

Optimal Control (And Estimation) of Dynamic Systems with State Delay

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Control of Differential Equations with State Delay

Consider a MIMO Linear Differential-Difference Equation.

$$\dot{x}(t) = A_0x(t) + \sum_i A_i x(t - \tau_i) + B_1w(t) + B_2u(t),$$

$$y(t) = Cx(t) + D_1w(t) + D_2u(t),$$

Stability Analysis is a **CLOSED PROBLEM**.

- SOS analysis is accurate to 6 decimal places

However,

- H_∞ -Optimal Controller Design is OPEN
- H_∞ -Optimal Estimator Design is OPEN

In this talk, we will CLOSE the control problem.

- Also the estimator problem (MTNS 2018)

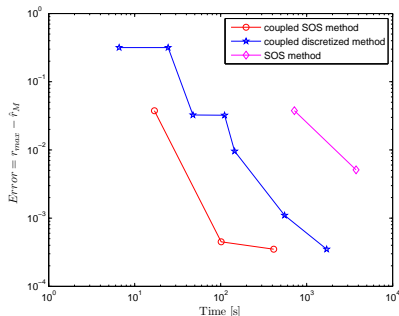


Figure: Comparison of asymptotic algorithms for maximum stable delay

H_∞ -Optimal Controller Synthesis Problem to be Solved

Consider solutions of

$$\begin{aligned}\dot{x}(t) &= A_0x(t) + \sum_i A_i x(t - \tau_i) + B_1w(t) + B_2u(t), \\ y(t) &= Cx(t) + D_1w(t) + D_2u(t).\end{aligned}$$

Problem Definition:

Minimize γ such that there exist K_0 , K_{1i} and $K_{2i}(s)$ such that if

$$u(t) = K_0x(t) + \sum_i K_{1i}x(t - \tau_i) + \sum_i \int_{-\tau_i}^0 K_{2i}(s)x(t + s)ds$$

then for any $w \in L_2$, $\|y\|_{L_2} \leq \gamma\|w\|_{L_2}$.

Roadmap of the Talk

Find $\mathcal{K} : Z \rightarrow Z$ such that

$$\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t), \quad u(t) = \mathcal{K}\mathbf{x}, \quad y(t) = \mathcal{C}\mathbf{x}(t) + \mathcal{D}_1 w(t) + \mathcal{D}_2 u(t)$$

implies $\|y\|_{L_2} \leq \gamma \|w\|_{L_2}$

Step 1: Solve the problem as a abstract but convex Linear Operator Inequality.

Step 2: Parameterize All Operators using Matrices.

- Synthesis conditions now linear matrix constraints and operator positivity constraints
- $\mathcal{P}_{\{P, Q_i, S_i, R_{ij}\}}$ framework

Step 3: Enforce Operator Positivity using LMIs.

Step 4: Solve the LMI and Reconstruct the controller gains.

- Invert the operator using matrix manipulations.

An LMI for Optimal Control of **ODEs**

Get rid of the delays and we have

$$\dot{x}(t) = Ax(t) + B_1w(t) + B_2u(t), \quad y(t) = Cx(t) + D_1w(t) + D_2u(t).$$

Lemma 1 (Full-State Feedback Controller Synthesis).

Define:

$$\hat{G}(s) = \left[\begin{array}{c|c} A + B_2K & B_1 \\ \hline C + D_2K & D_1 \end{array} \right].$$

The following are equivalent.

- There exists a K such that $\|\hat{G}\|_{H_\infty} \leq \gamma$.
- There exists a $P > 0$ and Z such that

$$\left[\begin{array}{ccc} PA^T + AP + Z^T B_2^T + B_2 Z & B_1 & PC_1^T + Z^T D_{12}^T \\ & B_1^T & D_{11}^T \\ & C_1 P + D_{12} Z & D_{11} & -\gamma I \end{array} \right] < 0$$

The Controller is recovered as $K = ZP^{-1}$.

- $P > 0$ ensures P is invertible.

Make a DDE look like an ODE: Put it in 1st-Order Form

Write the DDE as

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}_1w(t) + \mathcal{B}_2u(t), \quad y(t) = \mathcal{C}x(t) + \mathcal{D}_1w(t) + \mathcal{D}_2u(t).$$

where

$$\mathcal{A} \begin{bmatrix} x \\ \phi_i \end{bmatrix} (s) := \begin{bmatrix} A_0x + \sum_{i=1}^K A_i\phi_i(-\tau_i) \\ \dot{\phi}_i(s) \end{bmatrix}, \quad \left(\mathcal{C} \begin{bmatrix} \psi \\ \phi_i \end{bmatrix} \right) := [C_0\psi + \sum_i C_i\phi_i(-\tau_i)]$$

$$(\mathcal{B}_1w)(s) := \begin{bmatrix} B_1w \\ 0 \end{bmatrix}, \quad (\mathcal{B}_2u)(s) := \begin{bmatrix} B_2u \\ 0 \end{bmatrix},$$

$$(\mathcal{D}_1w)(s) := [D_1w], \quad (\mathcal{D}_2u)(s) := [D_2u]$$

Details: $\mathcal{A} : X \rightarrow Z_{n,K}$, $\mathcal{B}_1 : \mathbb{R}^m \rightarrow Z_{n,K}$, $\mathcal{B}_2 : \mathbb{R}^p \rightarrow Z_{n,n,K}$, $\mathcal{D}_1 : \mathbb{R}^m \rightarrow \mathbb{R}^q$, $\mathcal{D}_2 : \mathbb{R}^p \rightarrow \mathbb{R}^q$, and $\mathcal{C} : Z_{n,n,K} \rightarrow \mathbb{R}^p$ where

$$Z_{m,n,K} := \{\mathbb{R}^m \times L_2^n[-\tau_1, 0] \times \cdots \times L_2^n[-\tau_K, 0]\}$$
$$\left\langle \begin{bmatrix} y \\ \psi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{m,n,K}} := \tau_K y^T x + \sum_{i=1}^K \int_{-\tau_i}^0 \psi_i(s)^T \phi_i(s) ds$$
$$X := \left\{ \begin{bmatrix} x \\ \phi_i \end{bmatrix} \in Z_{n,K} : \phi_i \in W_2^n[-\tau_i, 0] \text{ and } \phi_i(0) = x \text{ for all } i \in [K] \right\}.$$

The DPS/DDE Equivalent of the Synthesis LMI

LMI Version of Controller Synthesis: Minimize γ such that $\exists P > 0$ and

$Z \in \mathbb{R}^{p \times n}$ such that

$$\begin{bmatrix} z \\ w \\ v \end{bmatrix}^T \begin{bmatrix} YA^T + AY + Z^T B_2^T + B_2 Z & B_1 & YC_1^T + Z^T D_{12}^T \\ & -\gamma I & D_{11}^T \\ C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} z \\ w \\ v \end{bmatrix} \\ = z^T P A^T z + z^T A P z + z^T Z^T B_2^T z + z^T B_2 Z z + z^T B_1 w + w^T B_1^T z - \gamma w^T w \\ + v^T (C P z) + (C P z)^T v + v^T (D_2 Z z) + (D_2 Z z)^T v + v^T (D_1 w) + (D_1 w)^T v - \gamma v^T v \\ \leq 0$$

for all $z \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $v \in \mathbb{R}^q$

DPS Version of Controller Synthesis: Minimize γ such that $\exists \mathcal{P} : X \rightarrow X$ (coercive, $\mathcal{P} = \mathcal{P}^*$, $\mathcal{P}(X) = X$) and \mathcal{Z} such that

$$\langle \mathcal{A} \mathcal{P} \mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{A} \mathcal{P} \mathbf{z} \rangle_Z + \langle \mathcal{B}_2 \mathcal{Z} \mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{B}_2 \mathcal{Z} \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{B}_1 w \rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z} \rangle_Z - \gamma w^T w \\ + v^T (C \mathcal{P} \mathbf{z}) + (C \mathcal{P} \mathbf{z})^T v + v^T (D_2 \mathcal{Z} \mathbf{z}) + (D_2 \mathcal{Z} \mathbf{z})^T v + v^T (D_1 w) + (D_1 w)^T v - \gamma v^T v \leq -\epsilon \|z\|_Z^2$$

for all $\mathbf{z} \in Z$, $w \in \mathbb{R}^m$, $v \in \mathbb{R}^q$

An LMI for Optimal *Estimation* of **ODEs**

Get rid of the delays and we have

$$\dot{x}(t) = Ax(t) + B_1w(t), \quad y(t) = C_2x(t) + Dw(t)$$

Observer:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + L(C_2\hat{x}(t) - y(t)), \quad z_e(t) = C_1(\hat{x}(t) - x(t))$$

Lemma 2 (H_∞ -Optimal Observer Synthesis).

Define the map $w \mapsto z_e$:

$$\hat{G}(s) = \left[\begin{array}{c|c} A + LC_2 & -(B + LD) \\ \hline C_1 & 0 \end{array} \right].$$

The following are equivalent.

- There exists a L such that $\|\hat{G}\|_{H_\infty} \leq \gamma$.
- There exists a $P > 0$ and Z such that

$$\begin{bmatrix} A^T P + C_2^T Z^T + PA + ZC_2 & -(PB + ZD) \\ -(PB + ZD)^T & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C_1^T C_1 & 0 \\ 0 & 0 \end{bmatrix} < 0.$$

The Observer Gain is recovered as $L = P^{-1}Z$.

The DPS/DDE Equivalent of the Observer LMI

LMI Version of Observer Synthesis: Minimize γ such that $\exists P > 0$ and $Z \in \mathbb{R}^{p \times n}$ such that

$$\begin{aligned} & \begin{bmatrix} e \\ w \end{bmatrix}^T \left[\begin{bmatrix} A^T P + C_2^T Z^T + PA + ZC_2 & -(PB + ZD) \\ -(PB + ZD)^T & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C_1^T C_1 & 0 \\ 0 & 0 \end{bmatrix} \right] \begin{bmatrix} e \\ w \end{bmatrix} \\ & = (PAe)^T e + (PAe)^T e + (ZCe)^T e + (ZC_2e)^T e \\ & - e^T PBw - (PBw)^T e - \gamma w^T w + \frac{1}{\gamma} (C_1e)^T (C_1e) < 0 \end{aligned}$$

for all $e \in \mathbb{R}^n$, $w \in \mathbb{R}^m$

DPS Version of Observer Synthesis: Minimize γ such that $\exists P > 0$ and Z such that

$$\begin{aligned} & \langle PAe, e \rangle_{L_2} + \langle e, PAe \rangle_{L_2} + \langle ZC_2e, e \rangle_{L_2} + \langle e, ZC_2e \rangle_{L_2} \\ & - \langle e, PBw \rangle_{L_2} - \langle Bw, Pe \rangle_{L_2} - \gamma w^T w + \frac{1}{\gamma} (C_1e)^T (C_1e) < -\epsilon \|e\|^2 \quad \forall e \in X, w \in \mathbb{R}^m \end{aligned}$$

How to Solve these LOIs?

Enforce $\mathcal{P} \geq 0$ or equivalently $V(x) = \langle x, \mathcal{P}x \rangle \geq 0$

The Wrong Way: Project onto \mathbb{R}^n

- 1. Model Transformations:** $V = z^T M z$ where $z(t) = x(t - \tau) + \int_{-\tau}^t A_0 x(s) + A_1 x(s - \tau) ds$.
- 2. Jensen's Inequality:** $V = z^T M z$ where $z(t) = \int_{-\tau}^0 \phi(t, s) ds$.
- 3. Wirtinger/Legendre:** $V = z^T M z$ where $z_i(t) = \int_{-\tau}^0 L_i(s) \phi(t, s) ds$.

The Right Way: Lift LMIs to $\mathbb{R}^n \times L_2$. Let $V = \langle \mathbf{z}, M \mathbf{z} \rangle$ where $M > 0$ and

$$\mathbf{z}(s) = \begin{bmatrix} x \\ Z(s)\phi(s) \\ \int_{-\tau}^0 Z(s, \theta)\phi(\theta)d\theta. \end{bmatrix} \quad \text{Then } V(\mathbf{x}) := \int_{-\tau}^0 \begin{bmatrix} x \\ \phi(s) \end{bmatrix} \left(\mathcal{P} \begin{bmatrix} x \\ \phi \end{bmatrix} \right) (s) ds$$

where

$$\left(\mathcal{P} \begin{bmatrix} x \\ \phi \end{bmatrix} \right) (s) = \begin{bmatrix} Px + \int_{-\tau}^0 Q(\theta)\phi(\theta)d\theta \\ Q(s)^T x + S(s)\phi(s) + \int_{-\tau}^0 R(s, \theta)\phi(\theta)d\theta \end{bmatrix}$$

$$P = M_{11} \cdot \frac{1}{\tau} \int_{-\tau}^0 ds, \quad Q(s) = \frac{1}{\tau} \left(M_{12} Z(s) + \int_{-\tau}^0 M_{13} Z(\eta, s) d\eta \right), \quad S(s) = \frac{1}{\tau} Z(s)^T M_{22} Z(s),$$

$$R(s, \theta) = Z(s)^T M_{23} Z(s, \theta) + Z(\theta, s)^T M_{32} Z(\theta) + \int_{-\tau}^0 Z(\eta, s)^T M_{33} Z(\eta, \theta) d\eta$$

The PQRS Framework - Parametrization and Positivity

Parameterize all operators as

$$\left(\mathcal{P}_{\{P, Q_i, S_i, R_{ij}\}} \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right) (s) := \begin{bmatrix} Px + \sum_{i=1}^K \int_{-\tau_i}^0 Q_i(s) \phi_i(s) ds \\ \tau_K Q_i(s)^T x + \tau_K S_i(s) \phi_i(s) + \sum_{j=1}^K \int_{-\tau_j}^0 R_{ij}(s, \theta) \phi_j(\theta) d\theta \end{bmatrix}$$

Positivity: To Constrain $\mathcal{P}_{\{P, Q_i, S_i, R_{ij}\}} \geq 0$: Define $a_i = \frac{\tau_i}{\tau_K}$ and

$$\hat{Q}(s) := [\sqrt{a_1} Q_1(a_1 s) \quad \cdots \quad \sqrt{a_K} Q_K(a_K s)], \quad \hat{S}(s) := \begin{bmatrix} S_1(a_1 s) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & S_K(a_K s) \end{bmatrix},$$

$$\hat{R}(s, \theta) := \begin{bmatrix} \sqrt{a_1 a_1} R_{11}(sa_1, \theta a_1) & \cdots & \sqrt{a_1 a_K} R_{1K}(sa_1, \theta a_K) \\ \vdots & \ddots & \vdots \\ \sqrt{a_K a_1} R_{K1}(sa_K, \theta a_1) & \cdots & \sqrt{a_K a_K} R_{KK}(sa_K, \theta a_K) \end{bmatrix}.$$

Now constrain (using $g = 1$ and $g = -s(s + \tau_K)$)

$$P = M_{11} \cdot \frac{1}{\tau_K} \int_{-\tau_K}^0 g(s) ds, \quad \hat{S}(s) = \frac{1}{\tau_K} g(s) Z(s)^T M_{22} Z(s)$$

$$\hat{Q}(s) = \frac{1}{\tau_K} \left(g(s) M_{12} Z(s) + \int_{-\tau_K}^0 g(\eta) M_{13} Z(\eta, s) d\eta \right)$$

$$\hat{R}(s, \theta) = g(s) Z(s)^T M_{23} Z(s, \theta) + g(\theta) Z(\theta, s)^T M_{32} Z(\theta) + \int_{-\tau_K}^0 g(\eta) Z(\eta, s)^T M_{33} Z(\eta, \theta) d\eta$$

Matlab Command: `[P,Q,R,S]=sosjointpos_mat_ker_ndelay_PQRS`

How to work in the PQRS framework?

Take each term in the LOI and make it look like a PQRS operator

$$\langle \mathbf{z}, \mathcal{AP}_{\{P, Q_i, S_i, R_{ij}\}} \mathbf{z} \rangle = \text{bunch of terms} = \langle \tilde{\mathbf{z}}, \mathcal{P}_{\{D, E_i, F_i, G_{ij}\}} \tilde{\mathbf{z}} \rangle$$

What does a PQRS operator look like?

$$\begin{aligned} & \left\langle \underbrace{\begin{bmatrix} h \\ \phi_i \end{bmatrix}}_{\tilde{\mathbf{z}}}, \mathcal{P}_{\{D, E_i, F_i, G_{ij}\}} \underbrace{\begin{bmatrix} h \\ \phi_i \end{bmatrix}}_{\tilde{\mathbf{z}}} \right\rangle_{Z_{r, n, K}} \\ &= \tau_K h^T D h + \tau_K \sum_{i=1}^K \int_{-\tau_i}^0 h^T E_i(s) \phi_i(s) ds + \tau_K \sum_i \int_{-\tau_i}^0 \phi_i(s)^T E_i(s)^T h ds \\ &+ \tau_K \sum_i \int_{-\tau_i}^0 \phi_i(s)^T F_i(s) \phi_i(s) ds + \sum_{ij} \int_{-\tau_i}^0 \int_{-\tau_j}^0 \phi_i(s)^T G_{ij}(s, \theta) \phi_i(\theta) d\theta ds. \end{aligned}$$

Take each term in $\langle \mathbf{z}, \mathcal{AP}_{\{P, Q_i, S_i, R_{ij}\}} \mathbf{z} \rangle$ and associate it to a D , E_i , F_i or G_{ij} .

- Illustrated on the next few slides

Put Each Term in the PQRS Framework

Define $\mathbf{z} = \begin{bmatrix} x \\ \phi_i \end{bmatrix}$ and $h = [v^T \quad w^T \quad x^T \quad \phi_1(-\tau_1)^T \quad \dots \quad \phi_K(-\tau_K)^T]^T$.

H_∞ -optimal Controller Synthesis Condition: Let $\mathcal{P} = \mathcal{P}_{\{P, Q_i, S_i, R_{ij}\}}$

$$\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z} \rangle_Z + \langle \mathcal{B}_2\mathcal{Z}\mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{B}_2\mathcal{Z}\mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{B}_1w \rangle_Z + \langle \mathcal{B}_1w, \mathbf{z} \rangle_Z - \gamma w^T w + v^T(\mathcal{C}\mathcal{P}\mathbf{z}) + (\mathcal{C}\mathcal{P}\mathbf{z})^T v + v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}) + (\mathcal{D}_2\mathcal{Z}\mathbf{z})^T v + v^T(D_1w) + (D_1w)^T v - \gamma v^T v \leq -\epsilon \|z\|$$

$$\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z} \rangle_{Z_{n,K}} + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z} \rangle_{Z_{n,K}} = \left\langle \begin{bmatrix} h \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_1, E_{1i}, \dot{S}_i, G_{ij}\}} \begin{bmatrix} h \\ \phi_i \end{bmatrix} \right\rangle_{Z_{r,n,K}}$$

where

$$D_1 := \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & C_0 + C_0^T & C_1 & \dots & C_K \\ 0 & 0 & C_1^T & -S_1(-\tau_1) & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & C_K^T & 0 & 0 & -S_K(-\tau_K) \end{bmatrix}, \quad \begin{aligned} C_0 &:= A_0 P + \tau_K \sum_{i=1}^K (A_i Q_i (-\tau_i)^T + \frac{1}{2} S_i(0)), \\ C_i &:= \tau_K A_i S_i(-\tau_i), \end{aligned}$$

$$E_{1i}(s) := [0 \quad 0 \quad B_i(s)^T \quad 0 \quad \dots \quad 0]^T, \quad B_i(s) := A_0 Q_i(s) + \dot{Q}_i(s) + \sum_{j=1}^K A_j R_{ji}(-\tau_j, s),$$

$$G_{ij}(s, \theta) := \frac{\partial}{\partial s} R_{ij}(s, \theta) + \frac{\partial}{\partial \theta} R_{ji}(s, \theta)^T.$$

Put Each Term in the PQRS Framework

$$\left(\mathcal{Z} \begin{bmatrix} \psi \\ \phi_i \end{bmatrix} \right) := \left[Z_0 \psi + \sum_i Z_{1i} \phi_i(-\tau_i) + \sum_i \int_{-\tau_i}^0 Z_{2i}(s) \phi_i(s) ds \right]$$

$$\langle \mathcal{A}P\mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{A}P\mathbf{z} \rangle_Z + \langle \mathcal{B}_2 \mathcal{Z}\mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{B}_2 \mathcal{Z}\mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{B}_1 w \rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z} \rangle_Z - \gamma w^T w + v^T (C P \mathbf{z}) + (C P \mathbf{z})^T v + v^T (D_2 \mathcal{Z}\mathbf{z}) + (D_2 \mathcal{Z}\mathbf{z})^T v + v^T (D_1 w) + (D_1 w)^T v - \gamma v^T v \leq -\epsilon \|z\|^2$$

$$\langle \mathcal{B}_2 \mathcal{Z}\mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{B}_2 \mathcal{Z}\mathbf{z} \rangle_Z = 2\tau_K x^T \left[B_2 Z_0 x + \sum_i B_2 Z_{1i} \phi_i(-\tau_i) + \sum_i \int_{-\tau_i}^0 B_2 Z_{2i}(s) \phi_i(s) ds \right]$$

$$= \tau_K \underbrace{\begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K) \end{bmatrix}}_{h^T}^T \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ *^T & 0 & 0 & 0 & \dots & 0 \\ *^T & *^T & B_2 Z_0 + Z_0^T B_2^T & B_2 Z_{11} & \dots & B_2 Z_{1K} \\ *^T & *^T & *^T & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ *^T & *^T & *^T & *^T & \dots & 0 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K) \end{bmatrix}}_h$$

$$+ 2\tau_K \sum_{i=1}^K \int_{-\tau_i}^0 \underbrace{\begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K) \end{bmatrix}}_{h^T}^T \underbrace{\begin{bmatrix} 0 \\ 0 \\ B_2 Z_{2i}(s) \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{E_{2i}(s)} \phi_i(s) ds = \left\langle \begin{bmatrix} h \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_2, E_{2i}, 0, 0\}} \begin{bmatrix} h \\ \phi_i \end{bmatrix} \right\rangle_{Z_{r,n,K}}$$

Put Each Term in the PQRS Framework

$$\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z} \rangle_Z + \langle \mathcal{B}_2 \mathcal{Z}\mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{B}_2 \mathcal{Z}\mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{B}_1 w \rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z} \rangle_Z - \gamma w^T w + v^T (\mathcal{C}\mathcal{P}\mathbf{z}) + (\mathcal{C}\mathcal{P}\mathbf{z})^T v + v^T (\mathcal{D}_2 \mathcal{Z}\mathbf{z}) + (\mathcal{D}_2 \mathcal{Z}\mathbf{z})^T v + v^T (\mathcal{D}_1 w) + (\mathcal{D}_1 w)^T v - \gamma v^T v \leq -\epsilon \|\mathbf{z}\|$$

$$\langle \mathbf{z}, \mathcal{B}_1 w \rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z} \rangle_Z - \gamma w^T w + v^T (\mathcal{D}_1 w) + (\mathcal{D}_1 w)^T v - \gamma v^T v$$

$$= \tau_K x^T \mathcal{B}_1 w + \tau_K (\mathcal{B}_1 w)^T x - \gamma w^T w + v^T (\mathcal{D}_1 w) + (\mathcal{D}_1 w)^T v - \gamma v^T v$$

$$= \tau_K \underbrace{\begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K) \end{bmatrix}^T}_{h^T} \underbrace{\frac{1}{\tau_K} \begin{bmatrix} -\gamma I & D_1 & 0 & 0 & \dots & 0 \\ D_1^T & -\gamma I & \tau_K B_1^T & 0 & \dots & 0 \\ 0 & \tau_K B_1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}}_{D_3} \underbrace{\begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K) \end{bmatrix}}_h$$

$$= \left\langle \begin{bmatrix} h \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_3, 0, 0, 0\}} \begin{bmatrix} h \\ \phi_i \end{bmatrix} \right\rangle_{Z_{r,n,K}}$$

Put Each Term in the PQRS Framework

$$\langle \mathcal{A}P\mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{A}P\mathbf{z} \rangle_Z + \langle \mathcal{B}_2 \mathcal{Z}\mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{B}_2 \mathcal{Z}\mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{B}_1 w \rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z} \rangle_Z - \gamma w^T w \\ + v^T (\mathcal{C}P\mathbf{z}) + (\mathcal{C}P\mathbf{z})^T v + v^T (\mathcal{D}_2 \mathcal{Z}\mathbf{z}) + (\mathcal{D}_2 \mathcal{Z}\mathbf{z})^T v + v^T (D_1 w) + (D_1 w)^T v - \gamma v^T v \leq -\epsilon \|\mathbf{z}\|$$

$$v^T (\mathcal{C}P\mathbf{z}) + (\mathcal{C}P\mathbf{z})^T v = 2v^T \left[\left(C_0 P + \sum_i \tau_K C_i Q_i(-\tau_i)^T \right) x + \tau_K \sum_i C_i S_i(-\tau_i) \phi_i(-\tau_i) \right. \\ \left. + \sum_{i=1}^K \int_{-\tau_i}^0 \left(C_0 Q_i(s) + \sum_j C_j R_{ji}(-\tau_j, s) \right) \phi_i(s) ds \right]$$

$$= \tau_K \begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K) \end{bmatrix}^T \underbrace{\begin{bmatrix} 0 & 0 & \frac{C_0 P}{\tau_K} + \sum_i C_i Q_i(-\tau_i)^T & C_1 S_1(-\tau_1) & \dots & C_K S_K(-\tau_K) \\ *^T & 0 & 0 & 0 & \dots & 0 \\ *^T & *^T & 0 & 0 & \dots & 0 \\ *^T & *^T & *^T & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ *^T & *^T & *^T & *^T & \dots & 0 \end{bmatrix}}_{D_4} \begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K) \end{bmatrix}$$

$$+ 2\tau_K \sum_{i=1}^K \int_{-\tau_i}^0 \begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K) \end{bmatrix}^T \underbrace{\begin{bmatrix} C_0 Q_i(s) + \sum_j C_j R_{ji}(-\tau_j, s) \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{E_{4i}(s)} \frac{1}{\tau_K} \phi_i(s) ds = \left\langle \begin{bmatrix} h \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_4, E_{4i}, 0, 0\}} \begin{bmatrix} h \\ \phi_i \end{bmatrix} \right\rangle_{Z_{r,n,K}}$$

Put Each Term in the PQRS Framework

$$\langle \mathcal{A}P\mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{A}P\mathbf{z} \rangle_Z + \langle \mathcal{B}_2 \mathcal{Z}\mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{B}_2 \mathcal{Z}\mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{B}_1 w \rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z} \rangle_Z - \gamma - \gamma w^T w + v^T (\mathcal{C}P\mathbf{z}) + (\mathcal{C}P\mathbf{z})^T v + v^T (\mathcal{D}_2 \mathcal{Z}\mathbf{z}) + (\mathcal{D}_2 \mathcal{Z}\mathbf{z})^T v + v^T (D_1 w) + (D_1 w)^T v - \gamma v^T v \leq -\epsilon \|\mathbf{z}\|^2$$

$$v^T (\mathcal{D}_2 \mathcal{Z}\mathbf{z}) + (\mathcal{D}_2 \mathcal{Z}\mathbf{z})^T v = 2v^T \left[D_2 Z_0 x + \sum_i D_2 Z_{1i} \phi_i(-\tau_i) + \sum_i \int_{-\tau_i}^0 D_2 Z_{2i}(s) \phi_i(s) ds \right]$$

$$= \tau_K \begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K) \end{bmatrix}^T \underbrace{\frac{1}{\tau_K} \begin{bmatrix} 0 & 0 & D_2 Z_0 & D_2 Z_{11} & \dots & D_2 Z_{1K} \\ *^T & 0 & 0 & 0 & \dots & 0 \\ *^T & *^T & 0 & 0 & \dots & 0 \\ *^T & *^T & *^T & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ *^T & *^T & *^T & *^T & \dots & 0 \end{bmatrix}}_{D_5} \begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K) \end{bmatrix}$$

$$+ 2\tau_K \sum_{i=1}^K \int_{-\tau_i}^0 \begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K) \end{bmatrix}^T \underbrace{\frac{1}{\tau_K} \begin{bmatrix} D_2 Z_{2i}(s) \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{E_{5i}(s)} \phi_i(s) ds = \left\langle \begin{bmatrix} h \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_5, E_{5i}, 0, 0\}} \begin{bmatrix} h \\ \phi_i \end{bmatrix} \right\rangle_{Z_{r,n,K}}$$

Combine Terms and enforce Constraint

And, finally,

$$\epsilon \|z\|_Z^2 = \left\langle \begin{bmatrix} h \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{\hat{I}, 0, I, 0\}} \begin{bmatrix} h \\ \phi_i \end{bmatrix} \right\rangle_{Z_{r,n,K}} \quad \text{where } \hat{I} = \text{diag}(0_{q+m}, I_n, 0_{nK})$$

Suppose there exist $P, Q_i, S_i, R_{ij}, Z_0, Z_{1i}$, and Z_{2i} such that

$$\begin{aligned} & \langle \mathcal{A}Pz, z \rangle_Z + \langle z, \mathcal{A}Pz \rangle_Z + \langle \mathcal{B}_2 Zz, z \rangle_Z + \langle z, \mathcal{B}_2 Zz \rangle_Z + \langle z, \mathcal{B}_1 w \rangle_Z + \langle \mathcal{B}_1 w, z \rangle_Z - \gamma - \gamma w^T w \\ & + v^T (\mathcal{C}Pz) + (\mathcal{C}Pz)^T v + v^T (\mathcal{D}_2 Zz) + (\mathcal{D}_2 Zz)^T v + v^T (D_1 w) + (D_1 w)^T v - \gamma v^T v + \epsilon \|z\|_Z^2 \\ & = \left\langle \begin{bmatrix} h \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D+\hat{I}, E_i, \dot{S}_i+I, G_{ij}\}} \begin{bmatrix} h \\ \phi_i \end{bmatrix} \right\rangle_{Z_{r,n,K}} \leq 0, \end{aligned}$$

where $D = \sum_{i=1}^5 D_i$, and $E_i(s) = \sum_{j=1}^5 E_{ij}(s)$. Then there exists a feedback controller $u(t) = ZP^{-1}x(t)$ which achieves CL H_∞ norm γ .

Matlab Code:

```
[P,Q,R,S] = sosjointpos_mat_ker_ndelay_PQRS_vZ
...
[P2,Q2,R2,S2] = sosjointpos_mat_ker_ndelay_PQRS_vZ
sosomeq(prog,D+P2); someq(prog,Q2{i}+E{i});
sosomeq(prog,S2{i}+F{i}); someq(prog,R2{i,j}+G{i,j});
```

How to ensure $\mathcal{P}(X) = X$

Not Needed for Optimal Estimator Synthesis

Recall PQRS Operators have the form

$$\begin{aligned} \begin{bmatrix} x' \\ \phi_i' \end{bmatrix} (s) &= \left(\mathcal{P}_{\{P, Q_i, S_i, R_{ij}\}} \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right) (s) \\ &= \begin{bmatrix} Px + \sum_{i=1}^K \int_{-\tau_i}^0 Q_i(s) \phi_i(s) ds \\ \tau_K Q_i(s)^T x + \tau_K S_i(s) \phi_i(s) + \sum_{j=1}^K \int_{-\tau_j}^0 R_{ij}(s, \theta) \phi_j(\theta) d\theta \end{bmatrix} \end{aligned}$$

So to achieve $x' = \phi_i'(0)$, we need

$$Px + \sum_{i=1}^K \int_{-\tau_i}^0 Q_i(s) \phi_i(s) ds = \tau_K Q_i(0)^T x + \tau_K S_i(0) \phi_i(0) + \sum_{j=1}^K \int_{-\tau_j}^0 R_{ij}(0, \theta) \phi_j(\theta) d\theta$$

or equivalently

$$P = \tau_K (Q_i(0)^T + S_i(0)), \quad Q_j(s) = R_{ij}(0, s) \quad \forall i, j$$

These are linear constraints on P and the coefficients of the polynomials Q_i, S_i, R_{ij} .

Complexity and Accuracy of Dual Stability ($\mathcal{AP} < 0$)

$$\dot{x}(t) = -x(t - \tau)$$

d	1	2	3	4	analytic
τ_{\max}	1.408	1.5707	1.5707	1.5707	1.5707
CPU sec	.18	.21	.25	.47	

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & .1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t - \tau)$$

d	1	2	3	4	limit
τ_{\max}	1.6581	1.716	1.7178	1.7178	1.7178
τ_{\min}	.10019	.10018	.10017	.10017	.10017
CPU sec	.25	.344	.678	1.725	

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & .1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x(t - \tau/2) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t - \tau)$$

d	1	2	3	4	limit
τ_{\max}	1.33	1.371	1.3717	1.3718	1.372
CPU sec	2.13	6.29	24.45	79.0	

$$\dot{x}(t) = - \sum_{i=1}^K \frac{x(t - i/K)}{K}$$

$K \downarrow n \rightarrow$	1	2	3	5	10
1	.366	.094	.158	.686	12.8
2	.112	.295	1.260	10.83	61.05
3	.177	1.311	6.86	96.85	5223
5	.895	13.05	124.7	2014	200950
10	13.09	59.5	5077	200231	NA

Table: CPU sec indexed by # of states (n) and # of delays (K)

Complexity Scaling Results:

- Viable when $nK < 50$

Significant reduction possible using Differential-Difference Formulation.

Roadmap of the Talk

The goal is to find $K \in \mathbb{R}^{m \times n}$ such that

$$\dot{x} = Ax + Bu, \quad u = Kx \quad \text{is Stable}$$

Step 1: Solve the problem as a abstract but convex Linear Operator Inequality.

Step 2: Parameterize All Operators using Matrices.

- Synthesis conditions now linear matrix constraints and operator positivity constraints
- $\mathcal{P}_{\{P, Q_i, S_i, R_{ij}\}}$ framework

Step 3: Enforce Operator Positivity using LMIs.

Step 4: Reconstruct the controller gains.

- Invert the operator using matrix manipulations.

Analytic Formula for Operator Inversion

Suppose $\mathcal{P} := \mathcal{P}_{\{P, Q_i, S_i, R_{ij}\}}$, $Q_i(s) = H_i Z(s)$ and $R_{ij}(s, \theta) = Z(s)^T \Gamma_{ij} Z(\theta)$.
Then $\mathcal{P}^{-1} = \mathcal{P}_{\{\hat{P}, \hat{Q}_i, \hat{S}_i, \hat{R}_{ij}\}}$ where if we define

$$H = [H_1 \quad \dots \quad H_K] \quad \text{and} \quad \Gamma = \begin{bmatrix} \Gamma_{11} & \dots & \Gamma_{1K} \\ \vdots & & \vdots \\ \Gamma_{K,1} & \dots & \Gamma_{K,K} \end{bmatrix},$$

then

$$\hat{P} = (I - \hat{H}VH^T)P^{-1}, \quad \hat{Q}_i(s) = \frac{1}{\tau_K} \hat{H}_i Z(s) S_i(s)^{-1}$$

$$\hat{S}_i(s) = \frac{1}{\tau_K^2} S_i(s)^{-1} \quad \hat{R}_{ij}(s, \theta) = \frac{1}{\tau_K} S_i(s)^{-1} Z(s)^T \hat{\Gamma}_{ij} Z(\theta) S_i(\theta)^{-1},$$

where

$$[\hat{H}_1 \quad \dots \quad \hat{H}_K] = \hat{H} = P^{-1}H(VH^T P^{-1}H - I - V\Gamma)^{-1}$$

$$\begin{bmatrix} \hat{\Gamma}_{11} & \dots & \hat{\Gamma}_{1K} \\ \vdots & & \vdots \\ \hat{\Gamma}_{K,1} & \dots & \hat{\Gamma}_{K,K} \end{bmatrix} = \hat{\Gamma} = -(\hat{H}^T H + \Gamma)(I + V\Gamma)^{-1}, \quad V = \begin{bmatrix} V_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & V_K \end{bmatrix}$$

$$V_i = \int_{-\tau_i}^0 Z(s) S_i(s)^{-1} Z(s)^T ds$$

Reconstructing the Full-State Feedback Controller Gains

Finally, we recover the controller as

$$u(t) = K_0 x(t) + \frac{1}{\tau_K} \sum_i K_{1i} x(t - \tau_i) + \frac{1}{\tau_K} \sum_i \int_{-\tau_i}^0 K_{2i}(s) x(t + s) ds$$

where $(Z_0, Z_{1i}, Z_{2i}$ are variables, Z is a vector of monomials)

$$K_0 = Z_0 \hat{P} + \sum_j \left(Z_{1j} S_j(-\tau_j)^{-1} Z(-\tau_j)^T + O_j \right) \hat{H}_j^T$$

$$K_{1i} = Z_{1i} S_i(-\tau_i)^{-1}, \quad O_i = \int_{-\tau_j}^0 Z_{2j}(s) S_j(s)^{-1} Z(s)^T ds$$

$$K_{2i}(s) = \left(Z_0 \hat{H}_i Z(s) + Z_{2i}(s) + \sum_{j=1}^K \left(Z_{1j} S_j(-\tau_j)^{-1} Z(-\tau_j)^T + O_j \right) \hat{\Gamma}_{ji} Z(s) \right) S_i(s)^{-1}$$

Note: This is *Full-State* Feedback.

- Contrast with output feedback: $u(t) = Kx(t)$ or $u(t) = Ky(t - r)$.

Boring Numerical Examples

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -1 \\ 0 & -.9 \end{bmatrix} x(t - \tau) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ .1 \end{bmatrix} u(t)$$

d	1	2	3	Padé	Fridman 2003	Li 1997
$\gamma_{\min}(\tau = .999)$.10001	.10001	.10001	.1000	.22844	1.8822
$\gamma_{\min}(\tau = 2)$	1.43	1.36	1.341	1.340	∞	∞
CPU sec	.478	.879	2.48	2.78	N/A	N/A

$$\dot{x}(t) = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} x(t - \tau) + \begin{bmatrix} -.5 \\ 1 \end{bmatrix} w(t) + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & -.5 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

d	1	2	3	Padé
$\gamma_{\min}(\tau = .3)$.3953	.3953	.3953	.3953
CPU sec	.655	1.248	2.72	2.91

Interesting Numerical Example

Fixing the Athenaeum Showers [Peet Thesis, 2006]

Tracking Control with integral feedback

- T_{2i} is the water temperature
- T_{1i} is the tap position
- τ_i is the time for water to move from tap to showerhead
- w_i is the desired water temperature (Not available to controller!)
- Opening the tap by user i decreases the water temperature of users $j \neq i$
- Minimize tap action and controller interference.

$$\dot{T}_{1i}(t) = T_{2i}(t) - w_i(t)$$

$$\dot{T}_{2i}(t) = -\alpha_i (T_{2i}(t - \tau_i) - w_i(t)) + \sum_{j \neq i} \gamma_{ij} \alpha_j (T_j(t - \tau_j) - w_j(t)) + u_i(t)$$

$$y_i(t) = \begin{bmatrix} T_{1i}(t) \\ .1u_i(t) \end{bmatrix}.$$

Fixing the Athenaeum Showers

$$\dot{x}(t) = A_0x(t) + \sum_i A_i x(t - \tau_i) + B_1 w(t) + B_2 u(t), \quad y(t) = Cx(t) + D_1 w(t) + D_2 u(t)$$

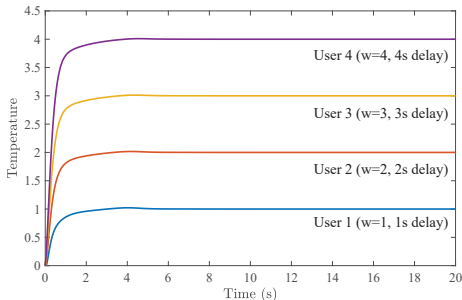
where

$$A_0 = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad A_i = \begin{bmatrix} 0 & 0 \\ 0 & \hat{A}_i \end{bmatrix}, \quad B_1 = \begin{bmatrix} -I \\ -\hat{\Gamma} + \text{diag}(\alpha_1 \dots \alpha_K) \end{bmatrix}$$

$$\hat{A}_i(:, i) = \alpha_i [\gamma_{i,1} \quad \dots \quad \gamma_{i,i-1} \quad -1 \quad \gamma_{i,i-1} \quad \dots \quad \gamma_{i,K}]^T$$

$$\hat{\Gamma}_{ij} = \alpha_j \gamma_{ij} = [q_1 \quad \dots \quad q_K], \quad B_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$C_0 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 \\ .1I \end{bmatrix}$$



Complexity: 8 states, 4 delays, 4 inputs, 4 disturbances, 8 regulated outputs

Results: A Matlab simulation of the step response of the closed-loop temperature dynamics ($T_{2i}(t)$) with 4 users (w_i and τ_i as indicated) coupled with the controller with closed-loop gain of .48

Conclusions:

- Extends the LMI Framework to DPS
 - ▶ Relies on a new Duality result
 - ▶ Other LK-based approaches can be used (But why?).
 - ▶ Most LMIs can be converted.
 - ▶ But be careful...
- Practical Implications
 - ▶ Solved the H_∞ -**optimal Full-State Feedback Synthesis Problem** for multi-state multi-delay systems.
 - ▶ Solved the H_∞ -**optimal Estimator Synthesis Problem** for multi-state single-delay systems.
 - ▶ Analytic Inverse allows controller and observer reconstruction.

Numerical Code Produced:

- LOI Toolbox
 - ▶ Packaged as DelayTools
 - ▶ Duality Test Now on CodeOcean
 - ▶ Both Papers on arXiv
- Next Talk:
 - ▶ Input Delay (Special Case of Observer Synthesis)
 - ▶ H_∞ optimal Dynamic Output Feedback Controller Synthesis

Available for download at
<http://control.asu.edu>

H_∞ -Optimal Observer Synthesis Problem to be Solved

Consider solutions of

$$\begin{aligned}\dot{x}(t) &= A_0x(t) + A_1x(t - \tau) + Bw(t) \\ y(t) &= C_2x(t)\end{aligned}$$

With a PDE observer (**observed errors**)(**nominal dynamics**)(**corrective gains**)

$$\begin{aligned}\dot{\hat{x}}(t) &= A_0\hat{x}(t) + A_1\hat{\phi}(t, -\tau) + L_1(C_2\hat{x}(t) - y(t)) + L_2(C_2\hat{\phi}(t, -\tau) - y(t - \tau)) \\ &\quad + \int_{-\tau}^0 L_3(\theta) (C_2\hat{\phi}(t, \theta) - y(t + \theta)) d\theta\end{aligned}$$

$$\begin{aligned}\partial_t\hat{\phi}(t, s) &= \partial_s\hat{\phi}(t, s) + L_4(s) (C_2\hat{x}(t) - y(t)) + L_5(s) (C_2\hat{\phi}(t, -\tau) - y(t - \tau)) \\ &\quad + L_6(s) (C_2\hat{\phi}(t, s) - y(t + s)) + \int_{-\tau}^0 L_7(s, \theta) (C_2\hat{\phi}(t, \theta) - y(t + \theta)) d\theta\end{aligned}$$

$$\hat{\phi}(t, 0) = \hat{x}(t)$$

Problem Definition:

Minimize γ such that there exist L_i such that if $z_e(t) = C_1(x(t) - \hat{x}(t))$, then for any $w \in L_2$, $\|z_e\|_{L_2} \leq \gamma\|w\|_{L_2}$.

An LMI for Optimal Estimation of **ODEs**

Get rid of the delays and we have

$$\dot{x}(t) = Ax(t) + B_1w(t), \quad y(t) = C_2x(t) + Dw(t)$$

Observer:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + L(C_2\hat{x}(t) - y(t)), \quad z_e(t) = C_1(\hat{x}(t) - x(t))$$

Lemma 3 (H_∞ -Optimal Observer Synthesis).

Define the map $w \mapsto z_e$:

$$\hat{G}(s) = \left[\begin{array}{c|c} A + LC_2 & -(B + LD) \\ \hline C_1 & 0 \end{array} \right].$$

The following are equivalent.

- There exists a L such that $\|\hat{G}\|_{H_\infty} \leq \gamma$.
- There exists a $P > 0$ and Z such that

$$\begin{bmatrix} A^T P + C_2^T Z^T + PA + ZC_2 & -(PB + ZD) \\ -(PB + ZD)^T & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C_1^T C_1 & 0 \\ 0 & 0 \end{bmatrix} < 0.$$

The Observer Gain is recovered as $L = P^{-1}Z$.

The DPS/DDE Equivalent of the Observer LMI

LMI Version of Observer Synthesis: Minimize γ such that $\exists P > 0$ and $Z \in \mathbb{R}^{p \times n}$ such that

$$\begin{aligned} & \begin{bmatrix} e \\ w \end{bmatrix}^T \left[\begin{bmatrix} A^T P + C_2^T Z^T + PA + ZC_2 & -(PB + ZD) \\ -(PB + ZD)^T & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C_1^T C_1 & 0 \\ 0 & 0 \end{bmatrix} \right] \begin{bmatrix} e \\ w \end{bmatrix} \\ & = (PAe)^T e + (PAe)^T e + (ZCe)^T e + (ZC_2e)^T e \\ & - e^T PBw - (PBw)^T e - \gamma w^T w + \frac{1}{\gamma} (C_1e)^T (C_1e) < 0 \end{aligned}$$

for all $e \in \mathbb{R}^n$, $w \in \mathbb{R}^m$

DPS Version of Observer Synthesis: Minimize γ such that $\exists P > 0$ and Z such that

$$\begin{aligned} & \langle PAe, e \rangle_{L_2} + \langle e, PAe \rangle_{L_2} + \langle ZC_2e, e \rangle_{L_2} + \langle e, ZC_2e \rangle_{L_2} \\ & - \langle e, PBw \rangle_{L_2} - \langle Bw, Pe \rangle_{L_2} - \gamma w^T w + \frac{1}{\gamma} (C_1e)^T (C_1e) < -\epsilon \|e\|^2 \quad \forall e \in X, w \in \mathbb{R}^m \end{aligned}$$

Observer: Put Each Term in the PQRS Framework

Define $\mathbf{z} = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix}$ and $h = [w^T \quad e_1^T \quad e_2(-\tau)^T]^T$.

$$\begin{aligned} & \langle \mathcal{P}A\mathbf{e}, \mathbf{e} \rangle_{L_2} + \langle \mathbf{e}, \mathcal{P}A\mathbf{e} \rangle_{L_2} + \langle \mathcal{Z}C_2\mathbf{e}, \mathbf{e} \rangle_{L_2} + \langle \mathbf{e}, \mathcal{Z}C_2\mathbf{e} \rangle_{L_2} \\ & - \langle \mathbf{e}, \mathcal{P}Bw \rangle_{L_2} - \langle Bw, \mathcal{P}\mathbf{e} \rangle_{L_2} - \gamma w^T w + \frac{1}{\gamma} (C_1\mathbf{e})^T (C_1\mathbf{e}) < -\epsilon \|\mathbf{e}\|^2 \quad \forall \mathbf{e} \in X, w \in \mathbb{R}^m, \end{aligned}$$

$$\langle \mathcal{A}P\mathbf{z}, \mathbf{z} \rangle_{Z_n} + \langle \mathbf{z}, \mathcal{A}P\mathbf{z} \rangle_{Z_n, K}$$

$$\begin{aligned} & = \int_{-\tau}^0 \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2(s) \end{bmatrix}^T \begin{bmatrix} D_1(s) & \tau E_1(s) \\ \tau E_1(s)^T & -\tau \dot{S}(s) \end{bmatrix} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2(s) \end{bmatrix} ds + \tau \int_{-\tau}^0 \int_{-\tau}^0 e_2(s)^T G(s, \theta) e_2(\theta) d\theta \\ & = \left\langle \begin{bmatrix} h \\ \mathbf{e}_2 \end{bmatrix}, \mathcal{P}_{\{D_1, E_1, -\dot{S}, G\}} \begin{bmatrix} h \\ \mathbf{e}_2 \end{bmatrix} \right\rangle_{L_2} \end{aligned}$$

where

$$\begin{aligned} D_1(s) &= \begin{bmatrix} 0 & * & * \\ 0 & PA_0 + A_0^T P + Q(0) + Q(0)^T + S(0) & * \\ 0 & A_1^T P - Q(-\tau)^T & -S(-\tau) \end{bmatrix} \\ E(s) &= \begin{bmatrix} 0 \\ A_0^T Q(s) + R(s, 0)^T - \dot{Q}(s) \\ A_1^T Q(s) - R(s, -\tau)^T \end{bmatrix} \quad G(s, \theta) = -R_\theta(s, \theta) - R_s(s, \theta). \end{aligned}$$

Observer: Put Each Term in the PQRS Framework

$$\langle \mathcal{P}Ae, e \rangle_{L_2} + \langle e, \mathcal{P}Ae \rangle_{L_2} + \langle \mathcal{Z}C_2e, e \rangle_{L_2} + \langle e, \mathcal{Z}C_2e \rangle_{L_2} \\ - \langle e, \mathcal{P}Bw \rangle_{L_2} - \langle Bw, \mathcal{P}e \rangle_{L_2} - \gamma w^T w + \frac{1}{\gamma} (C_1e)^T (C_1e) < -\epsilon \|e\|^2 \quad \forall e \in X, w \in \mathbb{R}^m,$$

$$\begin{aligned} & \langle \mathcal{Z}C_2e, e \rangle_{L_2} + \langle e, \mathcal{Z}C_2e \rangle_{L_2} = 2\tau e_1^T \left(Z_1 C_2 e_1 + Z_2 C_2 e_2(-\tau) + \int_{-\tau}^0 Z_3(\theta) C_2 e_2(\theta) d\theta \right) \\ & \quad + 2\tau \int_{-\tau}^0 e_2(s)^T \left(Z_4(s) C_2 e_1 + Z_5(s) C_2 e_2(-\tau) + Z_6(s) C_2 e_2(s) + \int_{-\tau}^0 Z_7(s, \theta) C_2 e_2(\theta) d\theta \right) \\ = & \tau \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \end{bmatrix}^T \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & Z_1 C_2 & Z_2 C_2 \\ 0 & C_2^T Z_2 & 0 \end{bmatrix}}_{D_2} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \end{bmatrix} + 2\tau \int_{-\tau}^0 \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \end{bmatrix}^T \underbrace{\begin{bmatrix} 0 \\ C_2^T Z_4(s)^T + Z_3(s) C_2 \\ C_2^T Z_5(s)^T \end{bmatrix}}_{E_2} e_2(s) ds \\ & + \tau \int_{-\tau}^0 e_2(s)^T \underbrace{(Z_6(s) C_2 + C_2^T Z_6(s)^T)}_{F_2} e_2(s) ds \\ & + \tau \int_{-\tau}^0 \int_{-\tau}^0 e_2(s)^T \underbrace{(Z_7(s, \theta) C_2 + C_2^T Z_7(\theta, s)^T)}_{G_2} e_2(\theta) d\theta \\ = & \left\langle \begin{bmatrix} h \\ e_2 \end{bmatrix}, \mathcal{P}_{\{D_2, E_2, F_2, G_2\}} \begin{bmatrix} h \\ e_2 \end{bmatrix} \right\rangle_{L_2} \end{aligned}$$

Observer: Put Each Term in the PQRS Framework

$$\langle \mathcal{P}Ae, e \rangle_{L_2} + \langle e, \mathcal{P}Ae \rangle_{L_2} + \langle \mathcal{Z}C_2e, e \rangle_{L_2} + \langle e, \mathcal{Z}C_2e \rangle_{L_2} \\ - \langle e, \mathcal{P}Bw \rangle_{L_2} - \langle Bw, \mathcal{P}e \rangle_{L_2} - \gamma w^T w + \frac{1}{\gamma} (C_1e)^T (C_1e) < -\epsilon \|e\|^2 \quad \forall e \in X, w \in \mathbb{R}^m,$$

$$- \langle e, \mathcal{P}Bw \rangle_{L_2} - \langle Bw, \mathcal{P}e \rangle_{L_2} = 2 \int_{-\tau}^0 e_1^T P B w ds + 2 \int_{-\tau}^0 e_2(s)^T \tau Q(s)^T B w ds \\ = \tau \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \end{bmatrix}^T \underbrace{\begin{bmatrix} 0 & -B^T P & 0 \\ -PB & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D_3} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \end{bmatrix} + 2\tau \int_{-\tau}^0 \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \end{bmatrix}^T \underbrace{\begin{bmatrix} -B^T Q(s) \\ 0 \\ 0 \end{bmatrix}}_{E_3} e_2(s) ds \\ = \left\langle \begin{bmatrix} h \\ e_2 \end{bmatrix}, \mathcal{P}_{\{D_3, E_3, 0, 0\}} \begin{bmatrix} h \\ e_2 \end{bmatrix} \right\rangle_{L_2}$$

Observer: Put Each Term in the PQRS Framework

$$\langle \mathcal{P}A\mathbf{e}, \mathbf{e} \rangle_{L_2} + \langle \mathbf{e}, \mathcal{P}A\mathbf{e} \rangle_{L_2} + \langle \mathcal{Z}C_2\mathbf{e}, \mathbf{e} \rangle_{L_2} + \langle \mathbf{e}, \mathcal{Z}C_2\mathbf{e} \rangle_{L_2} \\ - \langle \mathbf{e}, \mathcal{P}Bw \rangle_{L_2} - \langle Bw, \mathcal{P}\mathbf{e} \rangle_{L_2} - \gamma w^T w + \frac{1}{\gamma} (C_1\mathbf{e})^T (C_1\mathbf{e}) + \epsilon \|\mathbf{e}\|^2 \leq 0 \quad \forall \mathbf{e} \in X, w \in \mathbb{R}^m$$

$$- \gamma w^T w + \frac{1}{\gamma} (C_1\mathbf{e})^T (C_1\mathbf{e}) + \epsilon \|\mathbf{e}\|^2 \\ = \tau \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \end{bmatrix}^T \underbrace{\begin{bmatrix} -\frac{\gamma}{\tau} & 0 & 0 \\ 0 & \frac{1}{\gamma\tau} C_1^T C_1 + \epsilon I & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D_4} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \end{bmatrix} \\ = \left\langle \begin{bmatrix} h \\ \mathbf{e}_2 \end{bmatrix}, \mathcal{P}_{\{D_4, 0, 0, 0\}} \begin{bmatrix} h \\ \mathbf{e}_2 \end{bmatrix} \right\rangle_{L_2}$$

Combine Terms and enforce Constraint

Suppose there exist P, Q, S, R, Z_i such that

$$\begin{aligned} & \langle \mathcal{P}Ae, e \rangle_{L_2} + \langle e, \mathcal{P}Ae \rangle_{L_2} + \langle \mathcal{Z}C_2e, e \rangle_{L_2} + \langle e, \mathcal{Z}C_2e \rangle_{L_2} \\ & - \langle e, \mathcal{P}Bw \rangle_{L_2} - \langle Bw, \mathcal{P}e \rangle_{L_2} - \gamma w^T w + \frac{1}{\gamma} (C_1e)^T (C_1e) + \epsilon \|e\|^2 \\ & = \left\langle \begin{bmatrix} h \\ e_2 \end{bmatrix}, \mathcal{P}_{\{D, E, F, G\}} \begin{bmatrix} h \\ e_2 \end{bmatrix} \right\rangle_{L_2} \leq 0, \end{aligned}$$

where $D = \sum_{i=1}^5 D_i$, $E(s) = \sum_{j=1}^3 E_j(s)$ and $G(s, \theta) = \sum_{j=1}^2 G_j(s, \theta)$. Then if $\mathcal{L} = \mathcal{P}^{-1}\mathcal{Z}$ and

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= \mathcal{A}\hat{\mathbf{x}}(t) + \mathcal{L} (C_2\hat{\mathbf{x}}(t) - \mathbf{y}(t)), & \mathbf{y}(t)(s) &= \begin{bmatrix} C_2x(t) \\ C_2x(t+s) \end{bmatrix} \\ \hat{z}(t) &= C_1\mathbf{x}(t), & z_e(t) &= \hat{z}(t) - z(t), & \mathbf{x}(t)(s) &= \begin{bmatrix} x(t) \\ x(t+s) \end{bmatrix} \end{aligned} \quad (1)$$

and $z_e(t) = \hat{z}(t) - z(t)$, we have $\|z_e\|_{L_2} \leq \gamma \|w\|_{L_2}$

Observer Gains Reconstruction

Let $\mathcal{P}_{\{\hat{P}, \hat{Q}, \hat{S}, \hat{R}\}} = \mathcal{P}_{\{P, Q, S, R\}}^{-1}$. Then the observer dynamics are given by

$$\dot{\hat{x}}(t) = A_0 \hat{x}(t) + A_1 \hat{\phi}(t, -\tau) + L_1 (C_2 \hat{x}(t) - y(t)) + L_2 (C_2 \hat{\phi}(t, -\tau) - y(t - \tau)) \\ + \int_{-\tau}^0 L_3(\theta) (C_2 \hat{\phi}(t, \theta) - y(t + \theta)) d\theta, \quad \hat{\phi}(t, 0) = \hat{x}(t)$$

$$\partial_t \hat{\phi}(t, s) = \partial_s \hat{\phi}(t, s) + L_4(s) (C_2 \hat{x}(t) - y(t)) + L_5(s) (C_2 \hat{\phi}(t, -\tau) - y(t - \tau)) \\ + L_6(s) (C_2 \hat{\phi}(t, s) - y(t + s)) + \int_{-\tau}^0 L_7(s, \theta) (C_2 \hat{\phi}(t, \theta) - y(t + \theta)) d\theta$$

where

$$L_1 = \hat{P} Z_1 + \int_{-\tau}^0 \hat{Q}(\theta) Z_4(\theta) d\theta, \quad L_2 = \hat{P} Z_2 + \int_{-\tau}^0 \hat{Q}(\theta) Z_5(\theta) d\theta$$

$$L_3(\theta) = \hat{P} Z_3(\theta) + \hat{Q}(\theta) Z_6(\theta) + \int_{-\tau}^0 \hat{Q}(s) Z_7(s, \theta) ds$$

$$L_4(s) = \hat{Q}(s)^T Z_1 + \hat{S}(s) Z_4(s) + \int_{-\tau}^0 \hat{R}(s, \theta) Z_4(\theta) d\theta$$

$$L_5(s) = \hat{Q}(s)^T Z_2 + \hat{S}(s) Z_5(s) + \int_{-\tau}^0 \hat{R}(s, \theta) Z_5(\theta) d\theta, \quad L_6(s) = \hat{S}(s) Z_6(s)$$

$$L_7(s, \theta) = \hat{Q}(s)^T Z_3(\theta) + \hat{S}(s) Z_7(s, \theta) + \hat{R}(s, \theta) Z_6(\theta) + \int_{-\tau}^0 \hat{R}(s, \xi) Z_7(\xi, \theta) d\xi.$$

Boring Numerical Examples

Numerical Example 1 In this example, we consider the unstable system

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t - \tau) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t), \\ y(t) &= [0 \quad 7] x(t), \quad z(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t)\end{aligned}$$

Applying the Ricatti approach in [Fattouh 1998] with $\epsilon = .001$ we obtain a L_2 -gain of $\gamma = .580$. Applying the LOI, we obtain an L_2 -gain of .236. Of all the systems we tested, this one showed the least improvement in performance.

Numerical Example 2 A modified form of [Fridman 2001].

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -1 \\ 0 & -.9 \end{bmatrix} x(t - \tau) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t), \\ y(t) &= [0 \quad 1] x(t), \quad z(t) = [1 \quad 0] x(t)\end{aligned}$$

Using the original system with $\tau = 1$, a closed-loop gain of 22.8 was obtained in [Fridman 2001]. For this problem, [Fattouh 1998] was infeasible for any value of gain. Applying the LOI, we obtained a closed-loop gain of 2.33 using polynomials of degree 4.

Boring Numerical Examples

$$\dot{x}(t) = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t - \tau) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t),$$

$$y(t) = [0 \quad 7] x(t), \quad z(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t)$$

