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Synthesis of Full-State Observers for Time-delay Systems using SOS

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Abstract— **In this paper, we develop an SOS approach for design of observers for time-delay systems. The method is an extension of recently developed algorithms for control of infinite-dimensional systems. The observers we design are more general than the class of observers most commonly associated with time-delay systems in that they directly correct both the estimate of present state as well as the history of the state. The result is that the observer is itself a PDE. In this case the traditional notions of strong and weak observability do not apply and the resulting observer-based controllers can significantly outperform existing approaches.**

I. Introduction

In recent years, there have been considerable advances in the development of asymptotic algorithms for the analysis and control of systems with time delay. Roughly speaking, an asymptotic algorithm is defined as a sequence of algorithms, indexed by some metric of complexity, each instance of which provides a sufficient condition, is of polynomial-time complexity, and where as the sequence progresses, the complexity and accuracy of the algorithms increase - presumably to some notion of necessity. Examples of asymptotic algorithms for stability analysis of time-delay systems include the Piecewise linear approach of Gu [1], the Wirtinger approach of Seuret et al. [2] and the SOS approach as in, e.g. [3]. The SOS approach, in particular, has been extended to controller synthesis in [4].

The goal of this paper is to extend our recent success in the development of asymptotic algorithms for time-delay systems to the problem of H_∞-optimal state observer synthesis. An H_∞-optimal observer is itself a dynamical system which runs in parallel to the physical system being observed. The state of the observer is typically an estimate of the state of the physical system and is propagated using a set of dynamics similar to the physical system while also including a correction term based on measured outputs from the physical system. Within this framework, an optimal observer will minimize the effect of disturbances on the error between the estimate and the actual state.

When delay is included in the dynamics, the state of the system is a combination of the present state variable and its history over the period of delay. For this reason, the estimator dynamics for a time-delay system include both an estimate of the current state variable and its history. In the ideal case, real-time measurement errors should then be used to correct both the estimate of the history and the estimate of the current state variable. Until the 1990's, it was universally acknowledged [5], [6], [7], [8], [9], [10] that this was a natural framework for estimator design and that the resulting observer structure is a PDE and the observer gains are functions rather than matrices. However, the use of PDE observers for delay systems was mostly abandoned in the mid-1990s when Ricatti and LMI methods made it possible to search for quadratic Lyapunov functions and matrix-valued feedback. Referring to the observer structure in Eq. (10) of this paper, these earlier works set the gains $L_3 = L_4 = L_5 = L_6 = L_7 = 0$ - eliminating any correction to the history. Such works include [11], [12], [13], [14], [15], [16], [17], [18], [19], [20] and the overview in [21].

Unlike traditional LMI-based methods, asymptotic algorithms, while still using positive matrix variables, are able to search over function spaces. Most relevant to this paper, asymptotic algorithms based on SOS optimize polynomial variables which can then be used to parameterize Lyapunov functions of the "complete quadratic" type known to be necessary and sufficient for stability of a time-delay system. Recently, this approach has been formalized within the semigroup framework by using polynomial variables to parameterize the positive linear operator variables (See Eq. 9) foreseen in the early days of semigroup theory. In this context, then, we will use SOS to refer to the search for SOS *operators*, i.e. operators which have a square root which can be represented using a positive matrix as described in Theorem 3 - an approach inspired by the use SOS *polynomials* for analysis on nonlinear systems.

The representation of analysis and control problems as a search for operators subject to inequality constraints is sometimes termed a Linear Operator Inequality (LOI) for the purpose of denoting the obvious connection with LMIs - wherein the variable is a matrix. This parallelism is not superficial, however, as many LMI results have direct analogues as solveable LOIs. Indeed, the number of such parallels recently increased with the development of a duality theory for delayed and PDE systems which allowed the controller synthesis problem to be represented in such a form. While we will not belabor the point in this paper, the significance of these developments is such that the LMI for H_∞-optimal observer synthesis can now be represented as an LOI for which we have asymptotic algorithms. Furthermore, solution of this LOI yields a PDE observer. As will be seen in the numerical examples, the predicted H_∞ gain of such observers is very often an order of magnitude less than existing results and furthermore, simulation shows that these predicted gains are tight, indicating the conditions have little

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if any conservatism.

As the layout of this paper is somewhat unorthodox, we provide here an overview of the presentation. First, In Section III, we present the proof of an LMI condition for H_∞-optimal observer synthesis for ODEs. This result is not new. The purpose, however, is to establish a familiar baseline narrative which we will then follow and generalize in Section IV. The parallel between these sections is almost one-to-one. Specifically, in Section IV we use operators to define observer synthesis in as similar a form to the ODE case as possible. We then give the LOI version of the H_∞-optimal observer synthesis problem and prove its sufficiency using arguments almost identical to those in the ODE case.

II. Notation

The symmetric matrices are denoted $\mathbb{S}^n \subset \mathbb{R}^{n\times n}$. An element of a symmetric matrix which can be deduced from symmetry is denoted with a $*$. We use $L_2^n[T]$ to denote the vector-valued Lesbesque square integrable functions which map $T \to \mathbb{R}^n$. In this paper, either $T = [-\tau, 0]$ or $T = [0,\infty]$. We occasionally denote the Sobolev space $W_2^n[T] := \{x \in L_2^n[T] \;:\; \dot{x} \in L_2^n[T]\}$. We make frequent use of the direct product $Z_{m,n}[T] := \mathbb{R}^m \times L_2^n[T] \subset L_2^{m+n}[T]$ equipped with the L_2^{m+n} inner product $\langle\cdot,\cdot\rangle_{L_2}$. We also use the shorthand notation $Z_n := Z_{n,n}$. $\|\cdot\|_{L_2}$ and $\|\cdot\|_{Z_{m,n}}$ denote the norms on their respective spaces. We omit the domain and write $Z_n = Z_n[T]$ when it is clear from context.

III. The LMI for Observer Synthesis for ODEs

Our goal is to extend the LMI method for H_∞-optimal observer design to systems with time delay. To motivate our result, we first briefly define the LMI framework for H_∞-optimal observer synthesis. Our work on time-delay systems is then a straightforward generalization of this approach. Specifically, the H_∞ optimal filter design is based on the system dynamics

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bw(t) && (1)\\ z(t) &= C_1x(t), \qquad y(t) = C_2x(t) + Dw(t)\end{aligned}$$

where $w \in L_2$ is a disturbance, y is the measured output and z is the regulated output. The observer structure is given by

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + L(C_2\hat{x}(t) - y(t)),\\ \hat{z}(t) &= C_1\hat{x}(t), \quad z_e(t) = \hat{z}(t) - z(t) && (2)\end{aligned}$$

where $\hat{x}$ is the estimated state and z_e is the error in the estimated regulated output. The objective is to minimize the effect of w on the error z_e. The closed-loop dynamics of the error system with state $e(t) := \hat{x}(t) - x(t)$ are then determined by Eqs. (1) and (2) as

$$\dot{e}(t) = (A + LC_2)e(t) - (B + LD)w(t), \;\; z_e(t) = C_1e(t)$$

We may now apply the Kalman-Yakubovich-Popov (KYP) Lemma to obtain the LMI for optimal observer synthesis which minimizes the closed-loop map from w to z_e.

Theorem 1: Suppose there exist $P > 0$ and Z such that

$$\begin{bmatrix} A^TP + C_2^TZ^T + PA + ZC_2 & -(PB + ZD)\\ -(PB + ZD)^T & -\gamma I\end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} C_1^TC_1 & 0\\ 0 & 0\end{bmatrix} < 0.$$

Then for $L = P^{-1}Z$, $\hat{x}(0) = x(0)$, and any $w \in L_2$, the solution of Eqn. (1) coupled with (2) satisfies

$$\|z_e\|_{L_2} \le \gamma\,\|w\|_{L_2}. \tag{3}$$

Proof: Let $V(e) = e^TPe$. Then

$$\begin{aligned}\dot{V}(e(t)) &= ((A + LC_2)e(t))^TPe(t) + e(t)^TP(A + LC_2)e(t)\\ &\quad - e(t)^TP(B + LD)w(t) - ((B + LD)w(t))^TPe(t).\end{aligned}$$

Now since $Z = PL$ (and suppressing the time-dependency of $e(t)$ and $w(t)$) we have from the LMI:

$$\begin{aligned}&\begin{bmatrix} e\\ w\end{bmatrix}^T\begin{bmatrix} A^TP + C_2^TZ^T + PA + ZC_2 & -ZD - PB\\ -D^TZ^T - B^TP & -\gamma I\end{bmatrix}\begin{bmatrix} e\\ w\end{bmatrix}\\ &+ \frac{1}{\gamma}\begin{bmatrix} e\\ w\end{bmatrix}^T\begin{bmatrix} C_1^TC_1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} e\\ w\end{bmatrix} =\\ &\begin{bmatrix} e\\ w\end{bmatrix}^T\begin{bmatrix} (A + LC_2)^TP + P(A + LC_2) & -P(B + LD)\\ -(B + LD)^TP & -\gamma I\end{bmatrix}\begin{bmatrix} e\\ w\end{bmatrix}\\ &\quad + \frac{1}{\gamma}\begin{bmatrix} e\\ w\end{bmatrix}^T\begin{bmatrix} C_1^T\\ 0\end{bmatrix}\begin{bmatrix} C_1 & 0\end{bmatrix}\begin{bmatrix} e\\ w\end{bmatrix}\\ &= ((A + LC_2)e)^TPe + e^TP(A + LC_2)e - e^TP(B + LD)w\\ &\quad - ((B + LD)w)^TPe - \gamma w^Tw + \frac{1}{\gamma}(C_1e)^T(C_1e)\\ &= \dot{V}(e(t)) - \gamma\|w(t)\|^2 + \frac{1}{\gamma}\|z_e(t)\|^2 < 0\end{aligned}$$

for all t such that $[e(t)\; w(t)] \ne 0$. An integration yields

$$V(e(t)) - V(e(0)) + \frac{1}{\gamma}\int_0^t \|z_e(s)\|^2\,ds \le \gamma\int_0^t \|w(s)\|^2\,ds$$

Since $V(e(0)) = 0$ and $V(e(t)) \ge 0$, if we let $t \to \infty$, we obtain Eqn. (3). ■

IV. Problem Definition

We now extend this approach to time-delay systems of the form

$$\begin{aligned}\dot{x}(t) &= A_0x(t) + A_1x(t-\tau) + Bw(t) && (4)\\ z(t) &= C_1x(t), \qquad y(t) = C_2x(t) && (5)\end{aligned}$$

where $y(t) \in \mathbb{R}^q$ is the measured output, $w(t) \in \mathbb{R}^r$, $z(t) \in \mathbb{R}^p$ is the regulated output, $x(t) \in \mathbb{R}^n$ are the state variables and $\tau > 0$ is the delay. Before introducing our class of observers, we rewrite this system in semigroup format as

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}w(t)\\ z(t) &= \mathcal{C}_1\mathbf{x}(t), \qquad \mathbf{y}(t) = \mathcal{C}_2\mathbf{x}(t) && (6)\end{aligned}$$

where $\mathbf{x} = \begin{bmatrix} x_1 & x_2\end{bmatrix}^T \in Z_{n,n}$, x_1 is the current state and $x_2(s) = x_t(s)$ for $s \in [-\tau, 0]$ is the history. The infinitesimal generator $\mathcal{A} : X \to Z_n$ is defined as

$$(\mathcal{A}\mathbf{x})(s) = \left(\mathcal{A}\begin{bmatrix} x_1\\ x_2\end{bmatrix}\right)(s) := \begin{bmatrix} A_0x_1 + A_1x_2(-\tau)\\ \dot{x}_2(s)\end{bmatrix},$$

$$X := \left\{\begin{bmatrix} x_1\\ x_2\end{bmatrix} \in Z_n \;:\; \begin{smallmatrix} x_2\in W_2^n[-\tau,0] \text{ and}\\ x_2(0)=x_1\end{smallmatrix}\right\}.$$

and the operators $\mathcal{B} : \mathbb{R}^r \to Z_n$, $\mathcal{C}_1 : X \to \mathbb{R}^p$ and $\mathcal{C}_2 : X \to Z_q$ are defined as

$$(\mathcal{B}w)(s) := \begin{bmatrix} Bw\\ 0\end{bmatrix}$$

$$\left(\mathcal{C}_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) := \begin{bmatrix} C_1 x_1 \end{bmatrix}, \qquad \left(\mathcal{C}_2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)(s) := \begin{bmatrix} C_2 x_1 \\ C_2 x_2(s) \end{bmatrix}.$$

Note that in this problem formulation we set $D = 0$ as a realistic model of the effect of sensor noise requires the use of an auxiliary state and the resulting $\mathcal{D}$ operator is of integral form - thereby complicating the analysis. Note in addition, that this formulation assumes that the output history has been recorded for a period equal to the delay. While this second assumption is reasonable, it can be relaxed using a relatively simple modification of the theorem.

Now for a given operator $\mathcal{L} : Z_q \to Z_n$, we define the observer dynamics as follows

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= \mathcal{A}\hat{\mathbf{x}}(t) + \mathcal{L}\left(\mathcal{C}_2\hat{\mathbf{x}}(t) - \mathbf{y}(t)\right) \\ \hat{z}(t) &= \mathcal{C}_1\mathbf{x}(t), \qquad z_e(t) = \hat{z}(t) - z(t) \end{aligned} \tag{7}$$

where we constrain $\hat{\mathbf{x}} \in X$. Although we have not yet parameterized the operator $\mathcal{L}$, it is hopefully clear that this formulation is a straightforward extension of the LMI framework, only generalized to the operator setting. In this spirit, we give an operator-theoretic equivalent of Theorem 1.

Theorem 2: Suppose there exist bounded linear operators $\mathcal{P} : Z_n \to Z_n$ and $\mathcal{Z} : Y \to Z_n$, such that $\mathcal{P}$ is coercive and

$$\begin{aligned} &\langle (\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C}_2)\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, (\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C}_2)\mathbf{e}\rangle_{L_2} \\ &- \langle \mathbf{e}, \mathcal{P}\mathcal{B}w\rangle_{L_2} - \langle \mathcal{B}w, \mathcal{P}\mathbf{e}\rangle_{L_2} \\ &- \gamma \|w\|^2 + \frac{1}{\gamma}\|\mathcal{C}_1\mathbf{e}\|^2 < -\epsilon\|\mathbf{e}\|^2 \qquad \forall \mathbf{e} \in X,\ w \in \mathbb{R}^r, \end{aligned}$$

for some $\epsilon > 0$. Then for $\mathcal{L} = \mathcal{P}^{-1}\mathcal{Z}$, $\hat{\mathbf{x}}(0) = \mathbf{x}(0) = 0$ and any $w \in L_2^r$, the solution of Eqns. (6) and (7) satisfies

$$\|z_e\|_{L_2} \le \gamma \|w\|_{L_2}. \tag{8}$$

Proof: Let $\hat{\mathbf{x}}$ and $\mathbf{x}$ satisfy Eqns. (6) and (7). Define $\mathbf{e}(t) = \hat{\mathbf{x}}(t) - \mathbf{x}(t)$. Then $\mathbf{e}(t) \in X$ and by subtracting the corresponding equations in (6) and (7), we obtain

$$\dot{\mathbf{e}}(t) = (\mathcal{A} + \mathcal{L}\mathcal{C}_2)\mathbf{e}(t) - \mathcal{B}w(t).$$

We define the storage function $V(e) = \langle \mathbf{e}, \mathcal{P}\mathbf{e}\rangle \ge \delta\|\mathbf{e}\|^2$ which holds for some $\delta > 0$ since $\mathcal{P}$ is coercive. Then as in the proof of Theorem 1, we obtain

$$\dot{V}(t) - \gamma\|w(t)\|^2 + \frac{1}{\gamma}\|\mathcal{C}_1\mathbf{e}(t)\|^2 < 0$$

for all $[\mathbf{e}(t)\ w(t)] \neq 0$. Integration of this inequality yields

$$V(\mathbf{e}(t)) - V(\mathbf{e}(0)) + \frac{1}{\gamma}\int_0^t \|z_e(s)\|^2 ds \le \gamma \int_0^t \|w(s)\|^2 ds$$

As $V(e(0)) = 0$ and $V(\mathbf{e}(t)) \ge 0$, if we let $t \to \infty$, we see that the above implies (8). ∎

Note that the conditions of the theorem also establish $\lim_{t\to\infty} \mathbf{e}(t) = 0$ when $\lim_{t\to\infty} w(t) = 0$.

V. Parameterizing the Operators and Enforcing Positivity on $Z_{m,n}$

Having formulated the observer synthesis condition using operator inequalities in abstract form in Theorem 2, the next two sections are devoted to using matrices and matrix-valued functions to parameterize the operators $\mathcal{P}$ and $\mathcal{Z}$ and giving constraints which enforce positivity of operators. The first operator we will parameterize is $\mathcal{P}_{\{P,Q,R,S\}} : Z_{m,n} \to Z_{m,n}$

$$\begin{aligned} &\left(\mathcal{P}_{\{P,Q,R,S\}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)(s) \qquad\qquad (9) \\ &:= \begin{bmatrix} P x_1 + \int_{-\tau}^0 Q(\theta) x_2(\theta) d\theta \\ \tau\left(Q(s)^T x_1 + S(s) x_2(s) + \int_{-\tau}^0 R(s,\theta) x_2(\theta) d\theta\right) \end{bmatrix}. \end{aligned}$$

which is defined by matrix $P \in \mathbb{R}^{m\times m}$ and functions $Q : [-\tau, 0] \to \mathbb{R}^{m\times n}$, $S : [-\tau, 0] \to \mathbb{S}^n$, and $R : [-\tau, 0]^2 \to \mathbb{R}^{n\times n}$. Most operators variables discussed in this paper are of this form or some slight generalization thereof. The advantage of such operators are that they are sufficiently general while admitting a relatively simple LMI test for positivity. Specifically, the following theorem is taken from Theorem 9 in [4] and is a slight refinement of the results presented in [22].

Theorem 3: Suppose

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \ge 0$$

and

$$\begin{aligned} P &= M_{11} \cdot \frac{1}{\tau}\int_{-\tau}^0 g(s) ds \\ Q(s) &= \frac{1}{\tau}\left(g(s) M_{12} Y_1(s) + \int_{-\tau_K}^0 g(\eta) M_{13} Y_2(\eta, s) d\eta\right) \\ S(s) &= \frac{1}{\tau} g(s) Y_1(s)^T M_{22} Y_1(s) \\ R(s,\theta) &= g(s) Y_1(s)^T M_{23} Y_2(s,\theta) + g(\theta) Y_2(\theta, s)^T M_{32} Y_1(\theta) \\ &\quad + \int_{-\tau}^0 g(\eta) Y_2(\eta, s)^T M_{33} Y_2(\eta, \theta) d\eta \end{aligned}$$

where with $g(s) \ge 0$ for $s \in [-\tau_K, 0]$, $M_{11} \in \mathbb{S}^m$, $M_{22} \in \mathbb{S}^{d_1}$, $M_{33} \in \mathbb{S}^{d_2}$, $Z_1(s) \in \mathbb{R}^{d_2\times n}$, and $Z_2(s,\theta) \in \mathbb{R}^{d_2\times n}$ for any $d_1, d_2 \in \mathbb{N}$.

If $\mathcal{P} = \mathcal{P}_{\{P,Q,R,S\}}$, then $\mathcal{P} : Z_{m,n} \to Z_{m,n}$ is a bounded linear operator, $\mathcal{P} = \mathcal{P}^*$ and $\mathcal{P}_{\{P,Q,R,S\}} \ge 0$ on $Z_{m,n}$.

Wee take Z_1 and Z_2 to be a monomial basis for degree-d vector-valued polynomials and use $g(s) = 1$ and $g(s) = -s(s+\tau)$. We define the set of functions which define positive operators as

$$\Xi_{d,m,n} := \left\{ \{P,Q,R,S\} : \begin{array}{c} \{P,Q,S,R\} = \{P_1,Q_1,S_1,R_1\} + \{P_2,Q_2,S_2,R_2\}, \\ \text{where } \{P_1,Q_1,S_1,R_1\} \text{ and } \{P_2,Q_2,S_2,R_2\} \text{ satisfy} \\ \text{Thm. 3 with } g = 1 \text{ and } g = -s(s+\tau)\text{, respectively.} \end{array} \right\}$$

VI. Parametrization of the Operators $\mathcal{L}$ and $\mathcal{Z}$

Given our parametrization of $\mathcal{P}$, we treat the operator $\mathcal{L}$ in a similar manner and arrive at the well-known conclusion that the natural generalization of the observer to time-delay systems is NOT itself a time-delay system, but rather a PDE coupled with an ODE. Specifically, if we take the obvious parametrization of $\mathcal{L} : Z_q \to Z_n$ as

$$\begin{aligned} &(\mathcal{L}\mathbf{y})(s) = \\ &\begin{bmatrix} L_1 y_1 + L_2 y_2(-\tau) + \int_{-\tau}^0 L_3(\theta) y_2(\theta) d\theta \\ L_4(s) y_1 + L_5(s) y_2(-\tau) + L_6(s) y_2(s) + \int_{-\tau}^0 L_7(s,\theta) y_2(\theta) d\theta \end{bmatrix} \end{aligned}$$

where $L_1, L_2 \in \mathbb{R}^{n\times q}$, $L_3, L_4, L_5, L_6 : [-\tau, 0] \to \mathbb{R}^{n\times q}$, and $L_2 : [-\tau, 0]^2 \to \mathbb{R}^{n\times q}$, then $\dot{\hat{\mathbf{x}}}(t) = \mathcal{A}\hat{\mathbf{x}}(t) +$

$\mathcal{L}(\mathcal{C}_2\hat{\mathbf{x}}(t) - \mathbf{y}(t)))$ and $\hat{\mathbf{x}} \in X$ implies that the observer can be expressed in the following explicit form

$$\dot{\hat{x}}(t) = A_0\hat{x}(t) + A_1\hat{\phi}(t,-\tau) + L_1\left(C_2\hat{x}(t) - y(t)\right) + L_2\left(C_2\hat{\phi}(t,-\tau) - y(t-\tau)\right) + \int_{-\tau}^{0} L_3(\theta)\left(C_2\hat{\phi}(t,\theta) - y(t+\theta)\right)d\theta$$
$$\partial_t\hat{\phi}(t,s) = \partial_s\hat{\phi}(t,s) + L_4(s)\left(C_2\hat{x}(t) - y(t)\right) + L_5(s)\left(C_2\hat{\phi}(t,-\tau) - y(t-\tau)\right) + L_6(s)\left(C_2\hat{\phi}(t,s) - y(t+s)\right) + \int_{-\tau}^{0} L_7(s,\theta)\left(C_2\hat{\phi}(t,\theta) - y(t+\theta)\right)d\theta, \qquad \hat{\phi}(t,0) = \hat{x}(t) \tag{10}$$

Here $\hat{x}(t)$ represents the current estimate of the present state and $\hat{\phi}(t,s)$ represents the current estimate of the history of the state variable $x(t+s)$ for $s \in [-\tau, 0]$.

Given this structure on $\mathcal{P}$ and $\mathcal{L}$, it can be shown that the parametrization of the operator variable $\mathcal{Z} = \mathcal{P}\mathcal{L}$ is required to have the following form: $\mathcal{Z} : Z_q \to Z_n$ where

$$\mathcal{Z}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}(s) = \begin{bmatrix} Z_1y_1 + Z_2y_2(-\tau) + \int_{-\tau}^{0} Z_3(\theta)y_2(\theta)d\theta \\ \tau\left(Z_4(s)y_1 + Z_5(s)y_2(-\tau) + Z_6(s)y_2(s) + \int_{-\tau}^{0} Z_7(s,\theta)y_2(\theta)d\theta\right) \end{bmatrix} \tag{11}$$

and where the matrices and matrix-valued functions Z_i have the appropriate dimensions.

VII. Reformulation of the Synthesis Condition using $Z_{2n+r,n}$

In this section, we reformulate the conditions of Theorem 2 as a linear operator inequality where all operators are of the form of Equation (9). Specifically, we show that for $e \in X$,

$$\langle(\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C}_2)\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle\mathbf{e}, (\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C}_2)\mathbf{e}\rangle_{L_2} - \langle\mathbf{e}, \mathcal{P}\mathcal{B}w\rangle_{L_2} - \langle\mathcal{B}w, \mathcal{P}\mathbf{e}\rangle_{L_2} - \gamma\|w\|^2 + \frac{1}{\gamma}\|\mathcal{C}_1\mathbf{e}\|^2 = \langle\mathbf{v}, \mathcal{P}_{\{T,U,V,W\}}\mathbf{v}\rangle_{Z_{r+2n,n}}$$

where $\mathbf{v} = \begin{bmatrix} w^T & e_1^T & e_2(-\tau)^T & e_2^T \end{bmatrix}^T \in Z_{r+2n,n}$.

Theorem 4: Suppose there exists a matrix $P \in \mathbb{R}^{n\times n}$, polynomials $Q, R, S : [-\tau, 0] \to \mathbb{R}^{n\times n}$, matrix $Z_1 \in \mathbb{R}^{n\times q}$, polynomials $Z_2, Z_3, Z_4, Z_5, Z_6 : [-\tau, 0] \to \mathbb{R}^{n\times q}$ and polynomial $Z_y : [-\tau, 0] \times [-\tau, 0] \to \mathbb{R}^{n\times q}$ such that

$$\{P - \epsilon I, Q, R, S - \epsilon I\} \in \Xi_{d,n,n} \quad \text{and} \quad -\{T, U, V, W\} \in \Xi_{d,2n+r,n}$$

where

$$T := \begin{bmatrix} 0 & * & * \\ 0 & PA_0 + A_0^TP + Q(0) + Q(0)^T + S(0) & * \\ 0 & A_1^TP - Q(-\tau)^T & -S(-\tau) \end{bmatrix} + \begin{bmatrix} -\frac{\gamma}{\tau}I & * & * \\ -PB & \frac{1}{\gamma\tau}C_1^TC_1 + Z_1C_2 + C_2^TZ_1^T + \epsilon I & * \\ 0 & C_2^TZ_2^T & 0 \end{bmatrix}$$
$$U := \begin{bmatrix} -B^TQ(s) \\ C_2^TZ_4(s)^T + Z_3(s)C_2 \\ C_2^TZ_5(s)^T \end{bmatrix} + \begin{bmatrix} 0 \\ A_0^TQ(s) + R(s,0)^T - \dot{Q}(s) \\ A_1^TQ(s) - R(s,-\tau)^T \end{bmatrix}$$
$$V := -\dot{S}(s) + Z_6(s)C_2 + C_2^TZ_6(s)^T + \frac{\epsilon}{\tau}I$$
$$W := -R_\theta(s,\theta) - R_s(\theta,s)^T + Z_7(s,\theta)C_2 + C_2^TZ_7(\theta,s)^T.$$

Then if $\mathcal{L} = \mathcal{P}_{\{P,Q,R,S\}}^{-1}\mathcal{Z}$, where

$$\mathcal{Z}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}(s) = \begin{bmatrix} z_1 \\ \tau z_2 \end{bmatrix}$$
$$z_1 = Z_1y_1 + Z_2y_2(-\tau) + \int_{-\tau}^{0} Z_3(\theta)y_2(\theta)d\theta$$
$$z_2(s) = Z_4(s)y_1 + Z_5(s)y_2(-\tau) + Z_6(s)y_2(s) + \int_{-\tau}^{0} Z_7(s,\theta)y_2(\theta)d\theta,$$

Eqns (6) coupled with Eqns. (7) satisfy $\|z_e\|_{L_2} < \gamma\|w\|_{L_2}$.

Proof: First we note that

$$\langle\mathbf{e}, \mathcal{P}_{\{P,Q,R,S\}}\mathbf{e}\rangle = \langle\mathbf{e}, \mathcal{P}_{\{P-\epsilon I,Q,R,S-\epsilon I\}}\mathbf{e}\rangle + \epsilon\|\mathbf{e}\|_{Z_n}^2 \geq \epsilon\|\mathbf{e}\|_{Z_n}^2$$

Hence $\mathcal{P}_{\{P,Q,R,S\}}$ is coercive. Next, we show that for $\mathbf{e} \in X$,

$$\langle(\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C}_2)\mathbf{e}, \mathbf{e}\rangle + \langle\mathbf{e}, (\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C}_2)\mathbf{e}\rangle - \langle\mathbf{e}, \mathcal{P}\mathcal{B}w\rangle - \langle\mathcal{B}w, \mathcal{P}\mathbf{e}\rangle - \gamma\|w\|^2 + \frac{1}{\gamma}\|\mathcal{C}_1\mathbf{e}\|^2 + \epsilon\|\mathbf{e}\|_{Z_n}^2 = \left\langle \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2 \end{bmatrix}, \mathcal{P}_{\{T,U,V,W\}}\begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2 \end{bmatrix} \right\rangle \leq 0$$

and apply Theorem 2. We do this in parts by reformulating each element in isolation and then summing up. Specifically, we have the following parts

$$\langle\mathcal{P}\mathcal{A}\mathbf{e}, \mathbf{e}\rangle + \langle\mathbf{e}, \mathcal{P}\mathcal{A}\mathbf{e}\rangle + \langle\mathcal{Z}\mathcal{C}_2\mathbf{e}, \mathbf{e}\rangle + \langle\mathbf{e}, \mathcal{Z}\mathcal{C}_2\mathbf{e}\rangle - \langle\mathbf{e}, \mathcal{P}\mathcal{B}w\rangle - \langle\mathcal{B}w, \mathcal{P}\mathbf{e}\rangle - \gamma\|w\|^2 + \frac{1}{\gamma}\|\mathcal{C}_1\mathbf{e}\|^2 + \epsilon\|\mathbf{e}\|_{Z_n}^2$$

First, we have

$$-\gamma\|w\|^2 + \frac{1}{\gamma}\|\mathcal{C}_1\mathbf{e}\|^2 + \epsilon\|\mathbf{e}\|_{Z_n}^2 = -\gamma w^Tw + \frac{1}{\gamma}(C_1e_1)^TC_1e_1 + \epsilon\|\mathbf{e}\|_{Z_n}^2$$
$$= \int_{-\tau}^{0} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2(s) \end{bmatrix}^T \begin{bmatrix} -\frac{\gamma}{\tau}I & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma\tau}C_1^TC_1 + \epsilon I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon I \end{bmatrix} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2(s) \end{bmatrix} ds$$

Since $(\mathcal{PB}w)(s) = \begin{bmatrix} PBw \\ \tau Q(s)^T Bw \end{bmatrix}$, we have

$$\begin{aligned}
&\langle \mathbf{e}, \mathcal{PB}w \rangle \\
&= \int_{-\tau}^{0} e_1^T PBwds + \int_{-\tau}^{0} e_2(s)^T \tau Q(s)^T Bwds \\
&= \int_{-\tau}^{0} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2(s) \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ PB & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \tau Q(s)^T B & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2(s) \end{bmatrix} ds
\end{aligned}$$

Next we find

$$\begin{aligned}
&\mathcal{ZC}_2 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}(s) := \begin{bmatrix} v_1 \\ \tau v_2(s) \end{bmatrix} \\
&v_1 = Z_1 C_2 e_1 + Z_2 C_2 e_2(-\tau) + \int_{-\tau}^{0} Z_3(\theta) C_2 e_2(\theta) d\theta \\
&v_2(s) = Z_4(s) C_2 e_1 + Z_5(s) C_2 e_2(-\tau) + Z_6(s) C_2 e_2(s) \\
&\qquad + \int_{-\tau}^{0} Z_7(s,\theta) C_2 e_2(\theta) d\theta.
\end{aligned}$$

Thus

$$\begin{aligned}
&\langle \mathbf{e}, \mathcal{ZC}_2 \mathbf{e} \rangle_{L_2} \\
&= \tau e_1^T \left(Z_1 C_2 e_1 + Z_2 C_2 e_2(-\tau) + \int_{-\tau}^{0} Z_3(\theta) C_2 e_2(\theta) d\theta \right) \\
&\quad + \tau \int_{-\tau}^{0} e_2(s)^T \Big(Z_4(s) C_2 e_1 + Z_5(s) C_2 e_2(-\tau) \\
&\qquad + Z_6(s) C_2 e_2(s) + \int_{-\tau}^{0} Z_7(s,\theta) C_2 e_2(\theta) d\theta \Big) \\
&= \int_{-\tau}^{0} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2(s) \end{bmatrix}^T Y_1(s) \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2(s) \end{bmatrix} ds \\
&+ \tau \int_{-\tau}^{0} \int_{-\tau}^{0} e_2(s)^T Z_7(s,\theta) C_2 e_2(\theta) d\theta
\end{aligned}$$

where

$$Y_1(s) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & Z_1 C_2 & Z_2 C_2 & \tau Z_3(s) C_2 \\ 0 & 0 & 0 & 0 \\ 0 & \tau Z_4(s) C_2 & \tau Z_5(s) C_2 & \tau Z_6(s) C_2 \end{bmatrix}$$

Likewise,

$$\begin{aligned}
&\langle \mathcal{ZC}_2 \mathbf{e}, \mathbf{e} \rangle_{L_2} = \int_{-\tau}^{0} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2(s) \end{bmatrix}^T Y_1(s)^T \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2(s) \end{bmatrix} ds \\
&+ \tau \int_{-\tau}^{0} \int_{-\tau}^{0} e_2(s)^T C_2^T Z_7(\theta, s)^T e_2(\theta) d\theta
\end{aligned}$$

Combining all terms, we have

$$\begin{aligned}
&\langle \mathcal{ZC}_2 \mathbf{e}, \mathbf{e} \rangle + \langle \mathbf{e}, \mathcal{ZC}_2 \mathbf{e} \rangle - \langle \mathbf{e}, \mathcal{PB}w \rangle - \langle \mathcal{B}w, \mathcal{P}\mathbf{e} \rangle \\
&\qquad - \gamma \|w\|^2 + \frac{1}{\gamma} \|\mathcal{C}_1 \mathbf{e}\|^2 + \epsilon \|\mathbf{e}\|_{Z_n}^2 \\
&= \int_{-\tau}^{0} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2(s) \end{bmatrix}^T \begin{bmatrix} T_1(s) & \tau U_1(s) \\ \tau U_1(s)^T & \tau V_1(s) \end{bmatrix} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2(s) \end{bmatrix} ds \\
&\quad + \tau \int_{-\tau}^{0} \int_{-\tau}^{0} e_2(s)^T W_1(s,\theta) e_2(\theta) d\theta
\end{aligned}$$

where

$$\begin{aligned}
T_1(s) &= \begin{bmatrix} -\frac{\gamma}{\tau} I & * & * \\ -PB & \frac{1}{\gamma\tau} C_1^T C_1 + Z_1 C_2 + C_2^T Z_1^T + \epsilon I & * \\ 0 & C_2^T Z_2^T & 0 \end{bmatrix} \\
U_1(s) &= \begin{bmatrix} -B^T Q(s) \\ C_2^T Z_4(s)^T + Z_3(s) C_2 \\ C_2^T Z_5(s)^T \end{bmatrix} \\
V_1(s) &= Z_6(s) C_2 + C_2^T Z_6(s)^T + \frac{\epsilon}{\tau} I \\
W_1(s,\theta) &= Z_7(s,\theta) C_2 + C_2^T Z_7(\theta, s)^T.
\end{aligned}$$

The expansion of $\langle \mathcal{PA}\mathbf{e}, \mathbf{e} \rangle + \langle \mathbf{e}, \mathcal{PA}\mathbf{e} \rangle$ can be found in, e.g. [23], and is therefore omitted for brevity.

$$\begin{aligned}
&\langle \mathcal{PA}e, e \rangle + \langle e, \mathcal{PA}e \rangle \\
&= \int_{-\tau}^{0} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2(s) \end{bmatrix}^T \begin{bmatrix} T_2(s) & \tau U_2(s) \\ \tau U_2(s)^T & \tau V_2(s) \end{bmatrix} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2(s) \end{bmatrix} ds \\
&\quad + \tau \int_{-\tau}^{0} \int_{-\tau}^{0} e_2(s)^T W_2(s,\theta) e_2(\theta) d\theta
\end{aligned}$$

where

$$\begin{aligned}
T_2(s) &= \begin{bmatrix} 0 & * & * \\ 0 & PA_0 + A_0^T P + Q(0) + Q(0)^T + S(0) & * \\ 0 & A_1^T P - Q(-\tau)^T & -S(-\tau) \end{bmatrix} \\
U_2(s) &= \begin{bmatrix} 0 \\ A_0^T Q(s) + R(s,0)^T - \dot{Q}(s) \\ A_1^T Q(s) - R(s,-\tau)^T \end{bmatrix} \\
V_2(s) &= -\dot{S}(s) \\
W_2(s,\theta) &= -R_\theta(s,\theta) - R_s(s,\theta).
\end{aligned}$$

Since $T = T_1 + T_2$, $U = U_1 + U_2$, $V = V_1 + V_2$, and $W = W_1 + W_2$, we conclude that

$$\begin{aligned}
&\langle (\mathcal{PA} + \mathcal{ZC}_2)\mathbf{e}, \mathbf{e} \rangle + \langle \mathbf{e}, (\mathcal{PA} + \mathcal{ZC}_2)\mathbf{e} \rangle \\
&- \langle \mathbf{e}, \mathcal{PB}w \rangle - \langle \mathcal{B}w, \mathcal{P}\mathbf{e} \rangle - \gamma \|w\|^2 + \frac{1}{\gamma} \|\mathcal{C}_1 \mathbf{e}\|^2 + \epsilon \|\mathbf{e}\|_{Z_n}^2 \\
&= \left\langle \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2 \end{bmatrix}, \mathcal{P}_{\{T,U,V,W\}} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2 \end{bmatrix} \right\rangle \le 0
\end{aligned}$$

and hence the conditions of Theorem 2 are satisfied. ■

Numerical implementation of the conditions of Theorem 4 using the DELAYTOOLS mod pack for SOSTOOLS is relatively straightforward. An implementation of this test, the controller reconstruction, and simulations keyed to this paper can be found at [24].

VIII. Inverting the Operator

Now that we have an observer synthesis condition, we address the question of reconstructing the observer which attains the desired H_∞ gain condition. Recall this observer is of the form $\mathcal{L} = \mathcal{P}^{-1}\mathcal{Z}$. Clearly, we need an expression for $\mathcal{P}^{-1}$. Fortunately, there exist a closed-form expression for this inverse.

Lemma 5: Suppose $\mathcal{P} := \mathcal{P}_{\{P,Q,R,S\}}$ and $Q(s) = HZ(s)$ and $R(s,\theta) = Z(s)^T \Gamma Z(\theta)$ for some Z. Then if the matrices listed below are well defined,

$$\mathcal{P}^{-1} := \begin{bmatrix} \hat{P}x_1 + \frac{1}{\tau}\int_{-\tau}^0 \hat{Q}(\theta)x_2(\theta)d\theta \\ \hat{Q}(s)^T x_1 + \frac{1}{\tau}\hat{S}(s)x_2(s) + \frac{1}{\tau}\int_{-\tau}^0 \hat{R}(s,\theta)x_2(\theta)d\theta \end{bmatrix}$$

where

$$K = \int_{-\tau}^0 Z(s)S^{-1}(s)Z(s)^T ds,$$
$$T = (I + K\Gamma - KH^T P^{-1}H)^{-1},$$
$$\hat{H} = -P^{-1}HT, \qquad \hat{P} = \left[I + P^{-1}HTKH^T\right]P^{-1},$$
$$\hat{\Gamma} = \left[T^T H^T P^{-1} H - \Gamma\right](I + K\Gamma)^{-1},$$
$$\hat{Z}(s) = Z(s)S^{-1}(s), \qquad \hat{Q} = \hat{H}\hat{Z}(\theta),$$
$$\hat{S}(s) = S(s)^{-1}, \qquad \hat{R}(s,\theta) = \hat{Z}(s)^T\hat{\Gamma}\hat{Z}(\theta).$$

Proof: The proof is a minor modification of the result in [25]. ■

If Q and R are polynomials and Z is the monomial basis, the representations $Q(s) = HZ(s)$ and $R(s,\theta) = Z(s)^T\Gamma Z(\theta)$ are unique.

IX. Constructing the Observer Gains

Armed with this inverse, we may define the observer gains.

Lemma 6: If $\mathcal{L} = \mathcal{P}^{-1}_{\{P,Q,R,S\}}\mathcal{Z}$ where $\mathcal{Z}$ is as in Eqn. (11) and

$$\mathcal{P}^{-1} := \begin{bmatrix} \hat{P}x_1 + \frac{1}{\tau}\int_{-\tau}^0 \hat{Q}(\theta)x_2(\theta)d\theta \\ \hat{Q}(s)^T x_1 + \frac{1}{\tau}\hat{S}(s)x_2(s) + \frac{1}{\tau}\int_{-\tau}^0 \hat{R}(s,\theta)x_2(\theta)d\theta \end{bmatrix}$$

then

$$\mathcal{L}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}(s) = \begin{bmatrix} L_1 y_1 + L_2 y_2(-\tau) + \int_{-\tau}^0 L_3(\theta)y_2(\theta)d\theta \\ L_4(s)y_1 + L_5(s)y_2(-\tau) + L_6(s)y_2(s) + \int_{-\tau}^0 L_7(s,\theta)y_2(\theta)d\theta \end{bmatrix}$$

where

$$L_1 = \hat{P}Z_1 + \int_{-\tau}^0 \hat{Q}(\theta)Z_4(\theta)d\theta$$
$$L_2 = \hat{P}Z_2 + \int_{-\tau}^0 \hat{Q}(\theta)Z_5(\theta)d\theta$$
$$L_3(\theta) = \hat{P}Z_3(\theta) + \hat{Q}(\theta)Z_6(\theta) + \int_{-\tau}^0 \hat{Q}(s)Z_7(s,\theta)ds$$
$$L_4(s) = \hat{Q}(s)^T Z_1 + \hat{S}(s)Z_4(s) + \int_{-\tau}^0 \hat{R}(s,\theta)Z_4(\theta)d\theta$$
$$L_5(s) = \hat{Q}(s)^T Z_2 + \hat{S}(s)Z_5(s) + \int_{-\tau}^0 \hat{R}(s,\theta)Z_5(\theta)d\theta$$
$$L_6(s) = \hat{S}(s)Z_6(s)$$
$$L_7(s,\theta) = \hat{Q}(s)^T Z_3(\theta) + \hat{S}(s)Z_7(s,\theta) + \hat{R}(s,\theta)Z_6(\theta) + \int_{-\tau}^0 \hat{R}(s,\xi)Z_7(\xi,\theta)d\xi.$$

Proof: Through a simple expansion, we have

$$\mathcal{L}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}(s) = \mathcal{P}^{-1}_{\{P,Q,R,S\}}\mathcal{Z} = \begin{bmatrix} l_1 \\ l_2(s) \end{bmatrix}$$

where

$$\begin{aligned} l_1 &= \hat{P}\left(Z_1 y_1 + Z_2 y_2(-\tau) + \int_{-\tau}^0 Z_3(\theta)y_2(\theta)d\theta\right) \\ &\quad + \int_{-\tau}^0 \hat{Q}(\theta)\Big(Z_4(\theta)y_1 + Z_5(\theta)y_2(-\tau) \\ &\qquad + Z_6(\theta)y_2(\theta) + \int_{-\tau}^0 Z_7(\theta,\xi)y_2(\xi)d\xi\Big)d\theta \\ &= \left(\hat{P}Z_1 + \int_{-\tau}^0 \hat{Q}(\theta)Z_4(\theta)d\theta\right)y_1 \\ &\quad + \left(\hat{P}Z_2 + \int_{-\tau}^0 \hat{Q}(\theta)Z_5(\theta)d\theta\right)y_2(-\tau) \\ &\quad + \int_{-\tau}^0 \left(\hat{P}Z_3(\theta) + \hat{Q}(\theta)Z_6(\theta) + \int_{-\tau}^0 \hat{Q}(s)Z_7(s,\theta)ds\right)y_2(\theta)d\theta \\ &= L_1 y_1 + L_2 y_2(-\tau) + \int_{-\tau}^0 L_3(\theta)y_2(\theta)d\theta \end{aligned}$$

where

$$L_1 = \hat{P}Z_1 + \int_{-\tau}^0 \hat{Q}(\theta)Z_4(\theta)d\theta$$
$$L_2 = \hat{P}Z_2 + \int_{-\tau}^0 \hat{Q}(\theta)Z_5(\theta)d\theta$$
$$L_3(\theta) = \hat{P}Z_3(\theta) + \hat{Q}(\theta)Z_6(\theta) + \int_{-\tau}^0 \hat{Q}(s)Z_7(s,\theta)ds.$$

Likewise

$$\begin{aligned} l_2(s) &= \left(\hat{Q}(s)^T Z_1 + \hat{S}(s)Z_4(s) + \int_{-\tau}^0 \hat{R}(s,\theta)Z_4(\theta)d\theta\right)y_1 \\ &\quad + \left(\hat{Q}(s)^T Z_2 + \hat{S}(s)Z_5(s) + \int_{-\tau}^0 \hat{R}(s,\theta)Z_5(\theta)d\theta\right)y_2(-\tau) \\ &\quad + \hat{S}(s)Z_6(s)y_2(s) + \int_{-\tau}^0 \Big(\hat{Q}(s)^T Z_3(\theta) + \hat{S}(s)Z_7(s,\theta) \\ &\qquad + \hat{R}(s,\theta)Z_6(\theta) + \int_{-\tau}^0 \hat{R}(s,\xi)Z_7(\xi,\theta)d\xi\Big)y_2(\theta)d\theta \\ &= L_4(s)y_1 + L_5(s)y_2(-\tau) + L_6(s)y_2(s) + \int_{-\tau}^0 L_7(s,\theta)y_2(\theta)d\theta \end{aligned}$$

where

$$L_4(s) = \hat{Q}(s)^T Z_1 + \hat{S}(s)Z_4(s) + \int_{-\tau}^0 \hat{R}(s,\theta)Z_4(\theta)d\theta$$
$$L_5(s) = \hat{Q}(s)^T Z_2 + \hat{S}(s)Z_5(s) + \int_{-\tau}^0 \hat{R}(s,\theta)Z_5(\theta)d\theta$$
$$L_6(s) = \hat{S}(s)Z_6(s)$$
$$L_7(s,\theta) = \hat{Q}(s)^T Z_3(\theta) + \hat{S}(s)Z_7(s,\theta) + \hat{R}(s,\theta)Z_6(\theta) + \int_{-\tau}^0 \hat{R}(s,\xi)Z_7(\xi,\theta)d\xi$$

■

Theorem 7: Suppose that P, Q, R, S satisfy the conditions of Theorem 4 for $\gamma > 0$, $\hat{P}, \hat{Q}, \hat{R}, \hat{S}$ satisfy the conditions of Lemma 5, and $L_1, L_2, L_3, L_4, L_5, L_6, L_7$ satisfy the conditions of Lemma 6. Then if

$$\begin{aligned}
\dot{x}(t) &= A_0 x(t) + A_1 x(t-\tau) + Bw(t), \qquad y(t) = C_2 x(t) \\
\dot{\hat{x}}(t) &= A_0 \hat{x}(t) + A_1 \hat{\phi}(t,-\tau) + L_1 \left(C_2 \hat{x}(t) - y(t)\right) \\
&\quad + L_2 \left(C_2 \hat{\phi}(t,-\tau) - y(t-\tau)\right) \\
&\qquad + \int_{-\tau}^{0} L_3(\theta) \left(C_2 \hat{\phi}(t,\theta) - y(t+\theta)\right) d\theta \\
\partial_t \hat{\phi}(t,s) &= \partial_s \hat{\phi}(t,s) + L_4(s) \left(C_2 \hat{x}(t) - y(t)\right) \\
&\quad + L_5(s) \left(C_2 \hat{\phi}(t,-\tau) - y(t-\tau)\right) \\
&\qquad + L_6(s) \left(C_2 \hat{\phi}(t,s) - y(t+s)\right) \\
&\qquad\quad + \int_{-\tau}^{0} L_7(s,\theta) \left(C_2 \hat{\phi}(t,\theta) - y(t+\theta)\right) d\theta \\
z_e(t) &= C_1 (x(t) - \hat{x}(t)), \qquad \hat{\phi}(t,0) = \hat{x}(t) \qquad (12)
\end{aligned}$$

we have that $\|z_e\|_{L_2} \le \gamma \|w\|_{L_2}$.
Note that if we constrain $Q = 0$ and $Z_3 = Z + 4 = Z_5 = Z_6 = Z_7 = 0$, we recover an observer with corrections only to the present state.

X. Numerical Implementation

There are several numerical aspects which must be considered when constructing and implementing the observers defined above. The first is numerical computation of the inverse. The second is real-time simulation of the observer dynamics.

A. *Computing the Inverse*

There are several steps to computing the inverse which we address. The steps indicated here are contained in the m-file `P_PQRS_Inverse_joint_sep` in the Matlab package associated with this paper. The first step is to calculate the H and Γ matrices. Fortunately, both are uniquely defined and each element of these matrices is defined by a single coefficient in $Q(s)$ and $R(s,\theta)$. By calculating which elements of the matrix map to which coefficients, we may construct the matrices with minimal effort (See Code available from [24]).

The next step is to Calculate $S^{-1}(s)$. This is done by evaluating $S(s)$ at a number of discrete points, inverting the matrix at each point, and using a polynomial fit. In our code, we use the matlab function `polyfit`. Finally, the remaining numerical issue is calculation of K. For this we use the Matlab numerical integration tool `integral` which is based on a trapezoidal method. Given these values, we may readily obtain the observer gain parameters.

B. *Implementation of the Observer*

The PDE governing the observer dynamics is a generalization of the transport equation. Therefore, we use a central difference approximation based on a number of lumped states, N. Typically 20 states is more than sufficient to obtain accurate results. In the code associated with this paper, we verified the observers and H_∞-gain bounds using several different methods. A complete description is not given here due to space limitation, however, and hence we refer to that code for additional details.

XI. Numerical Examples

Significant care must be taken in the choice of numerical examples to correctly demonstrate the advantages and limitations of the proposed observer design. Specifically, most examples in the literature are 2-state and have disturbance inputs of the form $B = \begin{bmatrix}1\\1\end{bmatrix}$. That is, a single disturbance affects both states equally. In such cases our observers can achieve very small H_∞ norms – typically less than .001 (we do not test smaller gains due to potential numerical difficulties in verification). We can achieve these gains because the observers we design are highly optimized and can indirectly observe the disturbance through the measured output and use this information to correct the state estimate. However, we feel that this approach is not fair or realistic and hence use independent channels to disturb all states. For this reason, several of the examples given below have been modified from their original form. Because most codes are not available online, the result is that we only include numerical comparisons for the results in [21], for which we were able to reproduce the tests given in that paper. However, the readers should bear in mind that using the original systems and results from, e.g. [12], [26], the observers in this paper improve the achieved H_∞ gains by several orders of magnitude (Specifically, the H_∞ gains using our algorithm can be made *arbitrarily* ($<$ .001) small) and their omission is not due to poor performance with respect to these earlier works.

a) Estimating H_∞ Gain: In our first example, we show that the results of this paper are not conservative with respect to computing H_∞ gain of a time-delay system. Specifically, if the observed system is stable and $C_2 = 0$, then the observer synthesis condition yields the H_∞ gain of the closed-loop system. To illustrate this, we consider a simple system for which we estimate the true H_∞ gain graphically using the bode plot. Specifically,

$$\dot{x}(t) = -x(t) - x(t-\tau) + w(t), \qquad z(t) = x(t).$$

By graphical analysis of the Bode plot, for $\tau = 1$, we estimate the H_∞ gain to be .8913. Using an epsilon of .001 and degree 2 polynomials, the conditions of Theorem 1 are satisfied for $\gamma = .8915$. This indicates our estimate on H_∞ gain is accurate.

b) Numerical Example 1: In this example, we consider the unstable system

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix}-3 & 4\\ 2 & 0\end{bmatrix} x(t) + \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} x(t-\tau) + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} w(t), \\
y(t) &= \begin{bmatrix}0 & 7\end{bmatrix} x(t), \quad z(t) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} x(t)
\end{aligned}$$

Applying the Ricatti approach in [16] with $\epsilon = .001$ we obtain a L_2-gain of $\gamma = .580$. Applying the conditions of Theorem 1, we obtain an L_2-gain of .236. Of all the systems we tested, this one showed the least improvement in performance.

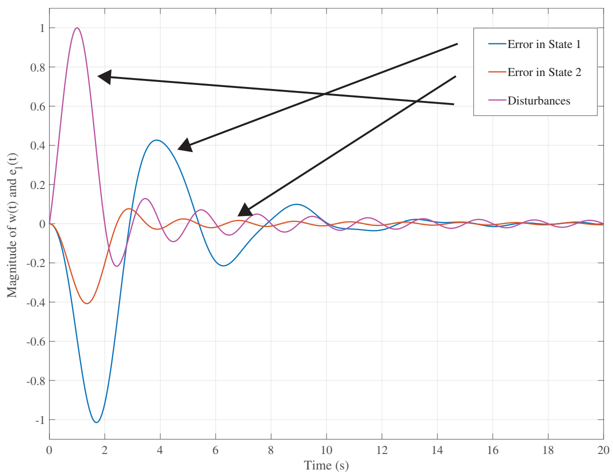

Fig. 1. A Matlab simulation of the error dynamics of System 13 coupled with the observer from Theorem 1 with gain 2.33 and delay $\tau = 1s$. The image displays $w(t)$ and $e_1(t) = \hat{x}(t) - x(t)$.

c) Numerical Example 2: In this example, we consider the following unstable system which is modified from the result in [12].

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -1 \\ 0 & -.9 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t),$$
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t), \quad z(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \tag{13}$$

Using the original system with $\tau = 1$, a closed-loop gain of 22.8 was obtained in [12]. For this problem, the Ricatti approach in [16] was infeasible for any value of gain. Applying the conditions of Theorem 1, we obtained a closed-loop gain of 2.33 using polynomials of degree 4. A Simulation of the error and disturbance dynamics is shown in Figure 1. Note that only the values of $w(t)$ and $e_1(t) = \hat{x}(t) - x(t)$ are shown in this figure. The input is a sinc function and the numerically calculated L_2 gain for this observer using the sinc function is 1.186.

XII. Conclusion

We have proposed an LMI approach to H_∞-optimal observer design for systems with time delay. These observers correct both the estimates of present state and history. Given a solution to the LMI, the observer gains can be reconstructed using algebraic techniques and implemented using discretization.

The Matlab code associated with this paper performs all these tasks and is freely available online. The numerical testing and validation indicates little if any conservatism in the H_∞ bound. The observers in this paper outperform existing observers, often by several orders of magnitude – to the extent that new test cases had to be created to fully understand the limitations of the approach.

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