A Convex Reformulation of the Controller Synthesis Problem for Infinite-Dimensional Systems using Linear Operator Inequalities (LOIs) with Application to MIMO Multi-Delay Systems.

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***Abstract*— In this paper, we propose a formal duality and operator-based framework for controller synthesis of infinite-dimensional systems using convex optimization. Specifically, we propose a class of what we call Linear Operator Inequalities (LOIs) and give conditions under which LOIs are solvable and under which they can be used for controller synthesis. Within this LOI framework, the first technical contribution of the paper is a new dual stability condition under which we can reformulate the controller synthesis problem as an LOI. The second technical contribution is LOIs for both primal and dual stability of systems with multiple delays. The third technical contribution is to show that both these LOIs are solvable. Finally, we use an LMI-based framework to solve these LOIs and demonstrate numerically that the primal and dual stability conditions are functionally equivalent for MIMO systems with multiple delays.**

I. Introduction

Research on controller synthesis for MIMO systems with multiple delays is underdeveloped. As is typical for controller synthesis problems, the fundamental problem is one of bilinearity. Specifically, all controller synthesis problems (even for ODEs) are bilinear. That is, for a given feedback controller, the search for a stability proof (e.g. using Lyapunov functions) is convex and relatively tractable (e.g. the set of Lyapunov functions is a convex cone). Furthermore, given a fixed Lyapunov function, the search for a controller is likewise tractable. However, if we are looking for both a controller and a stability proof, then the resulting optimization problem is bilinear and non-convex. Some papers use iterative methods to overcome this bilinearity by alternately optimizing the Lyapunov functional and controller as in [1] or [2] (via a "tuning parameter"). However, this iterative approach is not guaranteed to converge. Meanwhile, approaches based on frequency-domain methods, discrete approximation, or Smith predictors result in controllers which are not provably stable or are sensitive to variations in system parameters or in delay. Another alternative for both PDE [3] and time-delay systems [4] is backstepping, which avoids the bilinearity by choosing a target system and using an analytic construction of the controller which transforms the closed-loop system to that stable target system - thereby avoiding the non-convex optimization problem. However, these methods are typically difficult to generalize, can be conservative, and cannot be adapted to robust or optimal control.

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In 1996, it was established that the question of stability of systems with multiple non-commensurate delays is NP-hard [5]. This result, indicating that LMI-based methods will never yield a necessary and sufficient stability condition, prompted a retrenchment and reformulation of research on computational methods for analysis and control of systems with delay. Specifically, starting with the work in [6], research began to focus on algorithms which are asymptotically accurate. That is, these algorithms have a form of scalable computational complexity consisting of sequential instances indexed by some parameter such as level of discretization, polynomial degree, etc. For each instance, the algorithm gives a sufficient condition for stability with associated polynomial-time complexity, finite-dimensional LMI, and degree of conservatism. However, as the sequence of instances progresses, the complexity increases, the LMIs become larger, and the conservatism decreases to the limit of a necessary and sufficient condition. This reformulation of the stability problem as the limit of a sequence of algorithms was adapted to the polynomial optimization framework (using Sum-of-Squares (SOS)) in 2004 [7], an approach which allows us to include nonlinearities, uncertainties in the delay, time-varying delays [8], and Sampled-Data Systems [9], [10]. This sequential approach has since been adopted a number of other authors including, e.g. the delay-partitioning approach of [11], the Wirtinger-based method of [12], and the matrix dilation approach of [13]. All these approaches appear to converge numerically and the SOS approach, in particular, has been applied to MIMO systems with 100+ states. Indeed, the quality of these algorithms is such that we may consider the problem of computational stability analysis of linear time-delay systems to be solved. Unfortunately, however, due to bilinearity, all attempts to develop asymptotic algorithms for *control* of delayed systems have failed.

In this paper, we propose a formal framework for the use of asymptotic algorithms in control of infinite-dimensional systems in general and MIMO systems with multiple delays in particular. This framework differs from classical results on control of infinite-dimensional systems in that it is not based on an ODE representation, such in POD [14] and Galerkin projection [15], or any other finite-dimensional reduction [16]. Rather, the method is based on feasibility of Linear Operator Inequalities (LOIs), which we will formally define in Section III. Within this framework, we define what it means for an LOI to be solvable - conditions based largely on reduction of the LOI to an equivalent LOI which has already been shown to be solvable. For a set of solvable LOIs, we refer primarily to [17]. Having given conditions under which LOIs are solvable, in Section IV we define the conditions under which controller synthesis of an infinite-dimensional system can be represented as an LOI feasibility problem. Among these conditions, the most technically challenging is the existence of a dual stability condition.

Having described the LOI framework for controller synthesis, we may now define the technical contribution of the paper, which is a set of dual LOI stability conditions which relies on the formal LOI definition and applies to all suitably well-posed infinite dimensional systems. Having defined the conditions under which the dual LOI implies feasibility of the primal, we propose a mathematical for MIMO multiple-delay systems under which the LOIs for both primal and dual stability are well-defined. Specifically, we first formulate the standard Lyapunov Krasovskii stability conditions as a primal LOI with a suitably defined Hilbert space, state space, and parameterized set of operators. We then show that this LOI is solvable. We then repeat this procedure for the dual stability condition, adding additional linear constraints on the class of operators in order to satisfy the conditions of the duality theorem. Finally, we obtain a solvable dual LOI and use numerical computation to show that there is little if any conservatism in the dual LOI stability conditions.

II. Semigroups and Linear Operator Inequalities

Semigroup theory is a well-developed, mathematically rigorous, state-space theory for what are known as *distributed-parameter systems*, with time-delay systems being a special case. Seminal references are given by [18], [19] and [20]. Consider the general form of infinite-dimensional system

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t)$$

where $\mathcal{A} : X \to Z$ and $\mathcal{B} : U \to X$. Z is an inner product space and $X \subset Z$ is the space wherein the solution map is a "strongly continuous semigroup", $S(t) : X \to X$ which satisfies $\frac{d}{dt}S(t)\phi = \mathcal{A}S(t)\phi + \mathcal{B}u(t)$ and $S(0)\phi = \phi$. X is the "domain" of the infinitesimal generator, $\mathcal{A}$, and characterizes properties of the state.

Within the semigroup framework is an operator-based version of Lyapunov theory for linear systems. In particular, we have that if $\mathcal{A}$ is linear and $S(t)$ is exponentially stable, then there exists a positive operator, $\mathcal{P} : Z \to Z$, such that $\langle \mathcal{A}z, \mathcal{P}z\rangle + \langle \mathcal{P}z, \mathcal{A}z\rangle = -\langle z, z\rangle$ for all $z \in X$ [18]. As an operator inequality, we represent this as $\mathcal{A}^*\mathcal{P}+\mathcal{P}\mathcal{A} < 0$. The question of stability, then, is equivalent to the search for a positive operator, $\mathcal{P}$, which satisfies this Lyapunov inequality.

Linear Operator Inequalities (LOIs) like $\mathcal{A}^*\mathcal{P} + \mathcal{P}\mathcal{A} < 0$, as applied to time-delay systems, are simply a shorthand representation of Lyapunov-Krasovskii stability conditions. For example, if we parameterize

$$\left(\mathcal{P}\begin{bmatrix} x \\ \phi_i \end{bmatrix}\right)(s) := \begin{bmatrix} Px + \sum_{i=1}^K \int_{-\tau_i}^0 Q_i(s)\phi_i(s)ds \\ \tau_K Q_i(s)^T x + \tau_K S_i(s)\phi_i(s) + \sum_{j=1}^K \int_{-\tau_j}^0 R_{ij}(s,\theta)\phi_j(\theta)\, d\theta. \end{bmatrix}$$

then $V(x) = \langle x, \mathcal{P}x\rangle_Z$ is the standard complete-quadratic Lyapunov functional (See Eqn. (2)) and $\dot{V} = \langle \mathcal{A}z, \mathcal{P}z\rangle + \langle \mathcal{P}z, \mathcal{A}z\rangle$ is the time-derivative of this functional. We conclude that, in this case, testing an LOI is equivalent to testing positivity or negativity of a Lyapunov functional. In the following section, we propose a formalization of the concept of LOI and conditions under which they can be solved.

III. A Formal Definition of an LOI and Conditions for Solvability

Now that we have motivated the use of LOIs, we give a formal definition of an LOI and the necessary steps to be able to solve one.

A. Definition of an LOI

A solvable LOI is defined by a tuple of the form ($\mathcal{H}$, $\mathcal{G}$, $\mathbb{P}$, X, and Z). The LOI is then a feasibility problem of the form $\mathcal{H}\mathcal{P}\mathcal{G} + (\mathcal{H}\mathcal{P}\mathcal{G})^* > 0$ where $\mathcal{P} \in \mathbb{P}$ is the variable. The LOI is feasible if there exists some $\mathcal{P} \in \mathbb{P}$ such that

$$\langle \mathcal{H}\mathcal{P}\mathcal{G}x, x\rangle_Z + \langle x, \mathcal{H}\mathcal{P}\mathcal{G}x\rangle_Z \geq 0 \qquad \text{for all } x \in X \subset Z$$

Note that this form can include any number of coupled LOI constraints. Furthermore, to be well-defined, the tuple ($\mathcal{H}$, $\mathcal{G}$, $\mathbb{P}$, X, and Z) must have the following elements

1. Inner Product Space Z is an inner-product space (defines the meaning of ≥ 0).

2. State Space $X \subset Z$ quantifies "for all $x \in X$".

3. Variables The operator $\mathcal{P}$ may represent a single or multiple operator variables and is constrained to lie in set $\mathbb{P}$, which is typically a subset of bounded linear operators. Additionally, the set $\mathbb{P}$ may be used to define equality constraints on $\mathcal{P}$.

4. Data $\mathcal{H}$ and $\mathcal{G}$ are operators and may be unbounded.

5. Well-posedness For given $\mathcal{H}$ and $\mathcal{G}$, the inner product $\langle x, \mathcal{H}\mathcal{P}\mathcal{G}x\rangle_Z$ must be well-defined for all $\mathcal{P} \in \mathbb{P}$ and $x \in X$. The well-posedness constraint often requires constraints on the set of operators $\mathbb{P}$ and a careful choice of inner-product space Z and, more significantly, choice of state-space, X.

Note that implicit in this definition is the assumption that X is an infinite-dimensional set. If this were not the case, the LOI would reduce to an LMI. By definition, then, feasibility or solvability of an LOI precludes discretization of X. That is, $\langle \mathcal{H}\mathcal{P}\mathcal{G}x, x\rangle_Z + \langle x, \mathcal{H}\mathcal{P}\mathcal{G}x\rangle_Z \geq 0$ must hold for all X and not some finite-dimensional subset. Having now given a definition for an LOI, we give the necessary conditions for an LOI to be solvable.

B. Sufficient Conditions for an LOI to be solvable

Recall that an LOI is an inequality defined on an infinite-dimensional set, X. Therefore, any feasible solution must simultaneously satisfy an infinite number of constraints - making computation a seemingly intractable problem. However, we propose in this paper a standardized approach for generating a sequence of computationally tractable sufficient conditions with the additional condition that the sequence asymptotically approach necessity. This standardized approach requires four elements.

1) A formal definition of the LOI.
2) An embedding of $\mathbb{P} \subset \mathbb{R}^m$.
3) A Class of LOIs which are solvable.

4) An Equivalent Representation as a Solvable LOI.

We briefly explain each of these four elements.

1. A Definition of the LOI: See above.

2. An embedding $\mathbb{P} \subset \mathbb{R}^m$: The set of operators must be finite-dimensional. Practically, we require a map from $\mathbb{R}^m$ to a suitably rich subset of the operators defined in $\mathbb{P}$. In this paper, we use the SOS approach, wherein $\mathbb{R}^m$ parameterizes the coefficients of polynomials; the polynomials then parameterize multipliers and kernels; the multipliers and kernels then parameterize multiplier and integral operators. Constraints on the operators must then be translated to constraints on the parameters in $\mathbb{R}^m$. For example, let $Z_d(x)$ be the vector of monomials for degree d or less. Then if $X = L_2$, we can map $c \in \mathbb{R}^{m_1}$ or $c_2 \in \mathbb{R}^{m_2}$ to $\mathcal{P} \in \mathbb{P}$ as

$$(\mathcal{P}x)(s) = c_1^T Z_d(s)x(s) \;\text{ or }\; (\mathcal{P}x)(s) = c_2^T \int_{-\tau}^{0} Z_d(s,\theta)x(\theta)d\theta$$

Here, $c_1^T Z_d(s)$ is a multiplier and $c_2^T Z_d(s,\theta)$ is a kernel. Now suppose we want to constrain $\mathcal{P} = \mathcal{D}$ where $(\mathcal{D}x)(s) = D(s)x(s)$. Then any polynomial D can be written as $D(s) = d^T Z_d(s)$ and so we simply constrain $c_2 = 0$ and $c_1 = d$. Other constraints involving substitution and integration can be likewise treated.

3. A Class of LOIs which are solvable To solve an LOI, we typically reduce the LOI to a special case of a known class of LOI for which asymptotic algorithms already exist. A summary of several such classes can be found in [17]. For example, in this paper, we will use the known class ($\mathcal{H}$, $\mathcal{G}$, $\mathbb{P}$, X, and Z), where $Z := L_2^{n+m}$, $X := \{\mathbb{R}^n \times L_2^m\}$, $\mathbb{P}$ is the set of multiplier and integral operators with piecewise-continuous kernels, and $\mathcal{H}, \mathcal{G} \in \mathbb{P}$.

4. An Equivalent Representation as a Solvable LOI If the LOI is not already in a solvable class, then the next step is to construct a new, well-defined LOI in a solvable class whose feasibility is equivalent to feasibility of the original LOI. Furthermore, if the solution to the original LOI is required (e.g. if we need the stabilizing controller) then this conversion must be reversible. One approach to reversability is to retain the original parameters $c \in \mathbb{R}^m \mapsto \mathbb{P}$ in the new LOI. For example, consider the multiplier operator discussed above and let $X = W_2$, $\mathcal{H} = I$ and $\mathcal{G} = \partial_s$. Then we have

$$\langle x, \mathcal{H}\mathcal{P}\mathcal{G}x\rangle_Z = \int_{-\tau}^{0} x(s)c^T Z_d(s)\dot{x}(s)ds.$$

To enforce this LOI, we may form an equivalent LOI on $X' = \mathbb{R}^2 \times L_2$, $Z = L_2^3$ as

$$\langle x, \mathcal{H}\mathcal{P}\mathcal{G}x\rangle_Z = \int_{-\tau}^{0} x(s)c^T Z_d(s)\dot{x}(s)ds =$$
$$= \int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(-\tau) \\ x(s) \end{bmatrix}^T \begin{bmatrix} \frac{1}{2\tau}c^T Z_d(0) & 0 & 0 \\ 0 & c^T Z_d(-\tau) & 0 \\ 0 & 0 & c^T \dot{Z}_d(s) \end{bmatrix} \begin{bmatrix} x(0) \\ x(-\tau) \\ x(s) \end{bmatrix}$$
$$= \langle x', \mathcal{D}x'\rangle_Z'$$

Since the operator $\mathcal{D} \in \mathbb{P}'$ is a bounded multiplier operator, as per [17] the new LOI is solvable and c is also feasible for the original LOI.

IV. Controller Synthesis using LOIs

Even the simplest form of controller synthesis is a bilinear operator inequality: $\mathcal{P}\mathcal{A} + \mathcal{P}\mathcal{B}\mathcal{K} < 0$ with variables $\{\mathcal{P}, \mathcal{K}\} \in \mathbb{P}$. To solve controller synthesis problems in the LOI framework, we propose the following steps.

1) Formulate the full-state feedback synthesis problem
2) Formulate a Dual LOI: $\mathcal{P} > 0$ and $\mathcal{A}\mathcal{P} + (\mathcal{A}\mathcal{P})^* < 0$.
3) Convexification via variable substitution.
4) Include closed-loop gain metrics (if desired).
5) Solve the LOI.
6) Invert the operator to obtain the controller

To illustrate, we briefly describe each step. Note that we do not consider observer-based controllers.

Formulating the Control Problem Full-state feedback has the form:

$$\dot{x} = \mathcal{A}x + \mathcal{B}u \qquad u = \mathcal{K}x, \quad x \in X, \quad \mathcal{K} \in \mathbb{K}$$

A well-posed synthesis problem specifies X, Z, $\mathbb{K}$ and operators $\mathcal{B}$ and $\mathcal{A}$ such that the closed loop is well-defined for any $\mathcal{K} \in \mathbb{K}$ and generates solutions $x(t) \in X$ to the system when $x(0) \in X$.

Formulate a Dual LOI This step is the focus of the paper and gives conditions on X, Z, and $\mathbb{P}$ under which feasibility of $\mathcal{P} > 0$ and $\mathcal{A}\mathcal{P} + (\mathcal{A}\mathcal{P})^* < 0$ implies feasibility of $\mathcal{P} > 0$ and $\mathcal{P}\mathcal{A} + (\mathcal{P}\mathcal{A})^* < 0$.

Convexification Given a valid dual LOI, an LOI for full-state feedback controller synthesis is $\{\mathcal{P}, \mathcal{Z}\} \in \mathbb{P}$, $\mathcal{P} > 0$ and

$$\mathcal{A}\mathcal{P} + \mathcal{B}\mathcal{Z} + (\mathcal{A}\mathcal{P} + \mathcal{B}\mathcal{Z})^* < 0$$

where we recover the controller as $\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1}$. If we require $\mathcal{K} \in \mathbb{K}$, then we must add constraints on $\mathbb{P}$ for which $\{\mathcal{P}, \mathcal{Z}\} \in \mathbb{P}$ implies $\mathcal{Z}\mathcal{P}^{-1} \in \mathbb{K}$.

Closed-Loop L_2 Gain Metrics L_2-gain metrics can be added as in [21].

Inverting the Operator, $\mathcal{P}$ To recover the controller $\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1}$ it must be possible to construct the inverse operator $\mathcal{P}^{-1}$. For single-delay systems such a closed-form exists [22]. When closed-form formulae for the inverse do not exist, one may use a power series expansion, a finite truncation of which will yield an approximate inverse operator (See [23], [24], [25]).

Having now given our general methodology for controller synthesis via LOIs (the key to which is a duality theorem), we proceed to give the main duality result and then apply it to systems with multiple delays.

V. A Dual Stability Theorem for Infinite-Dimensional Systems

We next present a general class of dual stability LOI as motivated in the preceding section.

Theorem 1: Suppose that $\mathcal{A}$ generates a strongly continuous semigroup on Hilbert space Z with domain X. Further suppose there exists an $\epsilon > 0$ and a bounded, coercive linear operator $\mathcal{P} : X \to X$ with $\mathcal{P}(X) = X$ and which is self-adjoint with respect to the Z inner product and satisfies

$$\langle \mathcal{A}\mathcal{P}z, z\rangle_Z + \langle z, \mathcal{A}\mathcal{P}z\rangle_Z \le -\epsilon \|z\|_Z^2$$

for all $z \in X$. Then $\dot{x}(t) = \mathcal{A}x(t)$ generates an exponentially stable semigroup.

Proof: Because $\mathcal{P}$ is coercive and bounded there exist $\gamma, \delta > 0$ such that $\langle x, \mathcal{P}x\rangle \geq \gamma \|x\|^2$ and $\|\mathcal{P}x\| \leq \delta \|x\|$. By the Lax-Milgren theorem [26], its inverse exists and is bounded and $\mathcal{P}(X) = X$ implies $\mathcal{P}^{-1} : X \to X$. The inverse is self-adjoint since $\mathcal{P}$ is self-adjoint and hence $\langle \mathcal{P}^{-1}x, y\rangle = \langle \mathcal{P}^{-1}x, \mathcal{P}\mathcal{P}^{-1}y\rangle = \langle x, \mathcal{P}^{-1}y\rangle$. Since $\sup_z \frac{\|\mathcal{P}z\|}{\|z\|} = \delta < \infty$, $\inf_y \frac{\|\mathcal{P}^{-1}y\|}{\|y\|} = \inf_x \frac{\|x\|}{\|\mathcal{P}x\|} = \frac{1}{\delta} > 0$ and hence $\langle y, \mathcal{P}^{-1}y\rangle = \langle \mathcal{P}\mathcal{P}^{-1}y, \mathcal{P}^{-1}y\rangle \geq \gamma \|\mathcal{P}^{-1}y\|^2 \geq \frac{\gamma}{\delta^2}\|y\|^2$. Hence $\mathcal{P}^{-1}$ is coercive.

Define the Lyapunov functional

$$V(y) = \langle y, \mathcal{P}^{-1}y\rangle \geq \frac{\gamma}{\delta^2}\|y\|_Z^2$$

If $y(t)$ satisfies $\dot{y}(t) = \mathcal{A}y(t)$, then V has time derivative

$$\begin{aligned}\frac{d}{dt}V(y(t)) &= \langle \dot{y}(t), \mathcal{P}^{-1}y(t)\rangle + \langle y(t), \mathcal{P}^{-1}\dot{y}(t)\rangle \\ &= \langle \mathcal{A}y(t), \mathcal{P}^{-1}y(t)\rangle + \langle y(t), \mathcal{P}^{-1}\mathcal{A}y(t)\rangle \\ &= \langle \mathcal{A}y(t), \mathcal{P}^{-1}y(t)\rangle + \langle \mathcal{P}^{-1}y(t), \mathcal{A}y(t)\rangle .\end{aligned}$$

Now define $z(t) = \mathcal{P}^{-1}y(t) \in X$ for all $t \geq 0$. Then $y(t) = \mathcal{P}z(t)$ and since $\mathcal{P}$ is bounded and $\mathcal{P}^{-1}$ is coercive,

$$\begin{aligned}\dot{V}(y(t)) &= \langle \mathcal{A}y(t), \mathcal{P}^{-1}y(t)\rangle + \langle \mathcal{P}^{-1}y(t), \mathcal{A}y(t)\rangle \\ &= \langle \mathcal{A}\mathcal{P}z(t), z(t)\rangle + \langle z(t), \mathcal{A}\mathcal{P}z(t)\rangle \\ &\leq -\epsilon \|z(t)\|^2 \leq -\frac{\epsilon}{\delta}\langle z(t), \mathcal{P}z(t)\rangle \\ &= -\frac{\epsilon}{\delta}\langle y(t), \mathcal{P}^{-1}y(t)\rangle \leq -\frac{\epsilon\gamma}{\delta^3}\|y(t)\|^2 .\end{aligned}$$

Negativity of the derivative of the Lyapunov function implies exponential stability in the square norm of the state by, e.g. [18] or by the invariance principle. ■

The constraint $\mathcal{P}(X) = X$ ensures $\mathcal{P}^{-1} : X \to X$ and is satisfied if X is a closed subspace of Z or if X is itself a Hilbert space contained in Z and $\mathcal{P}$ is coercive on the space X. For the case of time-delay systems, X is not a closed subspace and we do not wish to constrain $\mathcal{P}$ to be coercive on X, since this space uses the Sobolev inner product. Therefore, we will need to prove that that for our class of operators, $\mathcal{P}(X) = X$.

A natural question is whether the constraint $P(X) = X$ is conservative (Non-conservatism of other constraints is implied in proof o Thm. 5.1.3 in [18]). In the case when X is a closed subspace of Z, then a projection argument can be applied. However, when X is not a closed subspace, the degree of conservatism is unclear.

VI. Stability of Time-Delay Systems as an LOI

In this section, we pose an LOI for primal stability of linear discrete-delay systems of the form

$$\begin{aligned}\dot{x}(t) &= A_0x(t) + \sum_{i=1}^{K} A_i x(t - \tau_i) \quad \text{for all} \quad t \geq 0, \\ x(t) &= \phi(t) \qquad\qquad \text{for all} \quad t \in [-\tau_K, 0] \end{aligned} \tag{1}$$

where $A_i \in \mathbb{R}^{n\times n}$, $\phi \in \mathcal{C}[-\tau_K, 0]$, and for convenience $\tau_1 < \tau_2 < \cdots < \tau_K$. We associate with any solution x and any time $t \geq 0$, the 'state' of System (1), $x_t \in \mathcal{C}[-\tau_K, 0]$, where $x_t(s) = x(t + s)$. For linear discrete-delay systems of the Form (1), the system has a unique solution for any $\phi \in \mathcal{C}[-\tau_K, 0]$ and global, local, asymptotic and exponential stability are all equivalent.

Stability of Eqn. (1) may be certified through the use of Lyapunov-Krasovskii functionals. In particular, it has been shown [27] that stability of Eqn. (1) is equivalent to the existence of a positive, decreasing Lyapunov-Krasovskii functional of the form

$$\begin{aligned}V(\phi) &= \tau_K\phi(0)^T P\phi(0) + \tau_K \sum_{i=1}^{K}\int_{-\tau_i}^{0}\phi(0)^T Q_i(s)\phi(s)ds \\ &+ \tau_K \sum_{i=1}^{K}\int_{-\tau_i}^{0}\phi(s)^T Q_i(s)^T\phi(0)ds + \tau_K \sum_{i=1}^{K}\int_{-\tau_i}^{0}\phi(s)^T S_i(s)\phi(s) \\ &+ \sum_{i,j=1}^{K}\int_{-\tau_i}^{0}\int_{-\tau_j}^{0}\phi(s)^T R_{ij}(s,\theta)\phi(\theta)d\theta,\end{aligned} \tag{2}$$

where the functions Q_i, S_i and R_{ij} may be assumed continuous on their respective domains of definition. This stability condition may now be formulated as a solvable LOI using the following mathematical framework. Let $Z = Z_{m,n,K} := \{\mathbb{R}^m \times L_2^n[-\tau_1, 0] \times \cdots \times L_2^n[-\tau_K, 0]\}$ and for $\{x, \phi_1, \cdots, \phi_K\} \in Z_{m,n,K}$, we define the shorthand notation

$$\begin{bmatrix} x \\ \phi_i \end{bmatrix} := \{x, \phi_1, \cdots, \phi_K\},$$

which allows us to define the inner product on Z as

$$\left\langle \begin{bmatrix} y \\ \psi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{m,n,K}} = \tau_K y^T x + \sum_{i=1}^{K}\int_{-\tau_i}^{0}\psi_i(s)^T\phi_i(s)ds.$$

When $m = n$, we simplify the notation using $Z_{n,K} := Z_{n,n,K}$. We may now conveniently write the state-space as

$$X := \left\{ \begin{bmatrix} x \\ \phi_i \end{bmatrix} \in Z_{n,K} \ : \ \substack{\phi_i \in W_2^n[-\tau_i,0] \text{ and} \\ \phi_i(0)=x \text{ for all } i\in[K]} \right\}.$$

which is a subspace of $Z_{n,K}$. We now define the infinitesimal generator, $\mathcal{A}$, of Eqn. (1) as

$$\mathcal{A}\begin{bmatrix} x \\ \phi_i \end{bmatrix}(s) := \begin{bmatrix} A_0x + \sum_{i=1}^{K} A_i\phi_i(-\tau_i) \\ \dot{\phi}_i(s) \end{bmatrix}. \tag{3}$$

Using these definitions of $\mathcal{A}$, Z and X, for matrix P and sufficiently smooth functions Q_i, S_i, R_{ij}, we define a class of operators $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \in \mathbb{P}$ of the "complete-quadratic" type as

$$\left(\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}\begin{bmatrix} x \\ \phi_i \end{bmatrix}\right)(s) := \tag{4}$$

$$\begin{bmatrix} Px + \sum_{i=1}^{K}\int_{-\tau_i}^{0} Q_i(s)\phi_i(s)ds \\ \tau_K Q_i(s)^T x + \tau_K S_i(s)\phi_i(s) + \sum_{j=1}^{K}\int_{-\tau_j}^{0} R_{ij}(s,\theta)\phi_j(\theta)\, d\theta. \end{bmatrix}$$

Then the complete-quadratic functional in Eqn. (2) is simply

$$V(\phi) = \left\langle \begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}\begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} .$$

The LOI corresponding to the time-derivative of the complete-quadratic functional is $\mathcal{A}^*\mathcal{P} + \mathcal{P}\mathcal{A} < 0$. However, since $\mathcal{A}$ is an unbounded operator, we must reformulate this second condition as a solvable LOI by noting that for $x \in X$, $\langle x, \mathcal{P}\mathcal{A}x\rangle + \langle \mathcal{P}\mathcal{A}x, x\rangle_Z = \langle \phi, \mathcal{D}\phi\rangle_{Z'}$, where $\phi \in Z' := Z_{n(K+1),n,K}$ and

$$\mathcal{D} = \mathcal{P}_{\{D_1, V_i, \dot{S}_i, G_{ij}\}}$$

where

$$D_1 := \begin{bmatrix} \Delta_0 & \Delta_1 & \cdots & \Delta_K \\ \Delta_1^T & S_1(-\tau_1) & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \Delta_K^T & 0 & 0 & S_K(-\tau_K) \end{bmatrix},$$

$$\Delta_0 = PA_0 + A_0^T P + \sum_{k=1}^K Q_k(0) + Q_k(0)^T + S_k(0)$$

$$\Delta_j = PA_j - Q_j(-\tau_j) \qquad j \in 1, \cdots, K$$

$$V_i(s) = \begin{bmatrix} \Pi_{0,i}(s)^T & \Pi_{1,i}(s)^T & \ldots & \Pi_{K,i}(s)^T \end{bmatrix}^T$$

$$\Pi_{0j}(s) = A_0^T Q_j(s) + \sum_{k=1}^K R_{jk}^T(s,0) - \dot{Q}_j(s)$$

$$\Pi_{ij}(s) = A_i^T Q_j(s) - R_{ji}^T(s,-\tau_i), \quad i,j \in 1, \cdots, K$$

$$G_{ij}(s,\theta) = -\frac{\partial}{\partial s} R_{ij}(s,\theta) - \frac{\partial}{\partial \theta} R_{ij}(s,\theta).$$

Thus the LOI for stability is the coupled LOIs $\mathcal{P} > \epsilon I$ on $Z_{n,K}$ and $\mathcal{D} < 0$ on $Z_{n(K+1),n,K}$. As we will see, LOIs on $Z_{m,n,K}$ are solvable and hence the stability problem is a solvable LOI. In the following sections, we see the dual stability condition is also a solvable LOI.

VII. A Class of Operators, $\mathbb{P}$, Which Satisfy the Duality Theorem

In this section, we add constaints on the operators in Eqn. (4) in order to satisfy the conditions of Thm. 1.

Lemma 2: Suppose that $S_i \in W_2^{n\times n}[-\tau_i, 0]$, $R_{ij} \in W_2^{n\times n}[[-\tau_i, 0] \times [-\tau_j, 0]]$ and $S_i(s) = S_i(s)^T$, $R_{ij}(s,\theta) = R_{ji}(\theta, s)^T$, $P = \tau_K Q_i(0)^T + \tau_K S_i(0)$ and $Q_j(s) = R_{ij}(0,s)$ for all i,j. Then $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ as defined in Eqn. (4) is a bounded linear operator, maps $X \to X$, and is self-adjoint on $Z_{n,K}$.

Proof: To simplify the presentation, let $\mathcal{P} := \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$. We first establish that $\mathcal{P} : X \to X$. If $\begin{bmatrix} x \\ \phi_i \end{bmatrix} \in X$, then $\phi_i \in \mathcal{C}[-\tau_i, 0]$ and $\phi_i(0) = x$. Now if

$$\begin{bmatrix} y \\ \psi_i(s) \end{bmatrix} = \left(\mathcal{P}\begin{bmatrix} x \\ \phi_i \end{bmatrix}\right)(s)$$

then since $P = \tau_K Q_i(0)^T + \tau_K S_i(0)$ and $Q_j(s) = R_{ij}(0,s)$, we have that

$$\begin{aligned}
&\psi_i(0) \\
&= \tau_K Q_i(0)^T x + \tau_K S_i(0)\phi_i(0) + \sum_{j=1}^K \int_{-\tau_j}^0 R_{ij}(0,\theta)\phi_j(\theta)d\theta \\
&= \left(\tau_K Q_i(0)^T + \tau_K S_i(0)\right)x + \sum_{j=1}^K \int_{-\tau_j}^0 R_{ij}(0,\theta)\phi_j(\theta)d\theta \\
&= Px + \sum_{j=1}^K \int_{-\tau_j}^0 Q_j(s)\phi_j(s)ds = y.
\end{aligned}$$

Further, since $S_i \in W_2$, $R_{ij} \in W_2$, and $\phi_i \in W_2$, we have $\begin{bmatrix} y \\ \psi_i \end{bmatrix} \in X$ and conclude $\mathcal{P} : X \to X$. Boundedness of Q_i, S_i and R_{ij} implies boundedness of the linear operator $\mathcal{P}$. Now, to prove that the operator $\mathcal{P}$ is self-adjoint with respect to the inner product $\langle \cdot, \cdot\rangle_{Z_{n,K}}$, we show

$$\langle y, \mathcal{P}x\rangle_{Z_{n,K}} = \langle \mathcal{P}y, x\rangle_{Z_{n,K}}$$

for any $x, y \in X$. Using the properties $S_i(s) = S_i(s)^T$ and $R_{ij}(s,\theta) = R_{ji}(\theta, s)^T$, we have the following.

$$\begin{aligned}
&\left\langle \begin{bmatrix} y \\ \psi_i \end{bmatrix}, \mathcal{P}\begin{bmatrix} x \\ \phi_i \end{bmatrix}\right\rangle_{Z_{n,K}} = \tau_K y^T \left(Px + \sum_{i=1}^K \int_{-\tau_i}^0 Q_i(\theta)\phi_i(\theta)d\theta\right) \\
&\quad + \sum_{i=1}^K \int_{-\tau_i}^0 \psi_i(s)^T \Big(\tau_K Q_i(s)^T x + \tau_K S_i(s)\phi_i(s) \\
&\qquad + \sum_{j=1}^K \int_{-\tau_j}^0 R_{ij}(s,\theta)\phi_j(\theta)d\theta\Big) \\
&= \tau_K \left(Py + \sum_{j=1}^K \int_{-\tau_j}^0 Q_i(s)\psi_j(s)ds\right)^T x \\
&\quad + \sum_{i=1}^K \int_{-\tau_i}^0 \Big(\tau_K Q_i(s)^T y + \tau_K S_i(s)^T \psi_i(s) \\
&\qquad + \sum_{j=1}^K \int_{-\tau_j}^0 R_{ji}(\theta,s)^T \psi_j(\theta)d\theta\Big)^T \phi_i(s)\, ds \\
&= \left\langle \mathcal{P}\begin{bmatrix} y \\ \psi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix}\right\rangle_{Z_{n,K}}
\end{aligned}$$

■

Finally, we show that for this class of operators, if $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ is coercive with respect to the L_2 norm, then $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}(X) = X$.

Lemma 3: Suppose that P, Q_i, S_i and R_{ij} satisfy the conditions of Lemma 2. If $\langle x, \mathcal{P}x\rangle_{Z_{n,K}} \geq \epsilon \|x\|^2$ for all $x \in X$. Then $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}(X) = X$

Proof: By Lemma 2, $\mathcal{P}$ is self-adjoint and maps $X \to X$. Since $\mathcal{P}$ is coercive, bounded and self-adjoint, its inverse is coercive, bounded and self adjoint. To show $\mathcal{P}(X) = X$, we need only show that $y = \mathcal{P}x \in X$ implies that $x \in X$. First, we show that if $\mathbf{y} = \begin{bmatrix} y \\ \psi_i(\theta) \end{bmatrix} \in X$, then $\mathbf{x} = \begin{bmatrix} x \\ \phi_i(\theta) \end{bmatrix} = \mathcal{P}^{-1}\mathbf{y}$ satisfies $x = \phi(0)$. We proceed by contradiction. Suppose $x - \phi_i(0) \neq 0$ for some i. Then we have

$$y = P(\phi_i(0) + x - \phi_i(0)) + \sum_{i=1}^K \int_{-\tau_i}^0 Q_i(s)\phi_i(s)ds$$

Now, since $\mathbf{y} \in X$, $y = \psi_i(0)$ and hence

$$y = P\phi_i(0) + \sum_{j=1}^K \int_{-\tau_j}^0 R_{ij}(0,\theta)\phi_j(\theta)\, d\theta,$$

which implies $P(x - \phi_i(0)) = 0$. Now, $\langle x, \mathcal{P}x\rangle_{Z_{n,K}} \geq \epsilon \|x\|^2$ implies $P \geq \epsilon I$. Hence $x - \phi(0) \neq 0$ implies $P(x - \phi(0)) \neq 0$, which is a contradiction. We conclude that $x = \phi_i(0)$.

Next, we prove $\phi_i \in W_2$ by showing $\left\|\dot{\phi}_i\right\|_{L_2} < \infty$. For this, we differentiate ψ_i to obtain

$$\dot{\psi}_i(s) = \tau_K \dot{Q}_i(s)^T x + \tau_K \dot{S}_i(s)\phi_i(s) + \tau_K S_i(s)\dot{\phi}_i(s) + \sum_{j=1}^K \int_{-\tau_j}^0 \partial_s R_{ij}(s,\theta)\phi_j(\theta)\, d\theta.$$

which we reverse to obtain

$$\tau_K S_i(s)\dot{\phi}_i(s) = \dot{\psi}_i(s) - \tau_K \dot{Q}_i(s)^T x - \tau_K \dot{S}_i(s)\phi_i(s) - \sum_{j=1}^K \int_{-\tau_j}^0 \partial_s R_{ij}(s,\theta)\phi_j(\theta)\, d\theta$$

which is L_2-bounded since $\dot{\psi}_i, \phi_i, \dot{Q}_i \in L_2$, and $\dot{S}_i$ and $\partial_s R_{i.j}$ are continuous and thus bounded on $[-\tau_i, 0]$. Now, for $x = 0$ and $\phi_j = 0$ for $j \neq i$, the constraint $\langle x, \mathcal{P}x\rangle_{Z_{n,K}} \geq \epsilon \|x\|^2$, implies

$$\tau_K S_i(s)\phi_i(s) + \int_{-\tau_i}^0 R_{ii}(s,\theta)\phi_i(\theta)\, d\theta$$

is coercive. Thus, since integral operators cannot be coercive for L_2-bounded kernels R_{ii}, we have that $S_i(s) \geq \eta I$ for some $\eta > 0$. Therefore, for each i, we conclude $\left\|\dot{\phi}_i\right\|_{L_2} \leq \frac{1}{\eta}\left\|S_i(s)\dot{\phi}_i(s)\right\| < \infty$. Hence $\mathbf{x} \in X$. We conclude that $\mathcal{P}(X) = X$. ∎

VIII. Reformulation as a Solvable LOI

Now that we have given conditions under which the operator $\mathcal{P}$ satisfies the duality theorem, we have that $\mathcal{AP} + \mathcal{PA}^* < 0$ implies stability where $\mathcal{A}$ is as defined in (3). However, $\mathcal{A}$ contains differential operators and hence (as was done for the primal case) in the following theorem we reformulate this as a solvable LOI with multiplier and integral operators on $Z_{n(K+1),n,K}$.

Theorem 4: Suppose that there exist P, Q_i, S_i and R_{ij} which satisfy the conditions of Lemma 2. If $\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}x\rangle_{Z_{n,K}} \geq \epsilon \|x\|^2$ for all $x \in X$ and

$$\left\langle \begin{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\\ \phi_i\end{bmatrix}, \mathcal{D}\begin{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\\ \phi_i\end{bmatrix}\right\rangle_{Z_{n(K+1),n,K}} \leq -\left\|\begin{bmatrix}x_1\\ \phi_i\end{bmatrix}\right\|^2_{Z_{n,K}}$$

for all $\begin{bmatrix}x\\ \phi_i\end{bmatrix} \in Z_{n(K+1,n,K)}$ (here $x_1 \in \mathbb{R}^n$) where

$$\mathcal{D} = \mathcal{P}_{\{D_1, V_i, \dot{S}_i, G_{ij}\}}$$

where

$$D_1 := \begin{bmatrix} C_0 + C_0^T & C_1 & \cdots & C_k \\ C_1^T & -S_1(-\tau_1) & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ C_{k^T} & 0 & 0 & -S_k(-\tau_K)\end{bmatrix},$$

$$C_0 := A_0 P + \tau_K \sum_{i=1}^K (A_i Q_i(-\tau_i)^T + \frac{1}{2}S_i(0)),$$

$$C_i := \tau_K A_i S_i(-\tau_i),$$

$$V_i(s) := \begin{bmatrix} B_i(s)^T & 0 & \cdots & 0\end{bmatrix}^T$$

$$B_i(s) := A_0 Q_i(s) + \dot{Q}_i(s) + \sum_{j=1}^K R_{ji}(-\tau_j, s),$$

$$G_{ij}(s,\theta) := \frac{\partial}{\partial s}R_{ij}(s,\theta) + \frac{\partial}{\partial \theta}R_{ji}(s,\theta)^T,$$

then the system defined by Eqn. (1) is exponentially stable.

Proof: Define the operators $\mathcal{A}$ and $\mathcal{P}$ as above. By Lemma 2, $\mathcal{P}$ is self-adjoint and maps $X \to X$. Since $\mathcal{P}$ is coercive by assumption, this implies by Theorem 1 the system is exponentially stable if

$$\left\langle \mathcal{A}P\begin{bmatrix}x\\ \phi_i\end{bmatrix}, \begin{bmatrix}x\\ \phi_i\end{bmatrix}\right\rangle + \left\langle \begin{bmatrix}x\\ \phi_i\end{bmatrix}\phi_i, \mathcal{A}P\begin{bmatrix}x\\ \phi_i\end{bmatrix}\right\rangle \leq -\left\|\begin{bmatrix}x\\ \phi_i\end{bmatrix}\right\|^2$$

for all $\begin{bmatrix}x\\ \phi_i\end{bmatrix} \in X$. We begin by constructing $\mathcal{AP}\begin{bmatrix}x\\ \phi_i\end{bmatrix} := \begin{bmatrix}y\\ \psi_i(s)\end{bmatrix}$, where

$$y = A_0 P x + \sum_{i=1}^K \int_{-\tau_i}^0 A_0 Q_i(s)\phi_i(s) ds + \sum_{i=1}^K A_i\Big(\tau_K Q_i(-\tau_i)^T x + \tau_K S_i(-\tau_i)\phi_i(-\tau_i) + \sum_{j=1}^K \int_{-\tau_j}^0 R_{ij}(-\tau_i,\theta)\phi_j(\theta) d\theta\Big),$$

$$\psi_i(s) = \tau_K \dot{Q}_i(s)^T x + \tau_K \dot{S}_i(s)\phi_i(s) + \tau_K S_i(s)\dot{\phi}_i(s) + \sum_{j=1}^K \int_{-\tau_j}^0 \frac{d}{ds}R_{ij}(s,\theta)\phi_j(\theta) d\theta.$$

Thus

$$\left\langle \begin{bmatrix}x\\ \phi_i\end{bmatrix}, \mathcal{AP}\begin{bmatrix}x\\ \phi_i\end{bmatrix}\right\rangle := \tau_K x^T y + \sum_{i=1}^K \int_{-\tau_i}^0 \phi_i(s)^T \psi_i(s) ds.$$

Examining these terms separately and using $x = \phi_i(0)$, we have

$$x^T y = x^T A_0 P x + \sum_{i=1}^K \int_{-\tau_i}^0 x^T A_0 Q_i(s)\phi_i(s) ds + \sum_{i=1}^K \tau_K x^T A_i Q_i(-\tau_i)^T x + \sum_{i=1}^K \tau_K x^T A_i S_i(-\tau_i)\phi_i(-\tau_i) + \sum_{i=1}^K \int_{-\tau_i}^0 \sum_{j=1}^K x^T A_j R_{ji}(-\tau_j,\theta)\phi_i(\theta) d\theta$$

Next, we examine the second term and use integration by parts to eliminate $\dot{\phi}$.

$$\begin{aligned}\sum_{i=1}^K \int_{-\tau_i}^0 \phi_i(s)^T \psi_i(s) ds &= \sum_{i=1}^K \tau_K \int_{-\tau_i}^0 \phi_i(s)^T \dot{Q}_i(s)^T x\, ds \\ &+ \sum_{i=1}^K \tau_K \int_{-\tau_i}^0 \phi_i(s)^T \dot{S}_i(s)\phi_i(s) ds + \sum_{i=1}^K \tau_K \int_{-\tau_i}^0 \phi_i(s)^T S_i(s)\dot{\phi}_i(s) ds \\ &+ \sum_{i,j} \int_{-\tau_i}^0 \int_{-\tau_j}^0 \phi_i(s)^T \frac{\partial}{\partial s}R_{ij}(s,\theta)\phi_j(\theta)\, ds\, d\theta \\ &= \sum_{i=1}^K \tau_K \int_{-\tau_i}^0 \phi_i(s)^T \dot{Q}_i(s)^T x\, ds + \frac{\tau_K}{2}\sum_{i=1}^K \int_{-\tau_i}^0 \phi_i(s)^T \dot{S}_i(s)\phi_i(s) ds \\ &+ \frac{\tau_K}{2} x^T \sum_{i=1}^K S_i(0) x - \frac{\tau_K}{2}\sum_{i=1}^K \phi_i(-\tau_i)^T S_i(-\tau_i)\phi_i(-\tau_i) \\ &+ \sum_{i,j} \int_{-\tau_i}^0 \int_{-\tau_j}^0 \phi_i(s)^T \frac{\partial}{\partial s}R_{ij}(s,\theta)\phi_j(\theta)\, ds\, d\theta.\end{aligned}$$

Combining both terms,

$$\left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{AP} \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} = \tau_K x^T y + \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi_i(s)^T \psi_i(s) ds$$
$$= x^T \left(\tau_K A_0 P + \sum_{i=1}^{K} \tau_K^2 A_i Q_i(-\tau_i)^T + \frac{\tau_K}{2} \sum_{i=1}^{K} S_i(0) \right) x$$
$$+ \tau_K^2 \sum_{i=1}^{K} x^T A_i S_i(-\tau_i) \phi_i(-\tau_i) - \frac{\tau_K}{2} \sum_{i=1}^{K} \phi_i(-\tau_i)^T S_i(-\tau_i) \phi_i(-\tau_i)$$
$$+ \tau_K \sum_{i=1}^{K} \int_{-\tau_i}^{0} x^T \Big(A_0 Q_i(s) + \dot{Q}_i(s) + \sum_{j=1}^{K} A_j R_{ji}(-\tau_j, s) \Big) \phi_i(s) ds$$
$$+ \frac{\tau_K}{2} \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi_i(s)^T \dot{S}_i(s) \phi_i(s) ds$$
$$+ \sum_{i,j} \int_{-\tau_i}^{0} \int_{-\tau_j}^{0} \phi_i(s)^T \frac{\partial}{\partial s} R_{ij}(s, \theta) \phi_j(\theta) \, ds \, d\theta$$

Combining this term with its adjoint, we recover

$$\left\langle \mathcal{AP} \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} + \left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{AP} \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}}$$
$$= \left\langle \begin{bmatrix} \begin{bmatrix} x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_k(-\tau_K) \end{bmatrix} \\ \phi_i \end{bmatrix}, \mathcal{D} \begin{bmatrix} \begin{bmatrix} x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_k(-\tau_K) \end{bmatrix} \\ \phi_i \end{bmatrix} \right\rangle_{Z_{nK,n,K}} \leq - \left\| \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\|_{Z_{n,K}}^2.$$

We conclude that all conditions of Theorem 1 are satisfied and hence System (1) is exponentially stable. ■

IX. CONVERTING LOIS ON $Z_{m,n,K}$ TO LOIS ON $\mathbb{R}^m \times L_2^n$

In this section, we construct an LOI on $\mathbb{R}^m \times L_2^n$ whose feasibility is equivalent to positivity on $Z_{m,n,K}$. Specifically, for an operator of the form of Eqn (4), we define a linear transformation

$$\{M, N\} := \mathcal{L}_1(P, Q_i, S_i, R_{ij})$$

where if we let $a_i = \frac{\tau_i - \tau_{i-1}}{\tau_i}$, then

$$M(s) = \Big\{ M_i(s) \quad \text{if } \; s \in [-\tau_i, -\tau_{i-1}]$$
$$N(s,\theta) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & N_{ij}(s,\theta) \end{bmatrix} \right. \quad \text{if } \; {\scriptstyle s \in [-\tau_i, -\tau_{i-1}], \atop \theta [-\tau_j, -\tau_{j-1}]}$$
$$M_i(s) := \begin{bmatrix} P & \frac{\tau_K}{a_i} Q_i(\frac{s+\tau_{i-1}}{a_i}) \\ \frac{\tau_K}{a_i} Q_i(\frac{s+\tau_{i-1}}{a_i})^T & \frac{\tau_K}{a_i} S_i(\frac{s+\tau_{i-1}}{a_i}) \end{bmatrix}$$
$$N_{ij}(s,\theta) := R_{ij}(\frac{s+\tau_{i-1}}{a_i}, \frac{\theta + \tau_{j-1}}{a_j})$$

Lemma 5: Let $\{M, N\} := \mathcal{L}_1(P, Q_i, S_i, R_{ij})$ and

$$(\mathcal{P}_{M,N} x)(s) := M(s) x(s) + \int_{-\tau_K}^{0} N(s,\theta) x(\theta) d\theta.$$

If $\langle x, \mathcal{P}_{M,N} x \rangle_{L_2^{m+n}} \geq \alpha \|x\|^2_{L_2^{m+n}}$ for some $\alpha > 0$ and all $x \in \mathbb{R}^m \times L_2^n[-\tau_K, 0]$, then $\left\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} x \right\rangle_{Z_{m,n,K}} \geq \alpha \|x\|^2_{Z_{m,n,K}}$ for all $x \in Z_{m,n,K}$.

Proof: The proof follows directly from the observation that, for $\begin{bmatrix} x \\ \phi_i \end{bmatrix} \in X$, we have

$$\left\langle \begin{bmatrix} x \\ \phi_i(s) \end{bmatrix}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \begin{bmatrix} x \\ \phi_i(s) \end{bmatrix} \right\rangle_{Z_{m,n,K}}$$
$$= \int_{-\tau_K}^{0} \begin{bmatrix} x \\ \hat{\phi}(s) \end{bmatrix}^T M(s) \begin{bmatrix} x \\ \hat{\phi}(s) \end{bmatrix} ds + \int_{-\tau_K}^{0} \int_{-\tau_K}^{0} \hat{\phi}(s)^T N(s,\theta) \hat{\phi}(s) ds \, d\theta$$
$$= \left\langle \begin{bmatrix} x \\ \hat{\phi} \end{bmatrix}, \mathcal{P}_{M,N} \begin{bmatrix} x \\ \hat{\phi} \end{bmatrix} \right\rangle_{L_2^{m+n}} \geq \alpha \left\| \begin{bmatrix} x \\ \hat{\phi} \end{bmatrix} \right\|^2_{L_2^{m+n}} = \alpha \left\| \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\|^2_{Z_{m,n,K}}$$

where

$$\hat{\phi}(s) = \Big\{ \phi_i(\tfrac{s+\tau_{i-1}}{a_i}) \quad s \in [-\tau_i, -\tau_{i-1}].$$

■

Solvability of LOIs on $\mathbb{R}^{n(K+1)} \times L_2^n$ with piecewise-polynomial multipliers and kernels was established in [17] using LMIs. Specifically, in [17], Ξ was used to denote the piecewise-continuous multiplier and kernel pairs satisfying the LMI conditions of Theorem 8 in [17]. For convenience and due to space limitations, we do not repeat that result verbatim, simply adopting the notation

$$\Xi_{d,m,n} := \{(M,N) \; : \; {\scriptstyle M \text{ and } N \text{ satisfy the conditions of} \atop {\text{Thm. 8 in [17] with } Z_1 = Z_{1pc} \atop \text{and } Z_2 = Z_{2pc} \text{ of degree } d.}}\}$$

X. DUAL STABILITY OF MULTI-DELAY SYSTEMS

In this section, we summarize the main result as an LMI using the notation $\in \Xi$ to represent the LMI constraints associated with Thm. 8 in [17].

Theorem 6: Suppose there exist $d \in \mathbb{N}$, constant $\epsilon > 0$, matrix $P \in \mathbb{R}^{n \times n}$, polynomials $S_i, Q_i \in W_2^{n \times n}[-\tau_i, 0]$, $R_{ij} \in W_2^{n \times n}[[-\tau_i, 0] \times [-\tau_j, 0]]$, such that

$$\{M - \epsilon I_{2n}, N\} \in \Xi_{d,n,n}$$
$$\left\{ -D - \epsilon \hat{I}, -E \right\} \in \Xi_{d,n(K+1),n},$$

where $\{M, N\} := \mathcal{L}_1(P, Q_i, S_i, R_{ij})$, $\hat{I} = \operatorname{diag}(I_n, 0_{n(K)}, I_n)$ and

$$\{D, E\} := \mathcal{L}_1(D_1, V_i, \dot{S}_i, G_{ij})$$

where D_1, V_i, and G_{ij} are as defined in Thm. 4 and

$$P = \tau_K Q_i(0)^T + \tau_K S_i(0), \qquad S_i(s) = S_i(s)^T,$$
$$R_{ij}(s,\theta) = R_{ji}(\theta, s)^T, \qquad Q_j(s) = R_{ij}(0, s) \quad .$$

Then the system defined by Eqn. (1) is exponentially stable.

Proof: Consider the operator $\mathcal{P} := \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$. Since $\{M - \epsilon I_{2n}, N\} \in \Xi_{d,n,n}$, by Lemma 5, we have

$$\left\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} x \right\rangle_{Z_{n,K}} \geq \epsilon \|x\|^2$$

for all $x \in Z_{n,K}$. Similarly, if we examine the operator $\mathcal{D}$ from Thm. 4, then $\mathcal{D} = \mathcal{P}_{\{D_1, V_i, \dot{S}_i, G_{ij}\}}$. Now, since $\{D, E\} := \mathcal{L}_1(D_1, V_i, \dot{S}_i, G_{ij})$ and $\left\{ -D - \epsilon \hat{I}, -E \right\} \in \Xi_{d,n(K+2),n}$, we have by Lemma 5

$$- \langle x, \mathcal{D} x \rangle_{Z_{n(K+1),n,K}} =$$
$$\left\langle x, \mathcal{P}_{\{D_1, V_i, \dot{S}_i, G_{ij}\}} x \right\rangle_{Z_{n(K+1),n,K}} \geq \epsilon \left\langle x, \hat{I} x \right\rangle_{Z_{n(K+1),n,K}}$$

for all $x \in Z_{n(K+1),n,K}$. We conclude that

$$\left\langle \begin{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\\ \phi_i\end{bmatrix}, \mathcal{D}\begin{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\\ \phi_i\end{bmatrix}\right\rangle_{Z_{n(K+1),n,K}} \le -\left\|\begin{bmatrix}x_1\\ \phi_i\end{bmatrix}\right\|^2_{Z_{n,K}}$$

for all $\begin{bmatrix}\begin{bmatrix}x_1 & x_1\end{bmatrix} & \phi_i\end{bmatrix}^T \in X$. Since $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ satisfies the conditions of Lemmas 2 and 3, Thm. 4 proves exponential stability of Equation (1). ∎

XI. Numerical Analysis

To briefly illustrate asymptotic accuracy of the dual LOI, we solve this LOI using an associated Matlab toolbox which can be found online at `http://control.asu.edu`. We consider only a single well-studied two-delay system with $\tau_1 = \tau_2/2$ and use bisection to search for the maximum stable τ_2.

$$\dot{x}(t) = \begin{bmatrix}0 & 1\\ -1 & .1\end{bmatrix}x(t) + \begin{bmatrix}0 & 0\\ -1 & 0\end{bmatrix}x(t-\tau/2) + \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}x(t-\tau)$$

d	1	2	3	4	true
$\tau_{\max}$	1.33	1.371	1.3717	1.3718	1.372
CPU sec	2.13	6.29	24.45	79.0	

This and all other examples tested indicate little, if any conservatism compared to the true maximum stable delay. In addition, testing was conducted for non-trivial examples, solving a 20-state, single-delay problem in 951s.

XII. Conclusion

In this paper, we have presented a computational and mathematical framework for control of infinite-dimensional systems. Key aspects of this framework are a methodology for formulating and solving Linear Operator Inequalities (LOIs) and a duality result allowing the controller synthesis problem to be formulated as an LOI. We have motivated this framework by showing that both primal and dual stability can be tested in the the LOI framework using asymptotic algorithms and appear to be equivalent from a numerical standpoint. The LOI for controller synthesis of multi-delay systems has not been included in this presentation, primarily for lack of space, but also pending a closed form solution for positive operators of the form $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$. A Matlab toolbox for solving LOIs in the multi-delay case has also been developed and can be found online at `http://control.asu.edu`. Computational complexity of the algorithms may be reduced by reformulation of the delayed system as in the differential-difference framework, as was done for the single delay case in [22]. Finally we note that the dual stability conditions in Thm. 4 have a sparse structure as compared with the LOI for primal stability, as can be seen in the functions V_i. It is possible that this sparsity may be exploited in the future to further reduce computational complexity.

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