s)ds\\ rQ^T(s)\psi+\int_{-r}^{0}R(s,\theta)\phi(\theta)d\theta+S(s)\phi(s)\end{bmatrix},$$

Where:

- Polynomials $P, Q(s), S(s), R(s,\theta)$ parameterize the operator.
- Real numbers parameterize the polynomials if we restrict the degree to $\leq d$.

Steps To Solving an LOI:

1. Reduce your LOI to one which has already been solved.
2. Done

Question: Which LOIs are Solveable?

We CAN solve LOIs on $X = L_2[-r_K, 0]$ using SDP

We can solve tuples of the following form (Z, X, $\mathbb{P}$, $\mathcal{H}$, $\mathcal{G}$)

1. $Z = L_2$
2. $X = L_2$
3. $\mathcal{P} \in \mathbb{P} := \{\mathcal{P} : (\mathcal{P}_{M,N}x)(s) := M(s)x(s) + \int_{-r}^{0} N(s,\theta)x(\theta)d\theta.\}$ where M, N are piecewise Polynomial
4. $H, G \in \mathbb{P}$

Then $\mathcal{H}\mathcal{P}\mathcal{G} \in \mathbb{P}$, and we can test whether $\mathcal{P}_{M,N} > 0$

Theorem 2.

For any functions Y_1, Y_2, let

$$M(s) = Y_1(s)^T Q_{11} Y_1(s)$$

$$N(s,\theta) = Y_1(s)Q_{12}Y_2(s,\theta) + Y_2(\theta,s)^T Q_{12}^T Y_1(\theta) + \int_{-r}^{0} Y_2(\omega,s)^T Q_{22} Y_2(\omega,\theta)\, d\omega$$

where $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \geq 0$. *Then* $\langle \mathcal{P}_{M,N}x, x\rangle_{L_2} > 0$ *for all* $x \in L_2^n[-r, 0]$.

Equivalence: $(L_2,\ \mathbb{R}^m \times L_2^n,\ \mathcal{P}\in\mathbb{P},\ \mathcal{P}>0)$ is feasible iff

$$(L_2, L_2^{m+n}, (\mathcal{P}\in\mathbb{P},\ \mathcal{T}\in\mathbb{T}), \mathcal{P}+\mathcal{T}>0)$$

is feasible, where $\mathbb{T} := \{\mathcal{T} \,:\, \langle x, \mathcal{T}x\rangle = 0,\ \forall x \in \mathbb{R}^m \times L_2^n\}$.

$\mathbb{T}$ can be parameterized as:

$$\mathbb{T} := \Bigg\{\mathcal{P}_{F,H} \;:\; \text{such that for some functions } K, L_{11}, L_{12}, L_{21},$$

$$F(s) = \begin{bmatrix} K(s) + \int_{-r}^{0}\int_{-r}^{0} \frac{L_{11}(\omega,t)}{r} d\omega dt & \int_{-r}^{0} L_{12}(\omega,s)d\omega \\ \int_{-r}^{0} L_{21}(s,\omega)d\omega & 0 \end{bmatrix}$$

$$H(s,\theta) = -\begin{bmatrix} L_{11}(s,\theta) & L_{12}(s,\theta) \\ L_{21}(s,\theta) & 0 \end{bmatrix}, \qquad \int_{-r}^{0} K(s)ds = 0\Bigg\}$$

Illustration: Primal Stability of Time-Delay Systems

Theorem: Stability of

$$\dot{x} = Ax + By(t-r), \qquad y(t) = Cx(t) + Dy(t-r).$$

is equivalent to existence of P, Q, S, R such that

$$\mathcal{P}_{\{P-\epsilon I, Q, S, R\}} \geq 0 \qquad \text{and} \qquad \mathcal{P}_{\{D_1, V, -\dot{S}, G\}} < 0$$

where

$$D_1 := \begin{bmatrix} \Delta_0 + \epsilon I & \Delta_1 \\ \Delta_1^T & D^T S(0) D - S(-r) \end{bmatrix}$$

$$\Delta_0 = A^T P + PA + Q(0)C + C^T Q(0)^T + C^T S(0) C,$$

$$\Delta_1 = PB + Q(0)D - Q(-r) + C^T S(0) D,$$

$$V(s) = \begin{bmatrix} A^T Q(s) - \dot{Q}(s) + C^T R^T(s,0) \\ B^T Q(s) + D^T R^T(s,0) + R^T(s,-r) \end{bmatrix},$$

$$G(s,\theta) = -\frac{\partial}{\partial s} R_{ij}(s,\theta) - \frac{\partial}{\partial \theta} R_{ij}(s,\theta).$$

In Lyapunov Form: $V = \langle x, \mathcal{P}_{\{P,Q,S,R\}} x \rangle_{L_2} \geq 0$ for all $x \in X$ and $\dot{V}(x) = \langle z, \mathcal{P}_{\{D_1, V, -\dot{S}, G\}} z \rangle_{L_2} \leq 0$ for all $z \in \mathbb{R}^{n+m} \times L_2^n$.

How to ensure $\mathcal{P}(X) = X$

Recall we have operators of the form

$$\left(\mathcal{P}_{\{P,Q,S,R\}}\begin{bmatrix}x\\ \phi\end{bmatrix}\right)(s) := \begin{bmatrix} Px + \int_{-r}^0 Q(s)\phi(s)ds \\ rQ(s)^T x + rS(s)\phi(s) + \int_{-r}^0 R(s,\theta)\phi(\theta)\, d\theta. \end{bmatrix}$$

with

$$X := \left\{ \begin{bmatrix}x\\ \phi\end{bmatrix} \in \mathbb{R}^m \times L_2^n[-r,0] \,:\, \begin{matrix}\phi\in W_2^n[-r,0] \text{ and} \\ \phi(0)=Cx+D\phi(-r) \text{ for all}\end{matrix} \right\}.$$

Lemma 3.

Suppose that S, R are polynomial ,

$$rQ^T(0) + rS(0)C = CP + rDQ^T(-r), \tag{3}$$

$$R(0,s) = CQ(s) + DR(-r,s), \quad \forall s, \tag{4}$$

$$DS(-r) = rS(0)D. \tag{5}$$

Then $\mathcal{P}_{\{P,Q,S,R\}}(X) = X$.

Dual Stability Theorem for Time-Delay Systems

Theorem 4.

The system

$$\dot{x} = Ax + By(t-r), \qquad y(t) = Cx(t) + Dy(t-r)$$

is stable if there exist P, Q, S, R *such that* $\mathcal{P}_{\{P,Q,S,R\}}(X) = X$ *and*

$$\mathcal{P}_{\{P,Q,S,R\}} \geq 0 \qquad \textit{and} \qquad \mathcal{P}_{\{D_1,V,\dot{S},G\}} < 0.$$

Where

$$D_1 := \begin{bmatrix} D_{11} + D_{11}^T & D_{12} \\ D_{12}^T & -S(-r) + D^T S(0) D \end{bmatrix},$$

$$D_{11} := AP + r(BQ(-r)^T + \frac{1}{2}C^T S(0) C), \qquad D_{12} := rBS(-r) + C^T S(0) D,$$

$$V(s) := \begin{bmatrix} AQ(s) + \dot{Q}(s) + BR(-r,s) \\ 0 \end{bmatrix}, \qquad G(s,\theta) := \frac{\partial}{\partial s} R(s,\theta) + \frac{\partial}{\partial \theta} R(s,\theta)^T.$$

In Case you are NOT sold on LOIs

A Dual Lyapunov-Krasovskii (Old-School) Formulation for Single Delay Case

$\dot{x} = Ax + By(t-r),\ y(t) = Cx(t) + Dy(t-r)$ is stable if there exist P, Q, R, S such that $\mathcal{P}_{\{P,Q,S,R\}}(X) = X$ and

$$V(\phi) = \int_{-r}^{0} \begin{bmatrix} x \\ \phi(s) \end{bmatrix}^T \begin{bmatrix} P & rQ(s) \\ rQ(s) & rS(s) \end{bmatrix} \begin{bmatrix} x \\ \phi(s) \end{bmatrix} ds + \int_{-r}^{0}\int_{-r}^{0} \phi(s)^T R(s,\theta)\phi(\theta) d\theta ds \geq \left\| \begin{bmatrix} x \\ \phi \end{bmatrix} \right\|^2$$

and

$$V_D(\phi) = \int_{-r}^{0} \begin{bmatrix} x \\ \phi(-r) \\ \phi(s) \end{bmatrix}^T \begin{bmatrix} D_{11} + D_{11}^T & D_{12} & rD_{13}(s) \\ D_{12}^T & -S(-r) + D^T S(0) D & 0_n \\ rD_{13}(s)^T & 0_n & r\dot{S}(s) \end{bmatrix} \begin{bmatrix} x \\ \phi(-r) \\ \phi(s) \end{bmatrix} ds + \int_{-r}^{0}\int_{-r}^{0} \phi(s)^T \left(\frac{d}{ds} R(s,\theta) + \frac{d}{d\theta} R(s,\theta) \right) \phi(\theta) d\theta ds \leq -\epsilon \left\| \begin{bmatrix} x \\ \phi \end{bmatrix} \right\|.$$

where

$$D_{11} := AP + r(BQ(-r)^T + \frac{1}{2} C^T S(0) C),$$

$$D_{12} := rBS(-r) + C^T S(0) D, \qquad D_{13}(s) := AQ(s) + \dot{Q}(s) + BR(-r,s).$$

IMPORTANT: V_D is NOT the derivative of V!!!

Compare with the Primal L-K Formulation

Note Reduced Sparsity

$\dot x = Ax + By(t-r),\ y(t) = Cx(t) + Dy(t-r)$ is stable if there exist P, Q, R, S such that

$$V(\phi) = \int_{-r}^{0} \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix}^T \begin{bmatrix} P & rQ(s)\\ rQ(s)^T & rS(s)\end{bmatrix} \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix} ds + r\int_{-r}^{0}\int_{-r}^{0} \phi(s)^T R(s,\theta)\phi(\theta) d\theta ds \ge \left\| [\phi] \right\|^2$$

and

$$\dot V(\phi) = \int_{-r}^{0} \begin{bmatrix}\phi(0)\\ \phi(-r)\\ \phi(s)\end{bmatrix}^T \begin{bmatrix} D_{11}+D_{11}^T & D_{12} & rD_{13}(s)\\ D_{12}^T & D^T S(0) D - S(-r) & rD_{23}(s)\\ rD_{13}(s)^T & rD_{23}(s)^T & -r\dot S(s)\end{bmatrix} \begin{bmatrix}\phi(0)\\ \phi(-r)\\ \phi(s)\end{bmatrix} ds - r\int_{-r}^{0}\int_{-r}^{0} \phi(s)^T \left(\frac{d}{ds}N(s,\theta) + \frac{d}{d\theta}N(s,\theta)\right)\phi(\theta) d\theta ds \le -\epsilon \left\| [\phi(0)] \right\|^2 .$$

where

$$D_{11} = PA + Q(0)C + \frac{1}{2}C^T S(0) C, \quad D_{12} = PB - Q(-r) + Q(0)D + C^T S(0) D,$$

$$D_{23} = B^T Q(s) + D^T R(0,s) + R(-r,s), \quad D_{13} = A^T Q(s) - \dot Q(s) + C^T R(0,s).$$

Complexity and Accuracy of Dual Stability Conditions

$$\dot{x}(t) = -x(t-\tau)$$

d	1	2	3	4	analytic
τ_{max}	1.408	1.5707	1.5707	1.5707	1.5707
CPU sec	.18	.21	.25	.47	

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & .1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau)$$

d	1	2	3	4	limit
τ_{max}	1.6581	1.716	1.7178	1.7178	1.7178
τ_{min}	.10019	.10018	.10017	.10017	.10017
CPU sec	.25	.344	.678	1.725	

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & .1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x(t-\tau/2) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau)$$

d	1	2	3	4	limit
τ_{max}	1.33	1.371	1.3717	1.3718	1.372
CPU sec	2.13	6.29	24.45	79.0	

$$\dot{x}(t) = (A-BKC)x(t) + (A+BKC)x(t-\tau),$$

where $K = 1$, $\tau = 3$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & 10 & 0 & 0 \\ 5 & -15 & 0 & -.25 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

d	1	2	3	4
CPU sec	1.45	5.99	24.78	63.21

Complexity Scaling Results: Single Delay Case

- **10 State Example (d=2):** 22s
- **20 State Example (d=2):** 951s

Further reduction possible using Differential-Difference Formulation.

Now Recall Our ODE Roadmap

The goal is to find $K \in \mathbb{R}^{m\times n}$ such that

$$\dot{x} = Ax + Bu, \qquad u = Kx \qquad\qquad \text{is Stable}$$

Step 1: DUALITY says the following are equivalent for fixed A, B, K:

1. $\exists P > 0$ such that $P(A+BK) + (A+BK)^T P < 0$.
2. $\exists Q > 0$ such that $(A+BK)Q + Q(A+BK)^T < 0$.

Step 2: Variable Substitution - Define variable $Z = KQ$. The Synthesis condition becomes

$$AQ + BZ + QA^T + Z^T B^T < 0 \qquad Q > 0, \quad Z \in \mathbb{R}^{m\times n}$$

Step 3: Controller Reconstruction. Given solution Q, Z, the controller is

$$K = ZQ^{-1}$$

Recall the Controller Synthesis LOI

Find $\mathcal{P}$, $\mathcal{Z}$ such that $\mathcal{P}(X) = X$, $\mathcal{P} > 0$

$$\begin{aligned}\langle \mathcal{A}\mathcal{P}x, x\rangle + \langle x, \mathcal{A}\mathcal{P}x\rangle + \langle \mathcal{B}\mathcal{K}\mathcal{P}x, x\rangle + \langle x, \mathcal{B}\mathcal{K}\mathcal{P}x\rangle \\ = \langle \hat{x}, P_{\{D_1, V, \dot{S}, G\}}\hat{x}\rangle + \langle \mathcal{B}\mathcal{Z}x, x\rangle + \langle x, \mathcal{B}\mathcal{Z}x\rangle < 0\end{aligned}$$

We already discussed $\langle x, \mathcal{D}x\rangle$. Now examine the new variable $\mathcal{Z} = \mathcal{K}\mathcal{P}$.

- Since $\mathcal{B}$ is not differential, it helps to let $\mathcal{K}$ have the form

$$\left(\mathcal{K}\begin{bmatrix} x \\ \phi \end{bmatrix}\right)(s) = K_0 x + K_1\phi(-\tau) + \int_{-\tau}^{0} K_2(s)\phi(s)ds,$$

- Then if $\mathcal{Z} = \mathcal{K}\mathcal{P}$, we can prove that $\mathcal{Z}$ has the form

$$\left(\mathcal{Z}\begin{bmatrix} x \\ \phi \end{bmatrix}\right)(s) = Z_0 x + Z_1\phi(-\tau) + \int_{-\tau}^{0} Z_2(s)\phi(s)ds,$$

$\mathcal{B}$ is simply $(\mathcal{B}u)(s)$

$$\left(\mathcal{B}\mathcal{Z}\begin{bmatrix} x \\ \phi \end{bmatrix}\right)(s) = \begin{bmatrix} FZ_0 x + FZ_1\phi(-\tau) + \int_{-\tau}^{0} FZ_2(s)\phi(s)ds \\ 0 \end{bmatrix}$$

Full-State Feedback Controllers

Theorem 5.

The System

$$\dot{x}(t) = Ax(t) + By(t-r) + Fu(t), \qquad (6)$$

$$y(t) = Cx(t) + Dy(t-r), \qquad (7)$$

is full-state-feedback stabilizable if there exist P, Q, S, R, Z_0, Z_1 *and* Z_2 *such that*

$$\mathcal{P}_{\{P,Q,S,R\}} \geq 0 \qquad \text{and} \qquad \mathcal{P}_{\{D_1,V,\dot{S},G\}} + \mathcal{P}_{\{L_1,L_2,0,0\}} < 0$$

where D_1, V, G are as previously defined and

$$L_1 = \begin{bmatrix} FZ_0 + (FZ_0)^T & FZ_1 \\ (FZ_1)^T & 0 \end{bmatrix}, \qquad L_2 = \begin{bmatrix} rFZ_2(s) \\ 0 \end{bmatrix}$$

As a Lyapunov function

$$V_D(x) = \underbrace{\langle \mathcal{A}\mathcal{P}x, x\rangle + \langle x, \mathcal{A}\mathcal{P}x\rangle_{L_2}}_{\langle \hat{x}, \mathcal{P}_{\{D_1,V,\dot{S},G\}}\hat{x}\rangle} + \underbrace{\langle \mathcal{B}\mathcal{Z}x, x\rangle + \langle x, \mathcal{B}\mathcal{Z}x\rangle_{L_2}}_{\langle \hat{x}, \mathcal{P}_{\{L_1,L_2,0,0\}}\hat{x}\rangle_{L_2}}$$

$$= \langle \hat{x}, \mathcal{P}_{\{D_1,V,\dot{S},G\}}\hat{x}\rangle + \langle \hat{x}, \mathcal{P}_{\{L_1,L_2,0,0\}}\hat{x}\rangle_Z$$

Again Recall Our ODE Roadmap

The goal is to find $K \in \mathbb{R}^{m\times n}$ such that

$$\dot{x} = Ax + Bu, \qquad u = Kx \qquad\qquad \text{is Stable}$$

Step 1: DUALITY says the following are equivalent for fixed A, B, K:

1. $\exists P > 0$ such that $P(A+BK) + (A+BK)^T P < 0$.
2. $\exists Q > 0$ such that $(A+BK)Q + Q(A+BK)^T < 0$.

Step 2: Variable Substitution - Define variable $Z = KQ$. The Synthesis condition becomes

$$AQ + BZ + QA^T + Z^T B^T < 0 \qquad Q > 0, \quad Z \in \mathbb{R}^{m\times n}$$

Step 3: Controller Reconstruction. Given solution Q, Z, the controller is

$$K = ZQ^{-1}$$

Analytic Formula for Operator Inversion [Significant!!!]

Suppose $\mathcal{P} > 0$ where

$$\begin{aligned}\mathcal{P}\begin{bmatrix}\psi\\ \phi\end{bmatrix}(s) &= \begin{bmatrix} P\psi + \int_{-r}^{0} Q(\theta)\phi(\theta)d\theta \\ rQ^T(s)\psi + \int_{-r}^{0} R(s,\theta)\phi(\theta)d\theta + S(s)\phi(s)\end{bmatrix}\\ R(s,\theta) &= Y^T(s)\Gamma Y(\theta), \qquad Q(s) = HY(s),\end{aligned}$$

Then the inverse $\mathcal{P}^{-1}$ is given by

$$\mathcal{P}^{-1}\begin{bmatrix}\psi\\ \phi\end{bmatrix}(s) = \begin{bmatrix} \hat{P}\psi + \int_{-r}^{0} \hat{Q}(\theta)\phi(\theta)d\theta \\ r\hat{Q}^T(s)\psi + \hat{S}(s)\phi(s) + \int_{-r}^{0} \hat{R}(s,\theta)\phi(\theta)d\theta\end{bmatrix},$$

where $\hat{R}(s,\theta)$, $\hat{Q}(\theta)$ and $\hat{S}(s)$ are given as follows

$$\begin{aligned}\hat{R}(s,\theta) &= \hat{Y}^T(s)\hat{\Gamma}\hat{Y}(\theta),\\ \hat{Q}(\theta) &= \hat{H}\hat{Y}(\theta), \qquad \hat{S}(s) = S^{-1}(s), \qquad \hat{Y}(s) = Y(s)S^{-1}(s)\\ \hat{H} &= -P^{-1}HT, \qquad \hat{P} = [I + rP^{-1}HTKH^T]P^{-1},\\ \hat{\Gamma} &= [rT^TH^TP^{-1}H - \Gamma](I + K\Gamma)^{-1}, \quad T = (I + K\Gamma - rKH^TP^{-1}H)^{-1},\end{aligned}$$

where $K = \int_{-r}^{0} Y(s)S^{-1}(s)Y^T(s)ds$,

A Full-State Feedback Controller

Finally, we recover the controller as

$$u(t) = K_0 x(t) + K_1 y(t-r) + \int_{-r}^{0} K_2(s) y(t+s) ds$$

where

$$\begin{aligned}
K_0 &= Z_0 \hat{P} + r Z_1 \hat{Q}^T(-r) + r \int_{-r}^{0} Z_2(s) \hat{Q}^T(s) ds, \\
K_1 &= Z_1 \hat{S}(-r), \\
K_2(s) &= Z_0 \hat{Q}(s) + Z_1 \hat{R}(-r, s) + Z_2(s) \hat{S}(s) + \int_{-r}^{0} Z_2(\theta) \hat{R}(\theta, s) d\theta.
\end{aligned}$$

Note: This is *Full-State* Feedback.

- Contrast with output feedback: $u(t) = Kx(t)$ or $u(t) = Ky(t-r)$.

Response: Design an Observer.

- Ongoing Research.

Conclusion: YOU can do controller synthesis!!!

$\dot{x} = Ax + By(t-r) + Fu(t),\ y(t) = Cx(t) + Dy(t-r)$ is stabilizable if there exist $P, Q, R, S, Z_0, Z_1, Z_2$ such that

$$\int_{-r}^{0} \begin{bmatrix} x \\ \phi(s) \end{bmatrix}^T \begin{bmatrix} P & rQ(s) \\ rQ(s) & rS(s) \end{bmatrix} \begin{bmatrix} x \\ \phi(s) \end{bmatrix} ds + \int_{-r}^{0}\int_{-r}^{0} \phi(s)^T R(s,\theta)\phi(\theta) d\theta ds \geq \epsilon \left\| \begin{bmatrix} x \\ \phi \end{bmatrix} \right\|^2$$

and

$$\int_{-r}^{0} \begin{bmatrix} x \\ \phi(-r) \\ \phi(s) \end{bmatrix}^T \begin{bmatrix} D_{11} + D_{11}^T & D_{12} & rD_{13}(s) \\ D_{12}^T & -S(-r) + D^T S(0) D & 0_n \\ rD_{13}(s)^T & 0_n & r\dot{S}(s) \end{bmatrix} \begin{bmatrix} x \\ \phi(-r) \\ \phi(s) \end{bmatrix} ds$$

$$+ \int_{-r}^{0}\int_{-r}^{0} \phi(s)^T \left(\frac{d}{ds} R(s,\theta) + \frac{d}{d\theta} R(s,\theta) \right) \phi(\theta) d\theta ds \leq -\epsilon \left\| \begin{bmatrix} x \\ \phi \end{bmatrix} \right\|.$$

where

$$D_{11} := AP + r(BQ(-r)^T + \frac{1}{2} C^T S(0) C) + FZ_0,$$
$$D_{12} := rBS(-r) + C^T S(0) D + FZ_1,$$
$$D_{13}(s) := AQ(s) + \dot{Q}(s) + BR(-r,s) + FZ_2(s),$$
$$rQ^T(0) + rS(0)C = CP + rDQ^T(-r), \quad R(0,s) = CQ(s) + DR(-r,s),$$
$$DS(-r) = rS(0)D.$$

Full-state Feedback Controller: Numerical Example

Consider a numerical example.

$$\dot{x}(t) = \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0.1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} y(t-r)$$

$$+ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} -0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x(t),$$

Results: Unstable without controller for any delay.

- Computation time 3s.
- No delay for which we cannot find a controller.

Numerical Example

Using a value of $r = 1.6s$, we compute the following controller:

$$u(t) = \begin{bmatrix}-1.874 & 2.232 & -0.830 & 3.099 & 0.030 & -1.033\end{bmatrix} x(t) + \begin{bmatrix}-0.239 \\ -0.343\end{bmatrix}^T y(t-1.6)$$

$$+ \int_{-1.6}^{0} \begin{bmatrix}-0.246 + 0.221s + 0.122s^2 - 0.012s^3 - 0.032s^4 \\ 0.238 - 0.398s + +0.007s^2 + 0.037s^3 + 0.010s^4\end{bmatrix}^T y(t+s)ds$$

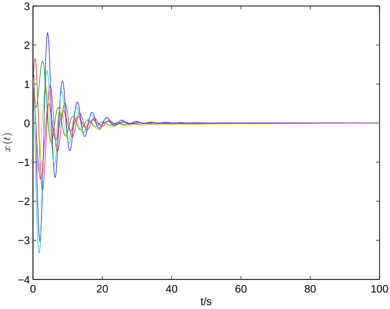

Figure: Trajectory of a delayed system ($r = 1.6s$) with full-state feedback

Conclusions:

- A Dual approach to controller synthesis
  - Convexifies the problem
  - Can be applied to any Lyapunov-Krasovskii-based approach.
  - NOT limited to SOS.
- Practical Implications
  - First numerical solution to **Full-State Feedback** of multi-state delayed systems.
  - No Analytic Solution to operator inversion in multi-delay case.

Numerical Code Produced:

- LOI Toolbox
  - Packaged as DelayTools
  - But limited Functionality
  - Can declare L_2-positive operator variables.

Available for download at
`http://control.asu.edu`

- Next Talk:
  - Observer-Based Controller Synthesis
  - Preliminary Work by Guoying