Chapter 1
Inversion of Separable Kernel Operator and Its Application in Control Synthesis

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Abstract In this chapter, we show how the problem of controller synthesis can be posed as a form of convex optimization in an operator-theoretic framework. Furthermore, we show how to: a) Parameterize the integral and multiplier operator-valued decision variables using finite-dimensional vectors; b) Verify and enforce positivity and negativity of multiplier and integral operators using positive matrices; c) Invert positive integral and multiplier operators through the use of a new formula based on algebraic manipulation. Finally, we show how these 3 parts can be combined into a computational procedure for finding a stabilizing state-feedback controller for systems defined by differential-difference equations - a class which includes differential systems with discrete delays. Finally, a numerical example is used to illustrate the form of the resulting stabilizing controller.

1.1 Introduction

A necessary and sufficient condition for stability of coupled differential-difference equations (of which delay-differential equations is a special case) is the existence of a so-called "complete quadratic" Lyapunov-Krasovskii functional. Such functionals have the form $V = \langle x, \mathscr{P}x \rangle$ where $\mathscr{P}$ is defined by a combination of multiplier and integral operators, the inner product is defined on L_2, and the state,

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$x \in \mathbb{R}^n \times L_2[-\tau,0]^m$ is a combination of the present state and memory of certain delayed channels. The first numerical algorithm to use SDP to parameterize and optimize over the set of complete-quadratic functionals was the discretized Lyapunov-Krasovskii functional method in [1], later refined in [2], wherein the multiplier and kernel of the Lyapunov-Krasovskii functional was assumed to be piecewise linear. A more recent SDP-based approach to parameterizing and optimizing there functionals is the Sum-Of-Squares (SOS) method presented in [3]. In the SOS method, the multiplier and kernel are parameterized by polynomials.

In this chapter, we focus on the problem of full-state feedback controller synthesis for differential-difference equations of the form

$$\dot{x} = Ax + By(t-r) + Fu(t),$$
$$y(t) = Cx(t) + Dy(t-r),$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $F \in \mathbb{R}^{n\times p}$, $D \in \mathbb{R}^{m\times m}$ and $r > 0$ is the delay. For most practical systems, the number of delayed channels is significantly smaller than the total number of state variables ($m << n$). For such systems, the complexity of the complete-quadratic functional associated with the differential-difference form is significantly lower than that associated with the delay-differential framework. By exploiting this reduced functional, the complexity of numerical algorithms such as the discretized and SOS approaches can be significantly reduced—resulting in more efficient and accurate tests for stability. Such reductions were explored and documented for the discretized Lyapunov-Krasovskii functional approach in [4], [5] and [12], and for the SOS formulation in [6].

Full-state feedback controller synthesis means that $u(t) = \mathscr{K}\begin{bmatrix} x(t) \\ y_t \end{bmatrix}$ where y_t is the history of $y(t)$ on the interval $[t-r,t]$ and $\mathscr{K} : \mathbb{R}^n \times L_2[-r,0]^m \to \mathbb{R}^p$. That is, the feedback controller uses knowledge of the entire state in determining the input. This is in contrast to most work on controller synthesis for time-delay systems which only use more directly measurable parts of the state such as $x(t)$ to determine the control input. The use of subsets of the state for controller synthesis are best classified as output feedback. However, the use of state-estimation for delayed systems in the output-feedback framework has not hitherto been explored.

More generally, conditions for control synthesis of delayed systems based on the complete quadratic Lyapunov-Krasovskii functional is still rare. An early example is [8], in which a more limited class of Lyapunov-Krasovskii functional is used, and some parameter constraints are imposed. Recently, a synthesis based on the inverse of kernel operator associated with the Lyapunov-Krasovskii functional for time-delay systems of retarded type in the SOS formulation was developed in [9] and [10]. This chapter extends this method to coupled differential-functional equations. The inverse operator is derived using a direct algebraic approach rather than the series expansion approach in [9] and [10]. The basic idea of such synthesis is outlined as follows.

Consider the coupled differential functional equations either in closed-loop or autonomous form:

$$\dot{x}(t) = Ax(t) + By(t-r) + \int_{-r}^{0} H(\theta)y(t+\theta)d\theta, \tag{1.1}$$

$$y(t) = Cx(t) + Dy(t-r), \tag{1.2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $H(\theta) \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{m\times m}$ and $r > 0$. Eqns. (1.1)-(1.2) have a unique solution $(x(t), y(t))$ for any initial condition

$$x(0) = \psi \in \mathbb{R}^n, \qquad y(t) = \phi(t) \quad \forall t \in [-r, 0] \qquad \phi = \phi \in \mathscr{PC}(r,m).$$

Here $\mathscr{PC}(r,m)$ represents the set of piecewise continuous functions from $[-r,0]$ to $\mathbb{R}^m$. For any piecewise-continuous function $y(t)$ and $\tau \in \mathbb{R}^+$, we define $y_\tau \in \mathscr{PC}(r,m)$ by $y_\tau(\theta) = y(\tau+\theta)$, $-r \le \theta \le 0$. Let

$$Z := \mathbb{R}^n \times \mathscr{PC}(r,m). \tag{1.3}$$

Then any initial condition $(\psi, \phi) \in Z$ uniquely defines a solution, which may be represented by a strongly continuous semigroup (C_0-semigroup) $\mathscr{S} : Z \to Z$,

$$z(t) = \mathscr{S}(t-\tau)z(\tau). \tag{1.4}$$

Solutions of System (1.1)-(1.2) satisfy an abstract differential equation on Z,

$$\dot{z} = \mathscr{A}z, \tag{1.5}$$

where $\mathscr{A}$ is the infinitesimal generator of the C_0-semigroup $\mathscr{S}$.

Stability of solutions of the system defined by Eqns. (1.1)-(1.2) is equivalent to the existence of a quadratic Lyapunov-Krasovskii functional of the form

$$V(z) = < z, \mathscr{P}z >, \tag{1.6}$$

where $\mathscr{P}$ is a self-adjoint operator, and $\langle \cdot, \cdot \rangle$ represents the inner product on L_2 [13]. The system is stable if $\mathscr{P}$ is coercive, and its derivative along the system trajectory

$$\dot{V}(z) = < z, (\mathscr{P}\mathscr{A} + \mathscr{A}^*\mathscr{P})z >,$$

is negative definite in the sense that $-\mathscr{P}\mathscr{A} - \mathscr{A}^*\mathscr{P}$ is coercive, where $\mathscr{A}^*$ is the adjoint operator of $\mathscr{A}$. Stability analysis is therefore equivalent to the existence of an operator $\mathscr{P}$ which satisfies the above conditions. Then, by using positive matrices to parameterize a suitably rich cone of positive operators, as described in [11] or [2], and by observing that the operator $\mathscr{P}$ appears linearly in $V(z)$ and $\dot{V}(z)$, the problem of stability analysis can be reduced to a semi-definite programming problem. Specifically, the methods defined in [11] consider the case where the operators are the combination of multiplier and integral operators with polynomial multipliers and kernels. The approach in [2] uses piecewise linear multipliers and kernels.

For the problem of controller synthesis, however, we are searching for both a Lyapunov operator $\mathscr{P}$ and a feedback operator $\mathscr{K}$. This poses a challenge in that, while we can parameterize both operators, the resulting expression for $\dot{V}(z)$ is bilinear in $\mathscr{K}$ and $\mathscr{P}$ and is hence non-convex. To illustrate this, consider a system with input $u(t)$ described by the abstract differential equation

$$\dot{z} = \mathscr{A}z + \mathscr{F}u. \tag{1.7}$$

If we want to design a linear feedback control in the form of

$$u = \mathscr{K}z, \tag{1.8}$$

such that the closed-loop system is stable, and use the Lyapunov-Krasovskii functional given in (1.6), then its derivative becomes

$$\dot{V}(z) =< z, (\mathscr{P}\mathscr{A} + \mathscr{A}^*\mathscr{P} + \mathscr{P}\mathscr{F}\mathscr{K} + (\mathscr{P}\mathscr{F}\mathscr{K})^*)z > . \tag{1.9}$$

Because we need to determine the feedback gain $\mathscr{K}$ in addition to the operator $\mathscr{P}$, $\dot{V}(z)$ becomes a bilinear function of the variables which define these operators. Since no reliable and efficient numerical method to solve bilinear matrix inequalities is currently available, we conclude that any solution to the controller synthesis problem must involve a reformulation or change of variables.

The approach we describe in this chapter is to use the transformation of variables

$$\mathscr{Q} = \mathscr{P}^{-1}, \tag{1.10}$$
$$\hat{\mathscr{K}} = \mathscr{K}\mathscr{P}^{-1}. \tag{1.11}$$

Clearly, given coercive $\mathscr{Q} > 0$ and $\hat{\mathscr{K}}$, and a procedure for finding the inverse $\mathscr{Q}^{-1}$, we can recover the original operators $\mathscr{P}$ and $\mathscr{K}$ as $\mathscr{P} = \mathscr{Q}^{-1}$ and $\mathscr{K} = \hat{\mathscr{K}}\mathscr{P}$. Now, by examination of V and $\dot{V}$ and by defining the transformed state $\hat{z} = \mathscr{P}z$, we obtain

$$V(z) = \langle \hat{z}, \mathscr{Q}\hat{z} \rangle, \tag{1.12}$$
$$\dot{V}(z) = \langle \hat{z}, (\mathscr{A}\mathscr{Q} + \mathscr{Q}\mathscr{A}^* + \mathscr{F}\hat{\mathscr{K}} + \hat{\mathscr{K}}^*\mathscr{F}^*)\hat{z} \rangle, \tag{1.13}$$

which are linear with respect to the new operator-variables $\mathscr{Q}$ and $\hat{\mathscr{K}}$. Therefore, if $\mathscr{P} : Z \to Z$ and we can parameterize operators which are positive on Z, then the controller synthesis problem can be represented as an SDP.

Critical to implementation of this approach, however, is to enforce the condition $\mathscr{P} : Z \to Z$ and the ability to invert the bounded linear operator $\mathscr{Q}$. While inversion of combined multiplier and integral operators is, in general, difficult, in the following sections we will show that it is possible to obtain an analytic expression for this inverse in the case where $\mathscr{Q}$ is separable, similar to the case discussed in [11]. Of course, numerical approximations to the inverse are possible through series expansion methods, as described in [10]. However, the existence of a closed-form analytic expression eliminates the approximation error due to finite truncation of the series and significantly reduces the complexity of the resulting inverse operator.

1.2 Preliminaries

Consider the coupled differential functional equations given in (1.1) and (1.2). If $\rho(D) < 1$, stability of this system is equivalent to the existence of a complete Lyapunov-Krasovskii functional of the following form,

$$V(\psi,\phi) = r\psi^T P\psi + 2r\psi^T \int_{-r}^{0} Q(\eta)\phi(\eta)d\eta + \int_{-r}^{0}\int_{-r}^{0} \phi^T(\xi)R(\xi,\eta)\phi(\eta)d\xi d\eta + \int_{-r}^{0} \phi^T(\eta)S(\eta)\phi(\eta)d\eta, \tag{1.14}$$

where

$$P = P^T \in \mathbb{R}^{n\times n}, \qquad Q(\eta) \in \mathbb{R}^{n\times m}, \tag{1.15}$$

$$R(\xi,\eta) = R^T(\eta,\xi) \in \mathbb{R}^{m\times m}, \qquad S(\eta) = S^T(\eta) \in \mathbb{S}^n, \tag{1.16}$$

where $\mathbb{S}^n \subset \mathbb{R}^{n\times n}$ represents the set of symmetric matrices. The use of complete Lyapunov-Krasovskii functionals of this form was described in [4] and [12], and these results imply the following lemma.

Lemma 1. *System (1.1)-(1.2) with $\rho(D) < 1$ is exponentially stable if and only if there exists a quadratic Lyapunov-Krasovskii functional of the Form (1.14)-(1.16), such that $\varepsilon||\psi||^2 \le V(\psi,\phi)$, and*

$$\dot{V}(\psi,\phi) \triangleq \limsup_{t\to 0^+} \frac{V(x(t,\psi,\phi),y_t(\psi,\phi)) - V(\psi,\phi)}{t} \tag{1.17}$$

satisfies $\dot{V}(\psi,\phi) \le -\varepsilon||\psi||^2$ for some $\varepsilon > 0$, where $(x(t,\psi,\phi),y_t(\psi,\phi))$ is the solution of (1.1) and (1.2) with initial condition (ψ,ϕ).

Define inner product on Z defined in (1.3),

$$\left\langle \begin{bmatrix}\psi_1\\ \phi_1\end{bmatrix}, \begin{bmatrix}\psi_2\\ \phi_2\end{bmatrix} \right\rangle = r\psi_1^T\psi_2 + \int_{-r}^{0} \phi_1^T(s)\phi_2(s)ds.$$

For matrix P and matrix functions Q,R,S that satisfy (1.15)-(1.16), we define the linear operator $\mathscr{P} : Z \to Z$ as follows

$$\mathscr{P}\begin{bmatrix}\psi\\ \phi\end{bmatrix}(s) = \begin{bmatrix} P\psi + \int_{-r}^{0} Q(\theta)\phi(\theta)d\theta \\ rQ^T(s)\psi + \int_{-r}^{0} R(s,\theta)\phi(\theta)d\theta + S(s)\phi(s)\end{bmatrix}. \tag{1.18}$$

Obviously, (1.15)-(1.16) implies that $\mathscr{P}$ is a bounded and self-adjoint linear operator. The complete Lyapunov-Krasovskii functional may now be expressed as

$$V(\psi,\phi) = \left\langle \begin{bmatrix}\psi\\ \phi\end{bmatrix}, \mathscr{P}\begin{bmatrix}\psi\\ \phi\end{bmatrix} \right\rangle.$$

System (1.1)-(1.2) defines a strongly C_0-semigroup $\mathscr{S}: Z \to Z$ that satisfies (1.4). The corresponding abstract differential equation is (1.5). Let the domain of definition of $\mathscr{A}$ be X. Then, $X := \left\{ \begin{bmatrix} \psi \\ \phi \end{bmatrix} \in Z \,\middle|\, \dot{\phi}(s) \in \mathscr{C}, \phi(0) = C\psi + D\phi(-r) \right\}$, where $\mathscr{C}$ represents the set of continuous functions. For controller synthesis, we would like to restrict $\mathscr{P}$ so that X is an invariant subspace of $\mathscr{P}$,

$$\mathscr{P}X \in X. \tag{1.19}$$

The conditions for $\mathscr{P}$ to satisfy (1.19) are as follows, which is a generalization of Theorem 3 in [7].

Lemma 2. *$\mathscr{P}$, as defined in* (1.18)*, satisfies (1.19) if and only if the following conditions are satisfied,*

$$rQ^T(0) + S(0)C = CP + rDQ^T(-r) \tag{1.20}$$

$$R(0,s) = CQ(s) + DR(-r,s), \quad \forall s, \tag{1.21}$$

$$DS(-r) = S(0)D. \tag{1.22}$$

Proof. Let $h(s) = rQ^T(s)\psi + \int_{-r}^{0} R(s,\theta)\phi(\theta)d\theta + S(s)\phi(s)$ and $g = P\psi + \int_{-r}^{0} Q(s)\phi(s)ds$. Then, $\mathscr{P}X \in X$ is equivalent to

$$h(0) = Cg + Dh(-r), \tag{1.23}$$

for all ψ and ϕ that satisfy

$$\phi(0) = C\psi + D\phi(-r). \tag{1.24}$$

Using (1.24), we have

$$\begin{aligned} h(0) &= rQ^T(0)\psi + S(0)\phi(0) + \int_{-r}^{0} R(0,\theta)\phi(\theta)d\theta \\ &= rQ^T(0)\psi + S(0)C\psi + \int_{-r}^{0} R(0,\theta)\phi(\theta)d\theta + S(0)D\phi(-r) \\ &= (rQ^T(0) + S(0)C)\psi + \int_{-r}^{0} R(0,\theta)\phi(\theta)d\theta + S(0)D\phi(-r), \end{aligned} \tag{1.25}$$

$$\begin{aligned} &Cg + Dh(-r) \\ &= CP\psi + \int_{-r}^{0} CQ(s)\phi(s)ds + rDQ^T(-r)\psi + \int_{-r}^{0} DR(-r,s)\phi(s)ds + DS(-r)\phi(-r) \\ &= (CP + rDQ^T(-r))\psi + DS(-r)\phi(-r) + \int_{-r}^{0} (CQ(s) + DR(-r,s))\phi(s)ds. \end{aligned} \tag{1.26}$$

Then the right-hand sides of (1.25)-(1.26) are equal for arbitrary ψ and ϕ if and only if (1.20)-(1.22) are satisfied.

1.3 Inverting Separable Operators

In this section, we present an analytical expression for the inverse of the operator $\mathscr{P}$ when it is separable. Similar to [10], such an analytic expression for the inverse operator can be used to expedite the construction of the stabilizing controller in the controller synthesis problem.

Definition 1. An operator $\mathscr{P}$, as defined in (1.18), is said to be separable if

$$R(s,\theta) = Z^T(s)\Gamma Z(\theta), \qquad Q(s) = HZ(s), \tag{1.27}$$

for some constant matrices $\Gamma = \Gamma^T$ and H, and matrix-valued function $Z(s)$. Note that a sufficient condition for $\mathscr{P}$ to be separable is that R and Q are polynomials.

Theorem 1. *Assume $\mathscr{P}$ in (1.18) is separable. Then, provided that all the inverse matrices below are well defined, its inverse may be expressed as*

$$\mathscr{P}^{-1}\begin{bmatrix}\psi\\ \phi\end{bmatrix}(s) = \begin{bmatrix}\hat{P}\psi + \int_{-r}^{0}\hat{Q}(\theta)\phi(\theta)d\theta\\ r\hat{Q}^T(s)\psi + \hat{S}(s)\phi(s) + \int_{-r}^{0}\hat{R}(s,\theta)\phi(\theta)d\theta\end{bmatrix}, \tag{1.28}$$

where $\hat{R}(s,\theta)$, $\hat{Q}(\theta)$ and $\hat{S}(s)$ are given as follows

$$\hat{R}(s,\theta) = \hat{Z}^T(s)\hat{\Gamma}\hat{Z}(\theta), \tag{1.29}$$

$$\hat{Q}(\theta) = \hat{H}\hat{Z}(\theta), \qquad \hat{S}(s) = S^{-1}(s), \qquad \hat{Z}(s) = Z(s)S^{-1}(s), \tag{1.30}$$

and $\hat{H}$, $\hat{P}$ and $\hat{\Gamma}$ are given below,

$$\hat{H} = -P^{-1}HT, \qquad \hat{P} = [I + rP^{-1}HTKH^T]P^{-1}, \tag{1.31}$$

$$\hat{\Gamma} = [rT^TH^TP^{-1}H - \Gamma](I + K\Gamma)^{-1}, \quad T = (I + K\Gamma - rKH^TP^{-1}H)^{-1}, \tag{1.32}$$

where $K = \int_{-r}^{0} Z(s)S^{-1}(s)Z^T(s)ds$, and I denotes the identity matrix with appropriate dimension.

Proof. Let the operator defined by the right hand side of (1.28) be denoted as $\hat{\mathscr{P}}$, then

$$\hat{\mathscr{P}}\mathscr{P}\begin{bmatrix}\psi\\ \phi\end{bmatrix}(s) = \begin{bmatrix}\Lambda_1\\ \Lambda_2\end{bmatrix},$$

$$\begin{aligned}\Lambda_1 &= \int_{-r}^{0}\left(\hat{P}Q(\theta) + \hat{Q}(\theta)S(\theta) + \int_{-r}^{0}\hat{Q}(\xi)R(\xi,\theta)d\xi\right)\phi(\theta)d\theta\\ &\quad + \left(\hat{P}P + \int_{-r}^{0} r\hat{Q}(\theta)Q^T(\theta)d\theta\right)\psi,\\ \Lambda_2 &= r\left(\hat{Q}^T(s)P + \hat{S}(s)Q^T(s) + \int_{-r}^{0}\hat{R}(s,\theta)Q^T(\theta)d\theta\right)\psi + \hat{S}(s)S(s)\phi(s)\\ &\quad + \int_{-r}^{0}\left(r\hat{Q}^T(s)Q(\theta) + \hat{S}(s)R(s,\theta) + \hat{R}(s,\theta)S(\theta)\right.\\ &\qquad \left. + \int_{-r}^{0}\hat{R}(s,\xi)R(\xi,\theta)d\xi\right)\phi(\theta)d\theta.\end{aligned}$$

Using (1.27) and (1.29)-(1.32), we obtain

$$
\begin{aligned}
&\hat{P}Q(\theta)+\hat{Q}(\theta)S(\theta)+\int_{-r}^{0}\hat{Q}(\xi)R(\xi,\theta)d\xi\\
&\quad=\left(\hat{P}H+\hat{H}+\hat{H}K\Gamma\right)Z(\theta)\\
&\quad=\left([I+rP^{-1}HTKH^{T}]P^{-1}H-P^{-1}HT-P^{-1}HTK\Gamma\right)Z(\theta)\\
&\quad=\left[P^{-1}H+P^{-1}HT(rKH^{T}P^{-1}H-I-K\Gamma)\right]Z(\theta)\\
&\quad=\left(P^{-1}H-P^{-1}H\right)Z(\theta)=0\\
&\hat{P}P+r\int_{-r}^{0}\hat{Q}(\theta)Q^{T}(\theta)d\theta\\
&\quad=\hat{P}P+r\hat{H}KH^{T}\\
&\quad=[I+rP^{-1}H(I+K\Gamma-rKH^{T}P^{-1}H)^{-1}KH^{T}]-rP^{-1}HTKH^{T}=I\\
&\hat{Q}^{T}(s)P+\hat{S}(s)Q^{T}(s)+\int_{-r}^{0}\hat{R}(s,\theta)Q^{T}(\theta)d\theta\\
&\quad=\hat{Z}^{T}(s)(\hat{H}^{T}P+H^{T}+\hat{\Gamma}KH^{T})\\
&\quad=\hat{Z}^{T}(s)\left\{-T^{T}H^{T}P^{-1}P+H^{T}+[rT^{T}H^{T}P^{-1}H-\Gamma](I+K\Gamma)^{-1}KH^{T}\right\}\\
&\quad=\hat{Z}^{T}(s)\left\{I-T^{T}(I-rH^{T}P^{-1}H(I+K\Gamma)^{-1}K)-\Gamma(I+K\Gamma)^{-1}K\right\}H^{T}\\
&\quad=\hat{Z}^{T}(s)\left\{I-T^{T}(I-rH^{T}P^{-1}HK(I+\Gamma K)^{-1})-\Gamma K(I+\Gamma K)^{-1}\right\}H^{T}\\
&\quad=\hat{Z}^{T}(s)\left\{I-T^{T}(I+\Gamma K-rH^{T}P^{-1}HK)(I+\Gamma K)^{-1}-\Gamma K(I+\Gamma K)^{-1}\right\}H^{T}\\
&\quad=\hat{Z}^{T}(s)[I-(I+\Gamma K)^{-1}-\Gamma K(I+\Gamma K)^{-1}]H^{T}=0\\
&r\hat{Q}^{T}(s)Q(\theta)+\hat{S}(s)R(s,\theta)+\hat{R}(s,\theta)S(\theta)+\int_{-r}^{0}\hat{R}(s,\xi)R(\xi,\theta)d\xi\\
&\quad=\hat{Z}^{T}(s)(r\hat{H}^{T}H+\Gamma+\hat{\Gamma}+\hat{\Gamma}K\Gamma)Z(\theta)\\
&\quad=\hat{Z}^{T}(s)\left\{-rT^{T}H^{T}P^{-1}H+\Gamma+[rT^{T}H^{T}P^{-1}H-\Gamma](I+K\Gamma)^{-1}(I+K\Gamma)\right\}Z(\theta)\\
&\quad=\hat{Z}^{T}(s)\left(-rT^{T}H^{T}P^{-1}H+\Gamma+rT^{T}H^{T}P^{-1}H-\Gamma\right)Z(\theta)=0
\end{aligned}
$$

Thus, we have shown

$$\hat{\mathscr{P}}\mathscr{P}\begin{bmatrix}\psi\\ \phi\end{bmatrix}=\begin{bmatrix}\psi\\ \phi\end{bmatrix}, \tag{1.33}$$

for all $\begin{bmatrix}\psi\\ \phi\end{bmatrix}\in Z$. Similarly, we can show

$$\mathscr{P}\hat{\mathscr{P}}\begin{bmatrix}\psi\\ \phi\end{bmatrix}=\begin{bmatrix}\psi\\ \phi\end{bmatrix}. \tag{1.34}$$

From (1.33) and (1.34), we conclude that $\hat{\mathscr{P}}=\mathscr{P}^{-1}$.

Theorem 2. *If the separable operator $\mathscr{P}$ satisfies $\mathscr{P}X\in X$, Then, $\mathscr{P}^{-1}X\in X$ holds.*

Proof. Let the linear operator $\mathscr{P}$ satisfy $\mathscr{P}X\in X$. By Lemma 2, this is equivalent to (1.20)-(1.22), from which, we obtain

$$CP^{-1}=rS^{-1}(0)(DZ^{T}(-r)-Z^{T}(0))H^{T}P^{-1}+S^{-1}(0)C, \tag{1.35}$$

$$CH=(Z^{T}(0)-DZ^{T}(-r))\Gamma, \tag{1.36}$$

$$S^{-1}(0)D=DS^{-1}(-r). \tag{1.37}$$

Applying (1.35)-(1.37) to the operator $\mathscr{P}^{-1}$ defined in (1.28), after tedious calculations, we can obtain the following equation,

$$r\hat{Q}^T(0)\psi+\hat{S}(0)\phi(0)+\int_{-r}^{0}\hat{R}(0,\theta)\phi(\theta)d\theta=C\left(\hat{P}\psi+\int_{-r}^{0}\hat{Q}(\theta)\phi(\theta)d\theta\right)$$
$$+D\left(r\hat{Q}^T(-r)\psi+\hat{S}(-r)\phi(-r)+\int_{-r}^{0}\hat{R}(-r,\theta)\phi(\theta)d\theta\right),$$

from which, we conclude that $\mathscr{P}^{-1}X \in X$.

1.4 Controller Synthesis

In this section, we consider a system with control input as follows

$$\dot{x}=Ax+By(t-r)+Fu(t), \tag{1.38}$$
$$y(t)=Cx(t)+Dy(t-r). \tag{1.39}$$

For this system, let us define the infinitesimal generator $\mathscr{A}$ as follows.

$$\left(\mathscr{A}\begin{bmatrix}x\\ y_t\end{bmatrix}\right)(s)=\begin{bmatrix}Ax+By(t-r)\\ \frac{d}{ds}y_t(s)\end{bmatrix}.$$

Likewise, we define the input operator $\mathscr{F}:\mathbb{R}^q\to X$ as

$$(\mathscr{F}u)(s):=\begin{bmatrix}Fu\\ 0\end{bmatrix}.$$

We define the controller synthesis problem as the search for matrices K_0, K_1 and matrix-valued function $K_2(s)$ such that if

$$u(t)=\mathscr{K}\begin{bmatrix}x(t)\\ y_t\end{bmatrix}, \tag{1.40}$$

where we define $\mathscr{K}:X\to\mathbb{R}^q$ as

$$\left(\mathscr{K}\begin{bmatrix}x\\ y_t\end{bmatrix}\right)(s)=K_0x(t)+K_1y(t-r)+\int_{-r}^{0}K_2(s)y(t+s)ds, \tag{1.41}$$

then the closed-loop system described by the equations (1.38)-(1.40) is stable.

Before we give the main result of the chapter, we briefly address SOS methods for enforcing joint positivity of coupled multiplier and integral operators using positive matrices. These methods have been developed in a series of papers, a summary of which can be found in the survey paper [11]. Specifically, for matrix-valued functions $M(s)$, $N(s,\theta)$, we say that$\{M,N\}\in\Xi$, if M and N satisfy the conditions of Theorem 8 in [11]. The constraint $\{M,N\}\in\Xi$ can be cast as an LMI using SOSTOOLS as described in [11] and this constraint ensures that the operator $\mathscr{P}$, defined as

$$\left(\mathscr{P}\begin{bmatrix}\psi\\ \phi\end{bmatrix}\right)(s)=\begin{bmatrix}M_{11}\psi+\int_{-r}^{0}M_{12}(\theta)\phi(\theta)d\theta\\ rM_{21}^{T}(s)\psi+\int_{-r}^{0}N(s,\theta)\phi(\theta)d\theta+M_{22}(s)\phi(s)\end{bmatrix},$$

is positive on X. Furthermore, we note that $\{M,N\}\in\Xi$ implies that $\mathscr{P}$ is separable and $P=\int M_{11}(s)ds$ and $S=M_{22}$ are invertible. We now state the main result.

Proposition 1. *Suppose there exist matrices M_0, M_1, $P=P^T$, matrix-valued functions $M_2(s)$, $Q(s)$, $R(s,\theta)$, $S(s)=S^T(s)\in\mathbb{S}^n$, and scalar $\varepsilon>0$ such that (1.20)-(1.22) are satisfied and the following conditions hold*

$$\{T,R\}\in\Xi, \tag{1.42}$$

$$\{-U,-W\}\in\Xi, \tag{1.43}$$

where

$$T(s)=\begin{bmatrix}P & rQ(s)\\ rQ^T(s) & S(s)\end{bmatrix}-\varepsilon I, \tag{1.44}$$

$$U(s)=\begin{bmatrix}\Gamma+\varepsilon I & BS(-r)+FM_1+\frac{1}{r}C^TS(0)D & \Upsilon\\ * & \frac{-1}{r}(S(-r)-D^TS(0)D) & 0\\ * & * & \dot{S}(s)\end{bmatrix}, \tag{1.45}$$

$$W(s,\theta)=\frac{\partial}{\partial s}R(s,\theta)+\frac{\partial}{\partial\theta}R(s,\theta), \tag{1.46}$$

where $*$ denotes entries in the matrix determined by symmetry,

$$\Gamma=AP+PA^T+r(BQ^T(-r)+Q(-r)B^T)+\tfrac{1}{r}C^TS(0)C+FM_0+M_0^TF,$$
$$\Upsilon=r[\dot{Q}(s)+BR(-r,s)+AQ(s)+FM_2(s)].$$

Then System (1.38)-(1.39) is stabilizable with a controller (1.40). In other words, let

$$u(t)=K_0x(t)+K_1y(t-r)+\int_{-r}^{0}K_2(s)y(t+s)ds, \tag{1.47}$$

where

$$K_0=M_0\hat{P}+rM_1\hat{Q}^T(-r)+r\int_{-r}^{0}M_2(s)\hat{Q}^T(s)ds, \tag{1.48}$$

$$K_1=M_1\hat{S}(-r), \tag{1.49}$$

$$K_2(s)=M_0\hat{Q}(s)+M_1\hat{R}(-r,s)+M_2(s)\hat{S}(s)+\int_{-r}^{0}M_2(\theta)\hat{R}(\theta,s)d\theta, \tag{1.50}$$

and $\hat{P}$, $\hat{Q}$, $\hat{R}$ and $\hat{S}$ are defined in Theorem 1. Then the closed-loop System (1.38)-(1.39) and (1.47) is stable.

Proof. Define $\mathscr{P}$ by (1.18). Then $\mathscr{P}$ is bounded and self-adjoint. Per Lemma 2, $\mathscr{P}: X\to X$. (1.43) implies $\mathscr{P}\geq\varepsilon I$. Per Theorem 1, the inverse $\mathscr{P}^{-1}$ can be expressed as in (1.28) and is likewise bounded and coercive with $\mathscr{P}^{-1}\geq\varepsilon' I$. Furthermore, from Theorem 2, $\mathscr{P}^{-1}: X\to X$ and $\mathscr{P}^{-1}=\mathscr{P}^{-*}$. In other words, the Lyapunov-Krasovskii functional satisfies

$$V = \left\langle \begin{bmatrix} \psi \\ \phi \end{bmatrix}, \mathscr{P}^{-1} \begin{bmatrix} \psi \\ \phi \end{bmatrix} \right\rangle \geq \varepsilon' \left\| \begin{bmatrix} \psi \\ \phi \end{bmatrix} \right\|^2 \tag{1.51}$$

for some $\varepsilon' > 0$ and all $\begin{bmatrix} \psi \\ \phi \end{bmatrix} \in X$. Furthermore,

$$\begin{aligned} &\left\langle \begin{bmatrix} \psi \\ \phi \end{bmatrix}, \mathscr{P}^{-1}\mathscr{A} \begin{bmatrix} \psi \\ \phi \end{bmatrix} \right\rangle + \left\langle \mathscr{A} \begin{bmatrix} \psi \\ \phi \end{bmatrix}, \mathscr{P}^{-1} \begin{bmatrix} \psi \\ \phi \end{bmatrix} \right\rangle \\ &= \left\langle \mathscr{P}^{-1} \begin{bmatrix} \psi \\ \phi \end{bmatrix}, \mathscr{A}\mathscr{P}\mathscr{P}^{-1} \begin{bmatrix} \psi \\ \phi \end{bmatrix} \right\rangle + \left\langle \mathscr{A}\mathscr{P}\mathscr{P}^{-1} \begin{bmatrix} \psi \\ \phi \end{bmatrix}, \mathscr{P}^{-1} \begin{bmatrix} \psi \\ \phi \end{bmatrix} \right\rangle. \end{aligned}$$

Next, we note that if we define $\mathscr{K}$ as in (1.41) and $\mathscr{M}$ as follows

$$\left(\mathscr{M} \begin{bmatrix} x \\ y_t \end{bmatrix} \right) = M_0 x(t) + M_1 y(t-r) + \int_{-r}^{0} M_2(s) y(t+s) ds.$$

Then (1.48)-(1.50) implies $\mathscr{K} := \mathscr{M}\mathscr{P}^{-1}$. Now we define a new state $\begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix} = \mathscr{P}^{-1} \begin{bmatrix} \psi \\ \phi \end{bmatrix} \in X$. Continuing, if $u = \mathscr{K} \begin{bmatrix} x \\ y_t \end{bmatrix} = \mathscr{K}\mathscr{P}\mathscr{P}^{-1} \begin{bmatrix} x \\ y_t \end{bmatrix} = \mathscr{M}\mathscr{P}^{-1} \begin{bmatrix} x \\ y_t \end{bmatrix}$, then the closed-loop system is stable if $\dot{V} < 0$, where

$$\begin{aligned} \dot{V} = &\left\langle \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix}, \mathscr{A}\mathscr{P} \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix} \right\rangle + \left\langle \mathscr{A}\mathscr{P} \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix}, \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix} \right\rangle \\ &+ \left\langle \mathscr{F}\mathscr{M} \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix}, \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix}, \mathscr{F}\mathscr{M} \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix} \right\rangle. \end{aligned}$$

To show that $\dot{V} < 0$, we examine $\mathscr{A}\mathscr{P}$ and $\mathscr{F}\mathscr{M}$ separately. First, we have

$$\mathscr{A}\mathscr{P} \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix} (s) = \begin{bmatrix} \Psi \\ \Phi(s) \end{bmatrix}, \tag{1.52}$$

where

$$\begin{aligned} \Psi &= AP\hat{\psi} + \int_{-r}^{0} AQ(s)\hat{\phi}(s) ds + BrQ^T(-r)\hat{\psi} + BS(-r)\hat{\phi}(-r) + \int_{-r}^{0} BR(-r,\theta)\hat{\phi}(\theta) d\theta, \\ \Phi(s) &= r\dot{Q}^T(s)\hat{\psi} + \dot{S}(s)\hat{\phi}(s) + S(s)\dot{\hat{\phi}}(s) + \int_{-r}^{0} \frac{d}{ds} R(s,\theta)\hat{\phi}(\theta) d\theta. \end{aligned}$$

Then,

$$\begin{aligned} &\left\langle \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix}, \mathscr{A}\mathscr{P} \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix} \right\rangle \\ &= \int_{-r}^{0} \hat{\psi}^T \Psi ds + \int_{-r}^{0} \hat{\phi}^T(s) \Phi(s) ds \\ &= r\hat{\psi}^T AP\hat{\psi} + r\int_{-r}^{0} \hat{\psi}^T AQ(s)\hat{\phi}(s) ds + r\hat{\psi}^T BrQ^T(-r)\hat{\psi} + r\hat{\psi}^T BS(-r)\hat{\phi}(-r) \\ &\quad + r\int_{-r}^{0} \hat{\psi}^T BR(-r,\theta)\hat{\phi}(\theta) d\theta + \int_{-r}^{0} r\hat{\phi}^T(s)\dot{Q}^T(s)\hat{\psi} ds + \int_{-r}^{0} \hat{\phi}^T(s)\dot{S}(s)\hat{\phi}(s) ds \\ &\quad + \int_{-r}^{0}\int_{-r}^{0} \hat{\phi}^T(s) \frac{d}{ds} R(s,\theta)\hat{\phi}(\theta) ds d\theta + \int_{-r}^{0} \hat{\phi}^T(s) S(s)\dot{\hat{\phi}}(s) ds \end{aligned}$$

$$= \int_{-r}^{0} \begin{bmatrix} \hat{\Psi} \\ \hat{\phi}(-r) \\ \hat{\phi}(s) \end{bmatrix}^T \begin{bmatrix} AP + rBQ^T(-r) & BS(-r) & \Theta \\ 0 & 0 & 0 \\ r\dot{Q}^T(s) & 0 & \dot{S}(s) \end{bmatrix} \begin{bmatrix} \hat{\Psi} \\ \hat{\phi}(-r) \\ \hat{\phi}(s) \end{bmatrix} ds$$
$$+ \int_{-r}^{0}\int_{-r}^{0} \hat{\phi}^T(s) \frac{\partial}{\partial s} R(s,\theta)\hat{\phi}(\theta) ds d\theta + \int_{-r}^{0} \hat{\phi}^T(s) S(s) \dot{\hat{\phi}}(s) ds,$$

where $\Theta = r(AQ(s) + BR(-r,s))$. Since $\begin{bmatrix} \hat{\Psi} \\ \hat{\phi}(s) \end{bmatrix} \in X$, we have $\hat{\phi}(0) = C\hat{\Psi} + D\hat{\phi}(-r)$. Then,

$$\begin{aligned}
& \int_{-r}^{0} \hat{\phi}^T(s) S(s) \dot{\hat{\phi}}(s) ds \\
&= \hat{\phi}^T(0)S(0)\hat{\phi}(0) - \hat{\phi}^T(-r)S(-r)\hat{\phi}(-r) - \int_{-r}^{0} \hat{\phi}^T(s)\dot{S}(s)\hat{\phi}(s)ds - \int_{-r}^{0} \dot{\hat{\phi}}^T(s)S(s)\hat{\phi}(s)ds \\
&= \frac{1}{2}\left(\hat{\phi}^T(0)S(0)\hat{\phi}(0) - \hat{\phi}^T(-r)S(-r)\hat{\phi}(-r)\right) - \frac{1}{2}\int_{-r}^{0} \hat{\phi}^T(s)\dot{S}(s)\hat{\phi}(s)ds \\
&= \frac{1}{2}\int_{-r}^{0} \begin{bmatrix} \hat{\Psi} \\ \hat{\phi}(-r) \\ \hat{\phi}(s) \end{bmatrix}^T \begin{bmatrix} \frac{1}{r}C^T S(0) C & \frac{1}{r}(C^T S(0) D) & 0 \\ \frac{1}{r}(D^T S(0) C) & -\frac{1}{r}(S(-r) - D^T S(0) D) & 0 \\ 0 & 0 & -\dot{S}(s) \end{bmatrix} \begin{bmatrix} \hat{\Psi} \\ \hat{\phi}(-r) \\ \hat{\phi}(s) \end{bmatrix} ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
\dot{V} &= \left\langle \begin{bmatrix} \hat{\Psi} \\ \hat{\phi} \end{bmatrix}, \mathscr{A}\mathscr{P}\begin{bmatrix} \hat{\Psi} \\ \hat{\phi} \end{bmatrix} \right\rangle + \left\langle \mathscr{A}\mathscr{P}\begin{bmatrix} \hat{\Psi} \\ \hat{\phi} \end{bmatrix}, \begin{bmatrix} \hat{\Psi} \\ \hat{\phi} \end{bmatrix} \right\rangle \\
&\quad + \left\langle \mathscr{F}\mathscr{M}\begin{bmatrix} \hat{\Psi} \\ \hat{\phi} \end{bmatrix}, \begin{bmatrix} \hat{\Psi} \\ \hat{\phi} \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} \hat{\Psi} \\ \hat{\phi} \end{bmatrix}, \mathscr{F}\mathscr{M}\begin{bmatrix} \hat{\Psi} \\ \hat{\phi} \end{bmatrix} \right\rangle. \\
&= \int_{-r}^{0} \begin{bmatrix} \hat{\Psi} \\ \hat{\phi}(-r) \\ \hat{\phi}(s) \end{bmatrix}^T \begin{bmatrix} \Gamma & BS(-r) + FM_1 + \frac{1}{r}(C^T S(0) D) & \Upsilon \\ * & -\frac{1}{r}(S(-r) - D^T S(0) D) & 0 \\ * & * & \dot{S}(s) \end{bmatrix} \begin{bmatrix} \hat{\Psi} \\ \hat{\phi}(-r) \\ \hat{\phi}(s) \end{bmatrix} ds \\
&\quad + \int_{-r}^{0}\int_{-r}^{0} \hat{\phi}^T(s) \left(\frac{\partial}{\partial s} R(s,\theta) + \frac{\partial}{\partial \theta} R(s,\theta) \right) \hat{\phi}(\theta) ds d\theta,
\end{aligned}$$

From conditions (1.43), (1.45) and (1.46), we have $\dot{V} < 0$, which, along with (1.51) means that the closed-loop system defined by (1.38)-(1.39) and (1.47) is stable.

Remark 1. When $D = 0$, System (1.38)-(1.39) may be written in the standard delay-differential framework studied in [7] and [10]:

$$\begin{aligned}
\dot{x}(t) &= A_0 x(t) + A_1 x(t - r_1) + F u(t) \\
x(t) &= \phi(t).
\end{aligned}$$

The primary computational advantage of the differential-difference framework over control of System (1.38)-(1.39) is that we can replace $A_1 \in \mathbb{R}^{n\times n}$ with BC where $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$ and m is typically strictly less than n. Because the dimension of the decision variables in the optimization problem defined in this paper scale as $n + 2m$ as opposed to $3n$ using the framework in [7] and [10], the complexity of the resulting algorithm is significantly reduced.

Remark 2. Although not explicitly stated, in order to use SOS to enforce the conditions of Theorem 1 and Proposition 1, we choose our decision variables to be polynomial and use SOSTOOLS and the Positivstellensatz to enforce positivity/negativity on the interval $[-r,0]$. This approach is described in more detail in [7] and [10].

In the following, we present a numerical example to illustrate the controller obtained from the condition in Proposition 1. We consider the following system with a feedback controller as follows

$$\dot{x}(t) = \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0.1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.9 \end{bmatrix} x(t) + By(t-r) + Fu(t), \tag{1.53}$$

$$y(t) = \begin{bmatrix} -0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x(t), \tag{1.54}$$

where $r = 1.6$s, $B = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$, $F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$. By using Proposition 1, together with the tools of MuPad, Matlab, SOSTOOLS and polynomials with degree 2, we obtain the controller

$$u(t) = K_0 x(t) + \begin{bmatrix} -0.239 \\ -0.343 \end{bmatrix}^T y(t-r) + \int_{-1.6}^{0} K_2 y(t+s) ds, \tag{1.55}$$

$$K_0 = \begin{bmatrix} -1.874 & 2.232 & -0.830 & 3.099 & 0.030 & -1.033 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -0.246 + 0.221s + 0.122s^2 - 0.012s^3 - 0.032s^4 \\ 0.238 - 0.398s + +0.007s^2 + 0.037s^3 + 0.010s^4, \end{bmatrix}^T.$$

Using Controller (1.55) coupled with System (1.53)-(1.54) we simulate the closed-loop system, which is illustrated in Fig.1.

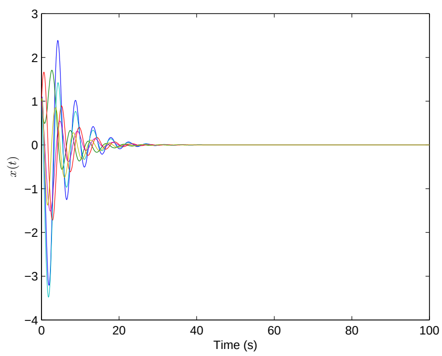

Fig. 1.1 States of System (1.53)-(1.54) coupled with stabilizing Controller from Prop. 1

1.5 CONCLUSIONS

In this chapter, we have obtained an analytic formulation for the inverse of jointly positive multiplier and integral operators as defined in [7]. This formulation has the advantage that it eliminates the need for either individual positivity of the multiplier and integral operators or the need to use a series expansion to find the inverse. This inversion formula is applied to controller synthesis of coupled differential-difference equations. The use of the differential-difference formulation has the advantage that the size of the resulting decision variables is reduced, thereby allowing for control of systems with larger numbers of states. These methods are illustrated by designing a stabilizing controller for a system with 6 states and a 2-dimensional delay channel.

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