A Universal State-Space Formulation of PDE Problems and Analysis using LMIs

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What is the "State" of a PDE?

Problems with Boundary Conditions and Well-Posedness

Definition 1 (The "state" is information).

The **State**, $\mathbf{x}(t)$ of a dynamic system is the **minimal information** at some time, t, needed to *Uniquely* determine the solution at all future times.

Definition 2 (The "state space" is a parameterization).

The **State Space**, X is the set of all valid states, **ANY** of which yields a **UNIQUE** solution.

Standard Definition: Its a state if a semigroup exists. $\mathbf{x} \in X$ if $\exists S$:

$$S(t)\mathbf{x} \in X,$$

$$\lim_{t \to 0^+} \partial_t S(t)\mathbf{x} = \mathcal{A}\mathbf{x}$$

Problems with state in PDEs:

- Too much information
  - No Solution
- Too little Information
  - Solution not Unique
- Hard to tell the difference

Euler-Bernoulli Beam:

$$u_{tt}(t,x) = -c u_{xxxx}(t,x)$$

Boundary Conditions:

$$u(0) = u_x(0) = u_{xx}(L) = u_{xxx}(L) = 0$$

State: $u_1 = u_t$, $u_2 = u_{xx}$

$$\mathbf{u}_t = \underbrace{\begin{bmatrix} 0 & -c\,\partial_x^2 \\ \partial_x^2 & 0 \end{bmatrix}}_{\mathcal{A} = \text{ Generator}} \mathbf{u}$$

State Space:

$$D(\mathcal{A}) = \{u_1, u_2 \in H^2 \times H^2 \ : \ u_1(0) = u_{1x}(0) = u_2(L) = u_{2x}(L) = 0\}$$

Looking For A Universal Formulation

"Problems" with the Semigroup Formulation

- Dynamics ($\mathcal{A}$) are hard to parameterize (e.g. Differential Operators)
- State Space ($D(\mathcal{A})$) is not minimal/Hilbert (e.g. $u \in H^1$ with $u(0) = 0$)

Dynamics are usually expressed in the Primal State $\mathbf{x}_p \in X_p$:

$$\mathbf{x}_p \in L^2_{n_1} \times H^1_{n_2} \times H^2_{n_3} := X_p$$

$$\begin{bmatrix} x_1(t,s) \\ x_2(t,s) \\ x_3(t,s) \end{bmatrix}_t = A_0(s) \underbrace{\begin{bmatrix} x_1(t,s) \\ x_2(t,s) \\ x_3(t,s) \end{bmatrix}}_{\mathbf{x}_p} + A_1(s) \begin{bmatrix} x_2(t,s) \\ x_3(t,s) \end{bmatrix}_s + A_2(s) \begin{bmatrix} x_3(t,s) \end{bmatrix}_{ss}$$

Boundary Conditions: $x \in X_p$:

$$B \begin{bmatrix} u_2(0) \\ u_2(L) \\ u_3(0) \\ u_3(L) \\ u_{3s}(0) \\ u_{3s}(L) \end{bmatrix} = 0, \quad \text{rank}(B) = n_2 + 2n_3$$

Euler-Bernoulli Beam:

$$\mathbf{u}_t = \underbrace{\begin{bmatrix} 0 & -c \\ 1 & 0 \end{bmatrix}}_{=A_2 \; (A_0 = A_1 = 0)} \mathbf{u}_{ss}$$

State Space: $x \in H^2_2$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} u(0) \\ u(L) \\ u_x(0) \\ u_x(L) \end{bmatrix} = 0$$

Problems with the Primal State

Simplify the dynamics

$$\dot{\mathbf{x}}(t,s) = A_0(s)\mathbf{x} + A_1(s)\mathbf{x}_s + A_2(s)\mathbf{x}_{ss}$$

Define a Lyapunov Function:

$$V(\mathbf{x}) = \int_0^L \mathbf{x}(s)^T M(s)\mathbf{x}(s)ds$$

Then $V(x) > 0$ if $M(s) \geq 0$ for all s. However,

$$\dot{V}(\mathbf{x}) = \int_0^L \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \\ \mathbf{x}_{ss} \end{bmatrix}(s)^T \underbrace{\begin{bmatrix} A_0(s)^T M(s) + M(s)A_0(s) & M(s)A_1(s) & M(s)A_2(s) \\ A_1(s)^T M(s) & 0 & 0 \\ A_2(s)^T M(s) & 0 & 0 \end{bmatrix}}_{D(s)} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \\ \mathbf{x}_{ss} \end{bmatrix}(s)ds$$

Problem: $D(s) \not< 0$ for ANY choice of A_i! Why?

Answer: $\mathbf{x}, \mathbf{x}_s, \mathbf{x}_{ss}$ are not independent states!

Solution: Express the dynamics using the **Fundamental State**

The **Fundamental State:** is the *minimal* part of $\mathbf{x}$ which is needed to define the dynamics

The boundary strongly influences the dynamics!

Extreme Example: $D(\mathcal{A}) = \{\mathbf{u} \in H^2 \;:\; \mathbf{u}(0) = w_1(t),\; \mathbf{u}_s(0) = w_2(t)\}$

$$\dot{\mathbf{u}}(t,s) = \mathbf{u}(t,s), \qquad \mathbf{u}(t,0) = w_1(t), \quad \mathbf{u}_s(t,0) = w_2(t)$$

By the Fundamental Theorem of Calculus:

$$\begin{aligned}\mathbf{u}(s) &= s\mathbf{u}(0) + \mathbf{u}_s(0) + \int_0^s (s-\eta)\mathbf{u}_{ss}(\eta)d\eta \\ &= sw_1(t) + w_2(t) + \int_0^s (s-\eta)\mathbf{u}_{ss}(\eta)d\eta\end{aligned}$$

Now rewrite the dynamics in terms of $\mathbf{u}_{ss}$:

$$\dot{\mathbf{u}}(t,s) = sw_1(t) + w_2(t) + \int_0^s (s-\eta)\mathbf{u}_{ss}(\eta)d\eta$$

Time-Delay System:

$$\begin{aligned}\dot{x}(t) &= -x(t) + u(t,\tau) \\ \mathbf{u}_t(t,s) &= \mathbf{u}_s(t,s), \quad u(t,0) = x(t)\end{aligned}$$

or completely eliminate BCs:

$$\int_0^s \dot{\mathbf{u}}_s(t,\eta)d\eta = \mathbf{u}_s(t,s) + \int_0^\tau \mathbf{u}_s(t,\eta)d\eta$$

Conclusion: The BCs *fundamentally alter* the structure of the dynamics!

What is the Fundamental State? (BCs force us to choose $\mathbf{x}_f = \mathbf{u}_{ss}$)

The M, N_1, N_2 Operator Framework

Motivated by the previous slide, we use functions M, N_1, N_2 to parameterize Multiplier/Integral Operators with Semiseparable kernels as follows

$$\left(\mathcal{P}_{\{M,N_1,N_2\}}\mathbf{x}\right)(s) := M(s)\mathbf{x}(s)ds + \int_a^s N_1(s,\theta)\mathbf{x}(\theta)d\theta + \int_s^b N_2(s,\theta)\mathbf{x}(\theta)d\theta$$

Property 1: Composition

$$\mathcal{P}_{\{R_0,R_1,R_2\}} = \mathcal{P}_{\{B_0,B_1,B_2\}}\mathcal{P}_{\{N_0,N_1,N_2\}}$$

where

$$R_0(s) = B_0(s)N_0(s)$$

$$R_1(s,\theta) = B_0(s)N_1(s,\theta) + B_1(s,\theta)N_0(\theta) + \int_a^\theta B_1(s,\xi)N_2(\xi,\theta)d\xi + \int_\theta^s B_1(s,\xi)N_1(\xi,\theta)d\xi + \int_s^b B_2(s,\xi)N_1(\xi,\theta)d\xi$$

$$R_2(s,\theta) = B_0(s)N_2(s,\theta) + B_2(s,\theta)N_0(\theta) + \int_a^s B_1(s,\xi)N_2(\xi,\theta)d\xi + \int_s^\theta B_2(s,\xi)N_2(\xi,\theta)d\xi + \int_\theta^b B_2(s,\xi)N_1(\xi,\theta)d\xi$$

Property 2: Adjoint

$$\langle \mathbf{x}, \mathcal{P}_{\{\hat{N}_0,\hat{N}_1,\hat{N}_2\}}\mathbf{y}\rangle_{L_2} = \langle \mathcal{P}_{\{N_0,N_1,N_2\}}\mathbf{x}, \mathbf{y}\rangle_{L_2}$$

where

$$\hat{N}_0(s) = N_0(s)^T, \quad \hat{N}_1(s,\eta) = N_2(\eta,s)^T, \quad \hat{N}_2(s,\eta) = N_1(\eta,s)^T$$

Conversion Between Primal and Fundamental States

For simplicity, only consider x_3.

Define the Primal State, $\mathbf{x}_p$ and Fundamental State, $\mathbf{x}_f$ as

$$\mathbf{x}_p(t,s) := \begin{bmatrix} x(t,s) \end{bmatrix}, \qquad \mathbf{x}_f(t,s) = \begin{bmatrix} x_{ss}(t,s) \end{bmatrix} \in L_2^n, \quad x_{bf} = \begin{bmatrix} x(0) \\ x(L) \\ x_s(0) \\ x_s(L) \end{bmatrix}, \quad x_{bs} = \begin{bmatrix} x(0) \\ x_s(0) \end{bmatrix}$$

Question: How to Convert? First note that

$$x_s(s) = x_s(0) + \int_a^s \mathbf{x}_{ss}(\eta)d\eta \qquad = \begin{bmatrix} 0 & I \end{bmatrix} x_{bs} + \mathcal{P}_{\{0,I,0\}}\mathbf{x}_{ss}$$

$$x(s) = x(0) + s x_s(0) + \int_0^s (s-\eta)\mathbf{x}_{ss}(\eta)d\eta \qquad = \begin{bmatrix} I & s \end{bmatrix} x_{bs} + \mathcal{P}_{\{0,s-\eta,0\}}\mathbf{x}_{ss}$$

This implies that ANY boundary condition can be represented as

$$Bx_{bf} = B\left(Kx_{bs} + \mathcal{P}_{\{0,T_1,T_2\}}\mathbf{x}_{ss}\right) = 0$$

For some fixed T_1, T_2. Hence

$$BKx_{bs} = -B\mathcal{P}_{\{0,T_1,T_2\}}\mathbf{x}_{ss}$$

Hence we can solve for x_{bs} in terms of $\mathbf{x}_{ss}$

$$x_{bs} = -(BK)^{-1}B\mathcal{P}_{\{0,T_1,T_2\}}\mathbf{x}_{ss}$$

Conclusion: Given $\mathbf{x}_{ss}$, we can reconstruct $\mathbf{x}$!

$$x(s) = \mathcal{P}_{\{0,G_1,G_2\}}\mathbf{x}_{ss}, \qquad x_s(s) = \mathcal{P}_{\{0,G_3,G_4\}}\mathbf{x}_{ss}$$

Expressing the Dynamics in Fundamental Form

We may now replace $Bb_{bf} = 0$ and

$$\dot{\mathbf{x}}_p = A_0(s)\mathbf{x}_p + A_1(s)\begin{bmatrix} x_2(t,s) \\ x_3(t,s) \end{bmatrix}_s + A_2(s)\begin{bmatrix} x_3(t,s) \end{bmatrix}_{ss}$$

with the more fundamental version:

$$\dot{\mathbf{x}}_p(t) = \mathcal{P}_{\{H_0,H_1,H_2\}}\mathbf{x}_f(t) \qquad \mathbf{x}_p(t,s) := \begin{bmatrix} x_1(t,s) \\ x_2(t,s) \\ x_3(t,s) \end{bmatrix}, \qquad \mathbf{x}_f(t,s) = \begin{bmatrix} x_1(t,s) \\ x_{2s}(t,s) \\ x_{3ss}(t,s) \end{bmatrix}$$

Where: A_0, A_1, A_2 and B come from problem definition and

$$H_0(s) = A_0(s)G_0(s) + A_1(s)G_3(s) + A_{20}(s)$$

$$H_1(s,\theta) = A_0(s)G_1(s,\theta) + A_1(s)G_4(s,\theta),$$

$$H_2(s,\theta) = A_0(s)G_2(s,\theta) + A_1(s)G_5(s,\theta), \qquad A_{20}(s) = \begin{bmatrix} 0 & 0 & A_2(s) \end{bmatrix}$$

$$G_0(s) = L_0, \quad G_1(s,\theta) = L_1(s,\theta) + G_2(s,\theta), \quad G_2(s,\theta) = -K(s)(BT)^{-1}BQ(s,\theta)$$

$$G_3(s) = F_0, \quad G_4(s,\theta) = F_1 + L_1(s,\theta) + G_5(s,\theta), \quad G_5(s,\theta) = -V(BT)^{-1}BQ(s,\theta)$$

where

$$T = \begin{bmatrix} I & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & I & (b-a)I \\ 0 & 0 & I \\ 0 & 0 & I \end{bmatrix}, \quad Q(s,\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (b-\theta)I \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$K(s) = \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & (s-a) \end{bmatrix}, \quad L_0 = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L_1(s,\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (s-\theta)I \end{bmatrix}$$

$$F_0 = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

Lyapunov Functions for PDEs

$$\dot{\mathbf{x}}_p(t) = \mathcal{P}_{\{H_0,H_1,H_2\}}\mathbf{x}_f(t)$$

We now propose a Lyapunov function of the form

$$V(\mathbf{x}_p) = \langle \mathbf{x}_p, \mathcal{P}_{\{M,N_1,N_2\}}\mathbf{x}_p\rangle$$

The time-derivative of the Lyapunov function is

$$\begin{aligned}
\dot{V}(\mathbf{x}_p(t)) &= \langle \dot{\mathbf{x}}_p, \mathcal{P}_{\{M,N_1,N_2\}}\mathbf{x}_p\rangle + \langle \mathbf{x}_p, \mathcal{P}_{\{M,N_1,N_2\}}\dot{\mathbf{x}}_p\rangle \\
&= \langle \mathcal{P}_{\{H_0,H_1,H_2\}}\mathbf{x}_f, \mathcal{P}_{\{M,N_1,N_2\}}\mathbf{x}_p\rangle + \langle \mathbf{x}_p, \mathcal{P}_{\{M,N_1,N_2\}}\mathcal{P}_{\{H_0,H_1,H_2\}}\mathbf{x}_f\rangle \\
&= \langle \mathcal{P}_{\{H_0,H_1,H_2\}}\mathbf{x}_f, \mathcal{P}_{\{M,N_1,N_2\}}\mathcal{P}_{\{G_0,G_1,G_2\}}\mathbf{x}_f\rangle + \langle \mathcal{P}_{\{G_0,G_1,G_2\}}\mathbf{x}_f, \mathcal{P}_{\{M,N_1} \\
&= \langle \mathbf{x}_f, \mathcal{P}^*_{\{H_0,H_1,H_2\}}\mathcal{P}_{\{M,N_1,N_2\}}\mathcal{P}_{\{G_0,G_1,G_2\}}\mathbf{x}_f\rangle + \langle \mathbf{x}_f, \mathcal{P}^*_{\{G_0,G_1,G_2\}}\mathcal{P}_{\{M,N_1} \\
&= \langle \mathbf{x}_f, \mathcal{P}_{\{K_0,K_1,K_2\}}\mathbf{x}_f\rangle + \langle \mathbf{x}_f, \mathcal{P}^*_{\{K_0,K_1,K_2\}}\mathbf{x}_f\rangle
\end{aligned}$$

Stability Condition: $\mathcal{P}_{\{M,N_1,N_2\}} > 0$ and

$$\mathcal{P}_{\{K_0,K_1,K_2\}} + \mathcal{P}^*_{\{K_0,K_1,K_2\}} \leq 0$$

Enforcing Positivity in the M, N_1, N_2 Framework

An LMI Condition

Theorem 3.

For any functions $Z(s)$ and $Z(s,\theta)$, and $g(s) \geq 0$ for all $s \in [a,b]$

$$\begin{aligned}
M(s) &= g(s)Z(s)^T P_{11} Z(s)\\
N_1(s,\theta) &= g(s)Z(s)^T P_{12} Z(s,\theta) + g(\theta)Z(\theta,s)^T P_{31} Z(\theta) + \int_a^\theta g(\nu)Z(\nu,s)^T P_{33} Z(\nu,\theta)d\nu\\
&\quad + \int_\theta^s g(\nu)Z(\nu,s)^T P_{32} Z(\nu,\theta)d\nu + \int_s^L g(\nu)Z(\nu,s)^T P_{22} Z(\nu,\theta)d\nu\\
N_2(s,\theta) &= g(s)Z(s)^T P_{13} Z(s,\theta) + g(\theta)Z(\theta,s)^T P_{21} Z(\theta) + \int_a^s g(\nu)Z(\nu,s)^T P_{33} Z(\nu,\theta)d\nu\\
&\quad + \int_s^\theta g(\nu)Z(\nu,s)^T P_{23} Z(\nu,\theta)d\nu + \int_\theta^L g(\nu)Z(\nu,s)^T P_{22} Z(\nu,\theta)d\nu,
\end{aligned}$$

where

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13}\\ P_{21} & P_{22} & P_{23}\\ P_{31} & P_{32} & P_{33}\end{bmatrix} \geq 0,$$

then $\mathcal{P}^*_{\{M,N_1,N_2\}} = \mathcal{P}_{\{M,N_1,N_2\}}$ and $\langle \mathbf{x}, \mathcal{P}_{\{M,N_1,N_2\}}\mathbf{x}\rangle_{L_2} \geq 0$ for all $\mathbf{x} \in L_2[a,b]$.

Proof: Let $P = \begin{bmatrix} Q_0\\ Q_1\\ Q_2\end{bmatrix}\begin{bmatrix} Q_0 & Q_1 & Q_2\end{bmatrix}$ and define

$$T_0(s) = Q_0\sqrt{g(s)}Z(s), \quad T_1(s,\theta) = Q_1\sqrt{g(s)}Z(s,\theta), \quad T_2(s,\theta) = Q_2\sqrt{g(s)}Z(s,\theta)$$

Then

$$\mathcal{P}_{\{M,N_1,N_2\}} = \mathcal{P}^*_{\{T_0,T_1,T_2\}}\mathcal{P}_{\{T_0,T_1,T_2\}} \geq 0.$$

Notations and associated Matlab Functions:

$$\{M, N_1, N_2\} \in \Phi_d \qquad \rightarrow \qquad \mathcal{P}_{\{M,N_1,N_2\}} \geq 0$$

```
[prog, M, N1, N2]= sosjointpos_mat_ker_semisep(prog,n,d,d,s,th,[a,b])
```

$$\{M, N_1, N_2\} = \{T_0, T_1, T_2\} \times \{R_0, R_1, R_2\} \quad \rightarrow \quad \mathcal{P}_{\{M,N_1,N_2\}} = \mathcal{P}_{\{T_0,T_1,T_2\}}\mathcal{P}_{\{R_0,R_1,R_2\}}$$

```
[M, N1, N2] = semisep_MN1N2_compose(T0,T1,T2,R0,R1,R2,s,th,[a,b])
```

$$\{M, N_1, N_2\} = \{T_0, T_1, T_2\}^* \quad \rightarrow \quad \mathcal{P}_{\{M,N_1,N_2\}} = \mathcal{P}^*_{\{T_0,T_1,T_2\}}$$

```
[M, N1, N2] = semisep_MN1N2_transpose(T0,T1,T2,s,th)
```

Almost Complete Matlab Code:

```
pvar s th
[prog, G0, G1, G2]=...
[prog, H0, H1, H2]=...
prog = sosprogram([s th])
[prog, M, N1, N2]= sosjointpos_mat_ker_semisep(prog,n,d,d,s,th,II)
[J0, J1, J2] = semisep_MN1N2_compose(M+ep*I,N1,N2,G0,G1,G2,s,th,II)
[H0s, H1s, H2s] = semisep_MN1N2_transpose(H0,H1,H2,s,th)
[K0, K1, K2] = semisep_MN1N2_compose(H0s,H1s,H2s,J0,J1,J2,s,th,II)
[K0s, K1s, K2s] = semisep_MN1N2_transpose(K0,K1,K2,s,th)
[prog, [],N1e, N2e] = sosjointpos_mat_ker_semisep(prog,n,d+2,d+2,s,th,II)
[prog, [],gN1e, gN2e] = sosjointpos_mat_ker_semisep_psatz(prog,n,d+2,d+2,s,th,II)
[prog] = sosmateq(prog,K1+K1s+N1eq+gN1eq)
prog = sossolve(prog,pars)
```

Stability Conditions:

$\{M, N_1, N_2\} \in \Phi_d$

$$\{K_0, K_1, K_2\} = \{G_0, G_1, G_2\}^* \times \{M + \varepsilon I, N_1, N_2\} \times \{H_0, H_1, H_2\}$$

$\{K_0, K_1, K_2\} + \{K_0, K_1, K_2\}^* \in \Phi_{d+2}$

Testing for Accuracy

Example 1: Adapted from Valmorbida, 2014:

$$\dot{x}(t,s) = \lambda x(t,s) + x_{ss}(t,s) \qquad x(0) = x(1) = 0$$

Stable iff $\lambda < \pi^2 \cong 9.8696$. For $d = 1$, we prove stability for $\lambda = 9.8696$.

Example 2: From Valmorbida, 2016,

$$\dot{x}(t,s) = \lambda x(t,s) + x_{ss}(t,s) \qquad x(0) = 0, \quad x_s(1) = 0$$

Is unstable for $\lambda > 2.467$. For $d = 1$, we prove stability for $\lambda = 2.467$.

Example 3: From Gahlawat, 2017:

$$\dot{x}(t,s) = (-.5s^3 + 1.3s^2 - 1.5s + .7 + \lambda)x(t,s) + (3s^2 - 2s)x_s(t,s) + (s^3 - s^2 + 2)x_{ss}(t,s)$$

with $x(0) = 0$ and $x_s(1) = 0$. Unstable for $\lambda > 4.65$. For $d = 1$, we prove stability for $\lambda = 4.65$.

Example 4: From Valmorbida,2014,

$$\dot{x}(t,s) = \begin{bmatrix} 1 & 1.5 \\ 5 & .2 \end{bmatrix} x(t,s) + R^{-1} x_{ss}(t,s), \qquad x(0) = x_s(1) = 0$$

With $d = 1$, we prove stability for $R = 2.93$ (improvement over $R = 2.45$).

Example 5: From Valmorbida,2016,

$$\dot{x}(t,s) = \begin{bmatrix} 0 & 0 & 0 \\ s & 0 & 0 \\ s^2 & -s^3 & 0 \end{bmatrix} x(t,s) + R^{-1} x_{ss}(t,s), \qquad x(0) = x_s(1) = 0$$

Using $d = 1$, we prove stability for $R = 21$ (and greater) with a computation time of $4.06s$.

Testing for Computational Complexity

We explore computational complexity using a simple n-dimensional diffusion equation

$$\dot{x}(t,s) = x(t,s) + x_{ss}(t,s)$$

where $x(t,s) \in \mathbb{R}^n$. We then evaluate the computation time for different size problems, from $n = 1$ to $n = 20$.

n	1	5	10	20
CPU sec	.54	37.4	745	31620

Illustration 1: The Euler-Bernoulli Beam

Consider a simple cantilevered E-B beam:

$$u_{tt}(t,x) = -c u_{xxxx}(t,x), \qquad \text{where} \quad u(0) = u_x(0) = u_{xx}(L) = u_{xxx}(L) = 0$$

Step 1: Eliminate the u_{tt} term - let $u_1 = u_t$
Step 2: Eliminate u_{xxxx} - let $u_2 = u_{xx}$

$$\dot{u}_1 = u_{tt} = -c u_{xxxx} = -c u_{2xx}, \qquad \dot{u}_2 = u_{txx} = u_{1xx}.$$

Universal Formulation:

$$\mathbf{x}_t = \underbrace{\begin{bmatrix} 0 & -c \\ 1 & 0 \end{bmatrix}}_{A_2} \mathbf{x}_{xx}$$

where $A_0 = A_1 = 0$, $n_3 = 2$, and $n_1 = n_2 = 0$.
Boundary Conditions:

$$u_{xx}(L) = u_2(L) = 0 \quad \text{and} \quad u_{xxx}(L) = u_{2x}(L) = 0.$$

Insufficient BCs! - $rank(B) = 2$. Differentiate BCs in time to get:

$$u_t(0) = u_1(0) = 0 \quad \text{and} \quad u_{tx}(0) = u_{1x}(0) = 0.$$

This yields $rank(B) = 4$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_1(L) \\ u_2(L) \\ u_{1x}(0) \\ u_{2x}(0) \\ u_{1x}(L) \\ u_{2x}(L) \end{bmatrix} = 0.$$

Conclusion: The E-B beam is exp. stable for any $c > 0$ w/r to u_t and u_{xx}.

Illustration 2: The Timoschenko Beam

Consider a simple Timoschenko beam model:

$$\ddot{w} = \partial_x(w_x - \phi) \qquad = -\phi_x + w_{xx}$$

$$\ddot{\phi} = \phi_{xx} + (w_x - \phi) \qquad = -\phi + w_x + \phi_{xx}$$

with boundary conditions

$$\phi(0) = 0, \quad w(0) = 0, \quad \phi_x(L) = 0, \quad w_x(L) - \phi(L) = 0$$

Step 1: Eliminate w_{tt} and ϕ_{tt} - $u_1 = w_t$ and $u_3 = \phi_t$.
Step 2: Use BCs to pick the state - $u_2 = w_x - \phi$ and $u_4 = \phi_x$.

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_t = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A_0} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{A_1} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_x$$

where $A_2 = []$ and $n_1 = n_3 = 0$ and $n_2 = 4$ - a purely "hyperbolic" form. We only need 4 BCs:

$$u_1(0) = 0, \quad u_3(0) = 0, \quad u_4(L) = 0, \quad u_2(L) = 0$$

This gives a B has row rank $n_2 = 4$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{B} \begin{bmatrix} u(0) \\ u(L) \end{bmatrix} = 0$$

Stable! However, not exponentially stable ($\dot{V} \not< 0$) in all the given states.

Illustration 2b: The Timoschenko Beam revisited

Consider a modification - naively choose $u_2 = w_x$ and $u_4 = \phi$. This leads to

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_t = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{A_0} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A_1} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{A_2} u_{4xx}$$

where $n_1 = 0$, $n_2 = 3$, and $n_3 = 1$ and with 5 boundary conditions

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}}_{B} \begin{bmatrix} u_{1-3}(0) \\ u_{1-3}(L) \\ u_4(0) \\ u_4(L) \\ u_{4x}(0) \\ u_{4x}(L) \end{bmatrix} = 0.$$

NOT Stable in the given states!

However: If we add a damping term $-cu_{4t} = -cu_3$ to $\dot{u}_3$, then the only change is

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -c & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Now Stable for any $c > 0$! Stability is sensitive to definition of states!

Illustration 3: The Tip-Damped Wave Equation

The simplest tip-damped wave equation is

$$u_{tt}(t,s) = u_{ss}(t,s) \qquad u(t,0) \quad = 0 \qquad u_s(t,L) = -k u_t(t,L).$$

Guided by the boundary conditions, we choose

$$u_1(t,s) = u_s(t,s)$$
$$u_2(t,s) = u_t(t,s)$$

This yields

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A_1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_s$$

where $A_0 = 0$, $A_2 = []$ $n_1 = n_3 = 0$ and $n_2 = 2$. The boundary conditions are now

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \end{bmatrix}}_{B} \begin{bmatrix} u(0) \\ u_(L) \end{bmatrix} = 0.$$

We find this formulation is exp stable in the given states u_t, u_x for $k > 0$.

Illustration 4: Non-"Hyperbolic" Damped Wave Equation

Add u to the dynamics (stable for $a, k \neq 0$)

$$\begin{aligned} u_{tt}(t,s)u_{ss}(t,s) - 2au_t(t,s) - a^2u(t,s) \qquad & s \in [0,1] \\ u(t,0) = 0, \qquad u_s(t,1) \qquad & = -ku_t(t,1) \end{aligned}$$

Must choose the variables $u_1 = u_t$ and $u_2 = u$. Yields the diffusive form:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t = \underbrace{\begin{bmatrix} -2a & -a^2 \\ 1 & 0 \end{bmatrix}}_{A_0} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{A_2} u_{2xx}$$

where $A_1 = 0$, $n_1 = 0$, $n_2 = 1$, and $n_3 = 1$. The BCs on u_1 make us consider this a hyperbolic state!

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_1(L) \\ u_2(0) \\ u_2(L) \\ u_{2x}(0) \\ u_{2x}(L) \end{bmatrix} = 0.$$

Stable!, but not exponentially stable in the given state (confirmed analytically).

Conclusion and Extensions (Thanks to ONR #N000014-17-1-2117)

$\mathcal{P}_{\{M,N_1,N_2\}}$ Framework extends LMI techniques to PDEs.

- $A^TP + PA < 0$ becomes

$$\underbrace{\mathcal{P}^*_{\{H_0,H_1,H_2\}}}_{A^T}\underbrace{\mathcal{P}_{\{M,N_1,N_2\}}}_{P}\mathcal{P}_{\{G_0,G_1,G_2\}}+\mathcal{P}^*_{\{G_0,G_1,G_2\}}\underbrace{\mathcal{P}_{\{M,N_1,N_2\}}}_{P}\underbrace{\mathcal{P}_{\{H_0,H_1,H_2\}}}_{A}\leq 0$$

Conclusions:

PROs:

- Computationally Efficient
- A more rational treatment of boundary conditions.
- No Conservatism (Almost N+S)
- Easily Extended to New Problems
  - e.g. higher order derivatives
  - e.g. distributed dynamics

CONs:

- Requires $n_2 + 2n_3$ BCs to be clearly specified
- PDE Must be Stable in all States

Extensions:

- Input-Output Properties (ACC, 2019)
  - H_∞ Gain
  - passivity
- ODEs coupled with PDEs (CDPS, 2019)
- Optimal Estimator Synthesis
- Optimal Controller Synthesis

Solvable (in order of difficulty)

- Extension to 3D
- Duality (Stability of $\mathcal{A}^*$)
- Inversion of the $\mathcal{P}_{\{M,N_1,N_2\}}$ Operator
  - Want an Analytic Formula